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MATRIX TRANSFORMS OF λ-BOUNDEDNESS DOMAINS OF THE ZWEIER METHOD

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Abstract. In this paper, we find necessary and sufficient conditions for the Zweier matrix method $Z_{1/2}$ to be transform from the spaces of λ-bounded and λ-convergent sequences into the spaces of μ-bounded and μ-convergent sequences, where λ and μ are monotonically increasing sequences with positive entries (i.e. speeds). Also we find necessary and sufficient conditions for a matrix M to be transform from the λ-boundedness domain of $Z_{1/2}$ into the μ-boundedness domain of a triangular matrix method B. In addition, we introduce one class of multiplicative matrices M satisfying these necessary and sufficient conditions.

Keywords: Matrix transforms, convergence and boundedness with speed, Zweier method, factorable matrices.

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1. Introduction

Let ω be the set of all sequences over real or complex numbers, and X, Y be some subsets of ω. Let $A = (a_{nk})$ be a matrix with real or complex entries and

$$A_n x := \sum_k a_{nk} x_k, \ Ax := (A_n x)$$

for every $x = (x_k) \in \omega$. Throughout this chapter we assume that all indices and summation indices run from 0 to ∞ unless otherwise specified. If $Ax \in Y$ for every $x = (x_k) \in X$, we write $A \in (X, Y)$. In that case we say that A transforms X into Y. A sequence $x \in \omega$ is said to be A-summable (or summable by the summability method A) if the sequence Ax is convergent. A method A is called regular if

$$\lim_n A_n x = \lim_k x_k$$

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for every $x = (x_k) \in c$, where c is the set of all convergent sequences. The set of all A-summable sequences is denoted by c_A. A method $A = (a_{nk})$ is said to be lower triangular if $a_{nk} = 0$ for $k > n$, and normal if A is lower triangular and $a_{nn} \neq 0$ for every n.

In [9]-[10] Kangro introduced the notions of convergence and boundedness with speed. Let $\lambda = (\lambda_k)$ be a monotonically increasing sequence with $\lambda_k > 0$ if not specified otherwise. A convergent sequence $x := (\xi_k)$ with

$$\lim_k \xi_k := \varsigma \text{ and } v_k := \lambda_k (\xi_k - \varsigma)$$

is called bounded with the speed λ (shortly, λ-bounded) if $v_k = O(1)$, and convergent with the speed λ (shortly, λ-convergent) if the limit $\lim_k v_k$ exists and finite. The set of all λ-bounded sequences we denote by m^λ and the set of all λ-convergent sequences by c^λ. It is easy to see that $c^\lambda \subset m^\lambda \subset c$, and if $\lambda_k = O(1)$, then $c^\lambda = m^\lambda = c$. A sequence x is said to be A^λ-bounded if $Ax \in m^\lambda$. The set of all A^λ-bounded sequences we denote by m_A^λ. Of course, $m_A^\lambda \subset c_A$, and if λ is a bounded sequence, then $m_A^\lambda = c_A$. An overview on convergence and boundedness with speed can be found in [2] and [11]. Some recent works on characterization of some matrix classes involving some sets of difference sequences with speed can be found in [7]. Also, one may refer to [6] and [8] for more recent topics on the subject.

The Zweier method $Z_{1/2}$ is defined by the lower triangular matrix $A = (a_{nk}$, where (see [5], p. 14) $a_{00} = 1/2$ and

$$a_{nk} = \begin{cases} \frac{1}{2}, & \text{if } k = n-1 \text{ and } k = n; \\ 0, & \text{if } k < n-1 \end{cases}$$

for $n \geq 1$. The method $A = Z_{1/2}$ is regular (see [5], p. 49).

In this paper, we find necessary and sufficient conditions for $Z_{1/2} \in (m^\lambda, m^\mu)$, $Z_{1/2} \in (c^\lambda, c^\mu)$ and $Z_{1/2} \in (c^\lambda, m^\mu)$, where $\mu = (\mu_k)$ is an another speed; i.e., a monotonically increasing sequence with $\mu_k > 0$. Also we present necessary and sufficient conditions for $M \in (m^\lambda_{Z_{1/2}}, m^\mu_B)$, where B is a triangular matrix method and M is an arbitrary matrix with real and complex entries.

The paper has been organized as follows. In Section 2, some auxiliary results have been introduced. In Section 3, necessary and sufficient conditions for $Z_{1/2} \in (m^\lambda, m^\mu)$, $Z_{1/2} \in (c^\lambda, c^\mu)$ and $Z_{1/2} \in (c^\lambda, m^\mu)$ have been studied. In Section 4, necessary and sufficient conditions for $M \in (m^\lambda_{Z_{1/2}}, m^\mu_B)$ have been found and one class of multiplicative matrices M satisfying these conditions have been presented.

2. Auxiliary results

Let $A = (a_{nk})$ be a matrix with real or complex entries, $e := (1, 1, ...)$, $e^k := (0, ..., 0, 1, 0, ...)$ (where 1 is in the k-th position), $\lambda := (\lambda_k)$, $\mu := (\mu_k)$ monotonically increasing sequences with $\lambda_k > 0$, $\mu_k > 0$ and $\lambda^{-1} := 1/\lambda_k$.

Lemma 2.1 (see [2], p. 159-160 or [9], Theorem 1). *A method* $A = (a_{nk}) \in \left(m^\lambda, m^\mu\right)$ *if and only if*

$$\lim_n a_{nk} := \delta_k; \tag{1}$$

$$Ae \in m^\mu, \tag{2}$$

$$\sum_k \frac{|a_{nk}|}{\lambda_k} = O(1), \tag{3}$$

$$\mu_n \sum_k \frac{|a_{nk} - \delta_k|}{\lambda_k} = O(1). \tag{4}$$

Besides, if $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then $O(1)$ in condition (4) it is necessary to replace by $o(1)$.

Lemma 2.2 (see [2], p. 161-162 or [10], Theorem 1)**.** *A method $A = (a_{nk}) \in \left(c^\lambda, c^\mu\right)$ if and only if conditions (2.3) and (2.4) are fulfilled and*

$$Ae^k \in c^\mu, \tag{5}$$

$$Ae \in c^\mu, \tag{6}$$

$$A\lambda^{-1} \in c^\mu. \tag{7}$$

If $A \in \left(c^\lambda, c^\mu\right)$, then

$$\lim_n \mu_n(A_n x - \phi) = \sum_k a_k^{\lambda,\mu}(v_k - \nu) + \lim_n \mu_n(\mathfrak{A}_{\mathrm{n}} - \delta)\varsigma + \lim_n \mu_n\left(\sum_k \frac{a_{nk}}{\lambda_k} - a^\lambda\right)\nu, \tag{8}$$

where

$$\phi := \lim_n A_n x,\ \nu := \lim_k v_k,\ \mathfrak{A}_n := \sum_k a_{nk},\ \delta := \lim_n \mathfrak{A}_n$$

and

$$a^\lambda := \lim_n \sum_k \frac{a_{nk}}{\lambda_k};\ a_k^{\lambda,\mu} := \lim_n \mu_n \frac{a_{nk} - \delta_k}{\lambda_k}.$$

Lemma 2.3 (see [2], Exercise 8.3 or [11], p. 138)**.** *A method $A = (a_{nk}) \in \left(c^\lambda, m^\mu\right)$ if and only if $A \in \left(m^\lambda, m^\mu\right)$.*

Let further $A = (a_{nk})$ be a normal matrix method with its inverse $A^{-1} := (\eta_{nk})$, $B = (b_{nk})$ a triangular method and $M = (m_{nk})$ an arbitrary matrix, Throughout the paper we use the following notations:

$$h_{jl}^n := \sum_{k=l}^{l+j} m_{nk}\eta_{kl},$$

$G = (g_{nk}) = BM$, that is,

$$g_{nk} := \sum_{l=0}^{n} b_{nl} m_{lk},$$

$$\gamma_{nl}^r = \sum_{k=l}^{l+r} g_{nk}\eta_{kl},$$

and

$$\gamma_{nl} := \lim_r \gamma_{nl}^r \text{ (if the finite limits exist).}$$

Lemma 2.4 (see [2], Proposition 8.1 or [4], Lemma 1)**.** *The transformation $y = Mx$ exists for every $x \in m_A^\lambda$ if and only if*

$$there\,exist\,finite\,limits\quad \lim_j h_{jl}^n := h_{nl}, \tag{9}$$

$$\text{there exist finite limits } \lim_j \sum_{l=0}^{j} h_{jl}^n, \tag{10}$$

$$\sum_l \frac{\left|h_{jl}^n\right|}{\lambda_l} = O_n(1), \tag{11}$$

$$\lim_j \sum_{l=0}^{j} \frac{\left|h_{jl}^n - h_{nl}\right|}{\lambda_l} = 0. \tag{12}$$

Besides, condition (2.11) can be replaced by condition

$$\sum_l \frac{|h_{nl}|}{\lambda_l} = O_n(1). \tag{13}$$

Lemma 2.5 (see [2], Theorem 8.4 or [4], Theorem 1)**.** *$M \in \left(m_A^\lambda, m_B^\mu\right)$ if and only if conditions (2.9) - (2.12) are satisfied and*

$$\text{there exist finite limits } \lim_n \gamma_{nl} := \gamma_l, \tag{14}$$

$$\sum_l \frac{|\gamma_{nl}|}{\lambda_l} = O(1), \tag{15}$$

$$\mu_n \sum_l \frac{|\gamma_{nl} - \gamma_l|}{\lambda_l} = O(1), \tag{16}$$

$$(\rho_n) \in m^\mu, \ \rho_n := \lim_r \sum_{l=0}^{r} \gamma_{nl}^r. \tag{17}$$

Also, condition (2.15) can be replaced by condition

$$\sum_l \frac{|\gamma_l|}{\lambda_l} < \infty, \tag{18}$$

and if $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then $O(1)$ in condition (2.16) it is necessary to replace by $o(1)$.

Remark 2.1. *The existence of finite limits $\lim_n \gamma_{nl}$ follows from conditions (2.9) - (2.12). If M is a lower triangular , then conditions (2.9) - (2.12) are redundant in Lemma 2.4.*

3. Necessary and sufficient conditions for $Z_{1/2} \in \left(m^\lambda, m^\mu\right)$, $Z_{1/2} \in \left(c^\lambda, c^\mu\right)$ and $Z_{1/2} \in \left(c^\lambda, m^\mu\right)$

Theorem 3.1. *$Z_{1/2} \in \left(m^\lambda, m^\mu\right)$ if and only if*

$$\frac{\mu_n}{\lambda_{n-1}} = O(1). \tag{19}$$

If $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then $O(1)$ in condition (16) it is necessary to replace by $o(1)$.

Proof. It is sufficient to show that all conditions of Lemma 2.1 are satisfied for $A = Z_{1/2}$. As $\delta_k = 0$ and $Z_{1/2}e = e \in m^\mu$, then conditions (1) and (2) are fulfilled. For $A = Z_{1/2}$ we can present conditions (3) and (4) correspondingly in the form

$$\frac{1}{2}\left(\frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n}\right) = O(1) \tag{20}$$

and

$$\frac{1}{2}\mu_n\left(\frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n}\right) = O(1). \tag{21}$$

As λ and μ are monotonically increasing with $\lambda_k > 0$, $\mu_k > 0$, then condition (20) is valid and condition (21) holds if and only if condition (19) is satisfied and $\mu_n/\lambda_n = O(1)$. As validity of the relation $\mu_n/\lambda_n = O(1)$ follows from (19), then (19) is equivalent to (4) for $A = Z_{1/2}$.

It follows from Lemma 2.1 that if $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then $O(1)$ in condition (19) it is necessary to replace by $o(1)$. □

From Lemma 2.3 immediately follows

Theorem 3.2 (see [2], Exercise 8.3 or [11], p. 138)**.** *The Zweier method* $Z_{1/2} \in \left(c^\lambda, m^\mu\right)$ *if and only if* $Z_{1/2} \in \left(m^\lambda, m^\mu\right)$.

Theorem 3.3. *Let* $\lambda_n \neq O(1)$. *Then* $Z_{1/2} \in \left(c^\lambda, c^\mu\right)$ *if and only if*

$$\textit{there exists the finite limit } \lim_n \mu_n\left(\frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n}\right). \tag{22}$$

If $Z_{1/2} \in \left(c^\lambda, c^\mu\right)$, *then for every* $x = (\xi_k) \in c^\lambda$ *with* $\lim_k \xi_k = \varsigma$ *we have*

$$\lim_n \mu_n((Z_{1/2})_n x - \phi) = \frac{1}{2}\nu \lim_n \mu_n\left(\frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n}\right), \tag{23}$$

where

$$\phi := \lim_n (Z_{1/2})_n x,\ \nu := \lim_k \lambda_k(\xi_k - \varsigma).$$

Proof. It is sufficient to show that all conditions of Lemma 2.2 are satisfied for $A = Z_{1/2}$. As $Z_{1/2}e = e \in c^\mu$,

$$\lim_n (Z_{1/2})_n e^k = 0 \text{ and } \lim_n \mu_n (Z_{1/2})_n e^k = 0$$

(since $(Z_{1/2})_n e^k = 0$ for $n > k$), then conditions (5) and (6) are fulfilled. As

$$(Z_{1/2})_n \lambda^{-1} = \frac{1}{2}\left(\frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n}\right),$$

then condition [22] is equivalent to (7) for $A = Z_{1/2}$. As (22) implies the validity of (21), then from (22) also follows the validity of (3) and (4) for $A = Z_{1/2}$ (see the proof of Theorem 3.1). As

$$\delta = \mathfrak{A}_n = 1,\ a^\lambda = a_k^{\lambda,\mu} = 0$$

for $A = Z_{1/2}$, then relation (23 holds by (8). □

Definition 3.1. *A method A is said to preserving the* λ*-boundedness if* $A \in \left(m^\lambda, m^\lambda\right)$, *and is said to be* λ*-conservative if* $A \in (c^\lambda, c^\lambda)$.

It is proved in [2] (Examples 8.1 and 8.2), that $Z_{1/2}$ preserves the λ-boundedness if and only if $\lambda_n/\lambda_{n-1} = O(1)$, and is λ-conservative if and only if there exist the finite limit $\lim_n(\lambda_n/\lambda_{n-1})$.

4. Necessary and sufficient conditions for $M \in \left(m^\lambda_{Z_{1/2}}, m^\mu_B\right)$

Let throughout this section $B = (b_{nk})$ be a triangular method, $M = (m_{nk})$ an arbitrary matrix and $\lambda := (\lambda_k)$, $\mu := (\mu_k)$ monotonically increasing sequences with $\lambda_k > 0$, $\mu_k > 0$. The inverse of $Z_{1/2}$ we denote by $Z^{-1}_{1/2} := (\eta_{nk})$, where (see [1] p. 13)

$$\eta_{nk} = 2(-1)^{n-k} \text{ for } k \le n, \text{ and } \eta_{nk} = 0 \text{ for } k > n. \tag{24}$$

Proposition 4.1. *The transformation $y = Mx$ exists for every $x \in m^\lambda_{Z_{1/2}}$ if and only if condition (5.14) is satisfied and*

$$\text{series } \sum_k (-1)^k m_{nk} \text{ are convergent for every } n, \tag{25}$$

$$\text{series } \sum_k m_{n,2k} \text{ are convergent for every } n, \tag{26}$$

$$\sum_{l=0}^{j} \frac{1}{\lambda_l} \left| \sum_{k=l}^{l+j} (-1)^{k-l} m_{nk} \right| = O_n(1), \tag{27}$$

$$\lim_j \sum_{l=0}^{j} \frac{1}{\lambda_l} \left| \sum_{k=l+j+1}^{\infty} (-1)^{k-l} m_{nk} \right| = 0. \tag{28}$$

Besides, condition (27) can be replaced by condition

$$\sum_l \frac{1}{\lambda_l} \left| \sum_{k=l}^{\infty} (-1)^{k-l} m_{nk} \right| = O_n(1). \tag{29}$$

Proof. It is sufficient to show that all conditions of Lemma 2.4 are satisfied for $A = Z_{1/2}$. Using (24), we obtain

$$h^n_{jl} = 2\sum_{k=l}^{l+j} (-1)^{k-l} m_{nk}$$

and

$$\sum_{l=0}^{j} h^n_{jl} = 2\sum_{l=0}^{j}\sum_{k=l}^{l+j} (-1)^{k-l} m_{nk} = 2\sum_{k=0}^{2j} m_{nk} \sum_{l=0}^{k} (-1)^{k-l} = 2\sum_{k=0}^{2j} m_{n,2k}.$$

Hence condition (25) is equivalent to (9), condition (26) to (10), and condition (27) to (11). As

$$h_{jl} = \sum_{k=l}^{\infty} (-1)^{k-l} m_{nk},$$

then condition [28] is equivalent to (12). Finally, from Lemma 2.4 we obtain that condition (27) can be replaceed by (29). □

From Proposition 4.1 we immediately get

Corollary 4.1. *If the rows of a matrix $M = (m_{nk})$ are positive and monotonically decreasing; i.e., the sequence (m_{nk}) for every n is positive and monotonically decreasing, then condition (27) is fulfilled if*

$$\sum_{l=0}^{j} \frac{m_{nl}}{\lambda_l} = O_n(1),$$

and condition (28) is fulfilled if

$$\lim_j \sum_{l=0}^{j} \frac{m_{n,l+j+1}}{\lambda_l} = 0.$$

Theorem 4.1. *A matrix $M \in \left(m^{\lambda}_{Z_{1/2}}, m^{\mu}_{B}\right)$ if and only if conditions (2.25) - (2.28) are satisfied and*

$$there\, exist\, the\, finite\, limits\ \lim_n g_{nl}, \tag{30}$$

$$there\, exists\, the\, finite\, limit\ G_0, \tag{31}$$

$$\sum_l \frac{1}{\lambda_l} \left| \sum_{k=l}^{\infty} (-1)^{k-l} g_{nk} \right| = O(1), \tag{32}$$

$$\mu_n \sum_l \frac{1}{\lambda_l} \left| \sum_{k=l}^{\infty} (-1)^{k-l} g_{nk} - G_l \right| = O(1), \tag{33}$$

$$(\rho_n) \in m^{\mu}, \tag{34}$$

where

$$G_l := \lim_n \sum_{k=l}^{\infty} (-1)^{k-l} g_{nk},\ \ \rho_n := 2 \sum_{i=0}^{n} b_{ni} \sum_k m_{i,2k}.$$

Also, condition (2.32) can be replaced by condition

$$\sum_l \frac{|G_l|}{\lambda_l} < \infty, \tag{35}$$

and if $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then $O(1)$ in condition (2.33) it is necessary to replace by $o(1)$.

Proof. It is sufficient to show that all conditions of Lemma 2.5 are satisfied for $A = Z_{1/2}$. First we see that conditions (2.25) - (2.28) are equivalent to conditions (2.14) - (2.17) by Lemma 2.5. Due to (24) we obtain

$$\gamma^r_{nl} = 2 \sum_{k=l}^{l+r} (-1)^{k-l} g_{nk}. \tag{36}$$

It follows from (2.25) - (2.28) By Remark 2.1 that the finite limits γ_{nl} exist. Hence

$$\gamma_{nl} = 2 \sum_{k=l}^{\infty} (-1)^{k-l} g_{nk}. \tag{37}$$

This implies that condition (32) is equivalent to condition (15), condition (33) to condition (16), and the existence of finite limits G_l is equivalent to condition (14). From the existence of finite limits G_l follows the validity of condition (31). As by (37) we get

$$g_{nl} = \frac{1}{2}\left(\gamma_{nl} + \gamma_{n,l+1}\right),$$

then condition (30) also follows from the existence of finite limits G_l. Conversly, (30) and (31) imply the existence of finite limits G_l. Therefore, (30) and (31) are equivalent to (14).

Using (36), we can write

$$\sum_{l=0}^{r} \gamma_{nl}^{r} = 2\sum_{l=0}^{r}\sum_{k=l}^{l+r}(-1)^{k-l} g_{nk} = 2\sum_{k=0}^{2r} g_{nk} \sum_{l=0}^{k}(-1)^{k-l} =$$

$$2\sum_{k=0}^{r} g_{n,2k} = 2\sum_{k=0}^{r}\sum_{i=0}^{n} b_{n,i} m_{i,2k} = 2\sum_{i=0}^{n} b_{n,i}\sum_{k=0}^{r} m_{i,2k}.$$

Consequently the finite limits ρ_n defined by (34) exist, due to (26), and hence condition (34) is equivalent to (17).

Finally, by Lemma 2.5 we conclude that condition (32) can be replaced by (35), and if $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then $O(1)$ in condition (33) it is necessary to replace by $o(1)$. □

Now we consider the case if $M = (m_{nk})$ is a multiplicative matrix; i.e.,

$$m_{nl} = t_n v_l; \; (t_n) \in \omega, \, (v_l) \in \omega. \tag{38}$$

Proposition 4.2. *Let M be defined by (38), where (v_l) is a positive monotonically decreasing sequence and the series $\sum_l v_l$ is convergent. Then, $M \in \left(m^{\lambda}_{Z_{1/2}}, m^{\mu}_{B}\right)$ if and only if $t := (t_n) \in m^{\mu}_{B}$.*

Proof. It is sufficient to show that all conditions of Theorem 4.1 are satisfied for M, defined by (38). As

$$\sum_k (-1)^k m_{nk} = t_n \sum_k (-1)^k v_k$$

and

$$\sum_k m_{n,2k} = t_n \sum_k v_{2k},$$

then conditions (25) and (26) are fulfilled, since $v_k \geq 0$ and $\sum_l v_l$ is convergent. Also conditions (27) and (28) hold. Indeed,

$$\sum_{l=0}^{j} \frac{1}{\lambda_l}\left|\sum_{k=l}^{l+j}(-1)^{k-l} m_{nk}\right| = t_n \sum_{l=0}^{j}\frac{1}{\lambda_l}\left|\sum_{k=l}^{l+j}(-1)^{k-l} v_k\right| < \frac{t_n}{\lambda_0}\sum_l v_l = O_n(1)$$

and

$$\sum_{l=0}^{j}\frac{1}{\lambda_l}\left|\sum_{k=l+j+1}^{\infty}(-1)^{k-l} m_{nk}\right| < \frac{t_n}{\lambda_0}\sum_l v_{l+j+1}.$$

Thus conditions (27) and (28) are fulfilled, since the remainder $\sum_l v_{l+j+1}$ of the convergent series is convergent to zero.

We can write that

$$g_{nl} = v_l \sum_{i=l}^{n} b_{ni} t_i = v_l B_n t,$$

$$\sum_k (-1)^k g_{nk} = \sum_k (-1)^k v_k B_n t$$

and

$$\rho_n := V B_n t, \;\; V := 2\sum_k v_{2k}.$$

Therefore conditions (30), (31) and (34) are satisfied if and only if $t := (t_n) \in m_B^\mu$. As $t \in m_B^\mu$ implies that the finite limit $\lim_n B_n t$ exists and $|B_n t| = O(1)$, then

$$\sum_l \frac{1}{\lambda_l} \left| \sum_{k=l}^{\infty} (-1)^{k-l} g_{nk} \right| = \sum_l \frac{1}{\lambda_l} \left| \sum_{k=l}^{\infty} (-1)^{k-l} v_k \right| |B_n t| < |B_n t| \sum_l \frac{v_l}{\lambda_l} = O(1)$$

and

$$\mu_n \sum_l \frac{1}{\lambda_l} \left| \sum_{k=l}^{\infty} (-1)^{k-l} g_{nk} - G_l \right| = \mu_n |B_n t - \lim_n B_n t| \sum_l \frac{v_l}{\lambda_l} = O(1);$$

i.e., conditions (32) and (33) are fulfilled. □

5. Conclusions

This work characterizes certain matrix classes involving some spaces with involvement of speeds. The findings should inspire to investigate for several other matrix classes characterization by assigning speeds to different classes of participating spaces. It may also be interesting to know the speed of convergence while studying a process for convergence.

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