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GENERALIZED ENTIRE SEQUENCE SPACES DEFINED BY FRACTIONAL DIFFERENCE OPERATOR AND SEQUENCE OF MODULUS FUNCTIONS

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Abstract. In this paper, we introduce some generalized entire sequence spaces and analytic sequence spaces defined by fractional difference operator and a sequence of modulus functions. We study some topological properties and give some inclusion relations among the spaces.

Keywords: Paranorm space, modulus function, entire sequences, fractional difference operator.

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1. Introduction and Preliminaries

A complex sequence, whose k^{th} term is x_k, is denoted by (x_k). Let φ be the set of all finite sequences. A sequence $x = (x_k)$ is said to be analytic if $\sup_k |x_k|^{\frac{1}{k}} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence $x = (x_k)$ is called entire sequence if $\lim_{k\to\infty} |x_k|^{\frac{1}{k}} = 0$. The vector space of all entire sequences will be denoted by Γ. A modulus function is a function $f : [0,\infty) \to [0,\infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,
(3) f is increasing
(4) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequentially, modulus

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function has been discussed in ([1, 2, 3, 4, 20, 28, 29, 30, 31]) and references therein. Let $F = (f_k)$ be a sequence of modulus functions.

The space consisting of all those sequences x in w such that $f_k\Big(\frac{|x_k|^{1/k}}{\rho}\Big) \to 0$ as $k \to \infty$ for some arbitrary fixed $\rho > 0$ is denoted by Γ_F and is known as a space of entire sequences defined by a sequence of modulus function. The space Γ_F is a metric space with the metric $d(x,y) = \sup_k f_k\Big(\frac{|x_k - y_k|^{1/k}}{\rho}\Big)$ for all $x = (x_k)$ and $y = (y_k)$ in Γ_F. The space consisting of all those sequences x in w such that $\Big(\sup_k \Big(f_k\Big(\frac{|x_k|^{1/k}}{\rho}\Big)\Big)\Big) < \infty$ for some arbitrarily fixed $\rho > 0$ is denoted by Λ_F and is known as a space of analytic sequences defined by a sequence of modulus function.

A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ (see [19]).

Let X be a linear metric space. A function $p : X \to, \mathbf{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$,
(2) $p(-x) = p(x)$, for all $x \in X$,
(3) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda \; as \; n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0 \; as \; n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0 \; as \; n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm.

In [18], Kızmaz defined the sequence spaces

$$Z(\Delta) = \Big\{x = (x_k) : (\Delta x_k) \in Z\Big\} \text{ for } Z = \ell_\infty, c \text{ and } c_0,$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. Et and Çolak [14] generalized the difference sequence spaces to the sequence spaces

$$Z(\Delta^n) = \Big\{x = (x_k) : (\Delta^n x_k) \in Z\Big\} \text{ for } Z = \ell_\infty, c \text{ and } c_0,$$

where $n \in \mathbb{N}$, $\Delta^0_x = (x_k)$, $\Delta x = (x_k - x_{k+1})$,

$$\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}).$$

The generalized difference sequence has the following binomial representation

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+v}.$$

Later, several authors studied difference sequence spaces in different setting, we refer to [6, 7, 15, 24, 26, 11, 16, 12, 10]. The notion of difference operator has been recently used to define statistical convergence (see [17, 23]) while for recent work on statistical convergence we refer to [8, 9, 13, 21, 22, 25, 27]. In the recent past, Baliarsingh [5] defined the fractional difference operator as follows: Let $x = (x_k) \in w$ and α be a real number, then the fractional difference operator $\Delta^{(\alpha)}$ is defined by

$$\Delta^{(\alpha)} x_k = \sum_{i=0}^{k} \frac{(-\alpha)_i}{i!} x_{k-i},$$

where $(-\alpha)_i$ denotes the Pochhammer symbol defined as:

$$(-\alpha)_i = \begin{cases} 1 \quad \text{if} \quad \alpha = 0 \ \text{ or } \ i = 0, \\ \\ \alpha(\alpha+1)(\alpha+2)...(\alpha+i-1), \quad \text{otherwise.} \end{cases}$$

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 \leq p_k \leq \sup p_k = G$, $K = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

Let $F = (f_k)$ be a sequence of modulus function and X be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous seminorms q. The symbols $\Lambda(X)$ and $\Gamma(X)$ denote the spaces of all analytic and entire sequences, respectively, defined over X. If $p = (p_k)$ be a bounded sequence of strictly positive real numbers, then we define the following sequence spaces:

$$\Lambda\_F(\Delta^{(\alpha)}, p, q) = \Big\{x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(q\Big(\frac{|(\Delta^{(\alpha)} x_k)^{1/k}|}{\rho}\Big)\Big)\Big]^{p_k} < \infty, \\ \text{for some} \ \ \rho > 0\Big\}$$

and

$$\Gamma_F(\Delta^{(\alpha)}, p, q) = \Big\{x \in \Gamma(X) : \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(q\Big(\frac{|(\Delta^{(\alpha)} x_k)^{1/k}|}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \ \text{ as } n \to \infty, \\ \text{for some} \ \ \rho > 0\Big\}.$$

If we take $p = (p\_k) = 1$, we get

$$\Lambda_F(\Delta^{(\alpha)}, q) = \Big\{x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \Big[f\_k\Big(q\Big(\frac{|(\Delta^{(\alpha)} x\_k)^{1/k}|}{\rho}\Big)\Big)\Big] < \infty, \\ \text{for some} \ \ \rho > 0\Big\}$$

and

$$\Gamma_F(\Delta^{(\alpha)}, q) = \Big\{x \in \Gamma(X) : \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(q\Big(\frac{|(\Delta^{(\alpha)} x_k)^{1/k}|}{\rho}\Big)\Big)\Big] \to 0 \ \text{ as } n \to \infty, \\ \text{for some} \ \ \rho > 0\Big\}.$$

2. Main Results

Here, we examine some topological properties and prove inclusion relation between the spaces defined in the previous section.

Theorem 2.1 *Let $F = (f_k)$ be a sequence of modulus function and $p = (p_k)$ be bounded sequence of strictly positive real numbers. Then $\Gamma_F(\Delta^{(\alpha)}, p, q)$ and $\Lambda_F(\Delta^{(\alpha)}, p, q)$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_k), y = (y_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result, we need to find some $\rho_3 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}(\beta x_k+\gamma y_k)|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n\to\infty. \tag{2}$$

Since $x = (x_k), y = (y_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$, there exist some positive ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n\to\infty \tag{3}$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n\to\infty. \tag{4}$$

Since $F = (f_k)$ is a non-decreasing function, q is a seminorm and $\Delta^{(\alpha)}$ is linear, then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}(\beta x_k+\gamma y_k)|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{|\beta|^{\frac{1}{k}}(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_3}+\frac{|\gamma|^{\frac{1}{k}}(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}$$

so that

$$\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}(\beta x_k+\gamma y_k)|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{|\beta|(\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_3}+\frac{|\gamma|(\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}.$$

Take $\rho_3 > 0$ such that $\frac{1}{\rho_3} = \min\Big\{\frac{1}{|\beta|\,\rho_1}, \frac{1}{|\gamma|\,\rho_2}\Big\}$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}(\beta x_k+\gamma y_k)|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}$$
$$\begin{aligned}
&\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1}+\frac{(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\\
&\leq \frac{1}{n}\sum_{k=1}^{n}\Big[\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}+\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\Big]\\
&\leq K\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}+K\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}|}{\rho_1}\Big)\Big)\Big]^{p_k}\\
&\longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence

$$\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\beta\Delta^{(\alpha)}x_k+\gamma\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n\to\infty.$$

This proves that $\Gamma_F(\Delta^{(\alpha)}, p, q)$ is a linear space. Similarly, we can prove that $\Lambda_F(\Delta^{(\alpha)}, p, q)$ is a linear space

Theorem 2.2 *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. Then $\Gamma_F(\Delta^{(\alpha)}, p, q)$ is a paranormed space with paranorm defined by*

$$g(x) = \inf\Big\{\rho^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq 1,\ \ \rho > 0,\ \ m \in \mathbb{N}\Big\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(x) \geq 0,\ g(x) = g(-x)$ and $g(\theta) = 0$, where θ is the zero sequence of X. Let $(x_k), (y_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$. Let $\rho_1,\ \rho_2 > 0$ be such that

$$\sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k} \leq 1$$

and

$$\sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned}\sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}(x_k+y_k)|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} &\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}\\ &+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\\ &\leq 1.\end{aligned}$$

Hence
$g(x+y)$

$$\begin{aligned}&\leq \inf\Big\{(\rho_1+\rho_2)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1+\rho_2}\Big)\Big)\Big]^{p_k} \leq 1, \rho_1,\ \rho_2 > 0,\ \ m \in N\Big\}\\ &\leq \inf\Big\{(\rho_1)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k} \leq 1, \rho_1 > 0,\ \ m \in \mathbb{N}\Big\}\\ &+ \inf\Big\{(\rho_2)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k} \leq 1, \rho_2 > 0,\ \ m \in \mathbb{N}\Big\}.\end{aligned}$$

Thus we have $g(x+y) \leq g(x) + g(y)$. Hence g satisfies the triangle inequality.

$$\begin{aligned}g(\lambda x) &= \inf\Big\{(\rho)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\lambda\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq 1, \rho > 0,\ \ m \in \mathbb{N}\Big\}\\ &= \inf\Big\{(r|\lambda|)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{r}\Big)\Big)\Big]^{p_k} \leq 1, r > 0,\ \ m \in \mathbb{N}\Big\},\end{aligned}$$

where $r = \frac{\rho}{|\lambda|}$. Hence $\Gamma_F(\Delta^{(\alpha)}, p, q)$ is a paranormed space.

Theorem 2.3 *Let $F' = (f'_k)$ and $F'' = (f''_k)$ be two sequences of modulus functions. Then*

$$\Gamma_{F'}(\Delta^{(\alpha)}, p, q) \cap \Gamma_{F''}(\Delta^{(\alpha)}, p, q) \subseteq \Gamma_{F'+F''}(\Delta^{(\alpha)}, p, q).$$

Proof. Let $x=(x_k)\in\Gamma_{F'}(\Delta^{(\alpha)},p,q)\cap\Gamma_{F''}(\Delta^{(\alpha)},p,q)$. Then there exist ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f'_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}\to 0 \text{ as } n\to\infty. \tag{5}$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f''_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\to 0 \text{ as } n\to\infty. \tag{6}$$

Let $\rho>0$ such that $\frac{1}{\rho}=\min\Big(\frac{1}{\rho_1},\frac{1}{\rho_2}\Big)$. Then, we have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\Big[(f'_k+f''_k)\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} &\leq K\Big[\frac{1}{n}\sum_{k=1}^{n}\Big[f'_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}\Big]\\ &+ K\Big[\frac{1}{n}\sum_{k=1}^{n}\Big[f''_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\Big]\\ &\to 0 \text{ as } n\to\infty\end{aligned}$$

by (5) and (6). Hence

$$\frac{1}{n}\sum_{k=1}^{n}\Big[(f'_k+f''_k)\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}\to 0 \text{ as } n\to\infty.$$

Therefore $x=(x_k)\in\Gamma_{F'+F''}(\Delta^{(\alpha)},p,q)$.

Theorem 2.4 *Let $\alpha\geq 1$. Then, we have the following inclusions:*
(i) $\Gamma_F(\Delta^{(\alpha-1)},p,q)\subseteq\Gamma_F(\Delta^{(\alpha)},p,q)$,
(ii) $\Lambda_F(\Delta^{(\alpha-1)},p,q)\subseteq\Lambda_F(\Delta^{(\alpha)},p,q)$.

Proof. Let $x=(x_k)\in\Gamma_F(\Delta^{(\alpha-1)},p,q)$. Then we have $\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha-1)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}\to 0$ as $n\to\infty$, for some $\rho>0$. Since $F=(f_k)$ is non-decreasing and q is a seminorm, we have

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}\\ &\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha-1)}x_k-\Delta^{(\alpha-1)}x_{k+1}|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}\\ &\leq K\Big\{\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha-1)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}+\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha-1)}x_{k+1}|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}\Big\}\\ &\longrightarrow 0 \ \text{ as } \ n\to\infty.\end{aligned}$$

Therefore $\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}\to 0$ as $n\to\infty$. Hence $x\in\Gamma_F(\Delta^{(\alpha)},p,q)$. This completes the proof of (i). Similarly, we can prove (ii).

Theorem 2.5 *Let $0\leq p_k\leq r_k$ and let $\{\frac{r_k}{p_k}\}$ be bounded. Then $\Gamma_F(\Delta^{(\alpha)},r,q)\subset$*

$\Gamma_F(\Delta^{(\alpha)}, p, q)$.

Proof. Let $x = (x_k) \in \Gamma_F(\Delta^{(\alpha)}, r, q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k} \to 0 \text{ as } n \to \infty. \tag{7}$$

Let $t_k = \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k}$ and $\lambda_k = \frac{p_k}{r_k}$. Since $p_k \leq r_k$, we have $0 \leq \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$. Define

$$u_k = \begin{cases} t_k & \text{if } t_k \geq 1 \\ 0 & \text{if } t_k < 1 \end{cases}$$

and

$$v_k = \begin{cases} 0 & \text{if } t_k \geq 1 \\ t_k & \text{if } t_k < 1 \end{cases}$$

$t_k = u_k + v_k$, $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. It follows that $u_k^{\lambda_k} \leq u_k \leq t_k$, $v_k^{\lambda_k} \leq v_k^{\lambda}$. Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$. Hence

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)^{r_k}\Big]^{\lambda_k} &\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k} \\
\Longrightarrow \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)^{r_k}\Big]^{p_k/r_k} &\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k} \\
\Longrightarrow \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} &\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k}.
\end{aligned}$$

But

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k} \to 0 \text{ as } n \to \infty \text{ (by(7))}.$$

Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Hence $x = (x_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$. From (7), we get $\Gamma_F(\Delta^{(\alpha)}, r, q) \subset \Gamma_F(\Delta^{(\alpha)}, p, q)$.

Theorem 2.6 *(i) Let* $0 < \inf p_k \leq p_k \leq 1$. *Then* $\Gamma_F(\Delta^{(\alpha)}, p, q) \subset \Gamma_F(\Delta^{(\alpha)}, q)$,
(ii) Let $1 \leq p_k \leq \sup p_k < \infty$. *Then* $\Gamma_F(\Delta^{(\alpha)}, q) \subset \Gamma_F(\Delta^{(\alpha)}, p, q)$.

Proof. (i) Let $x = (x_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty. \tag{8}$$

Since $0 < \inf p_k \leq p_k \leq 1$,

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big] \leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty. \tag{9}$$

From (8) and (9) it follows that, $x = (x_k) \in \Gamma_F(\Delta^{(\alpha)}, q)$. Thus $\Gamma_F(\Delta^{(\alpha)}, p, q) \subset \Gamma_F(\Delta^{(\alpha)}, q)$.
(ii) Let $p_k \geq 1$ for each k and $\sup p_k < \infty$ and let $x = (x_k) \in \Gamma_F(\Delta^{(\alpha)}, q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big] \to 0 \text{ as } n \to \infty. \tag{10}$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]$$

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

This implies that $x = (x_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$. Therefore $\Gamma_F(\Delta^{(\alpha)}, q) \subset \Gamma_F(\Delta^{(\alpha)}, p, q)$.

Theorem 2.7 *If* $\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq |x_k|^{1/k}$, *then* $\Gamma \subset \Gamma_F(\Delta^{(\alpha)}, p, q)$.

Proof. Let $x = (x_k) \in \Gamma$. Then we have,

$$|x_k|^{1/k} \to 0 \text{ as } k \to \infty. \tag{11}$$

But $\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq |x_k|^{1/k}$, by our assumption, implies that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty \quad \text{by(11)}$$

Then $x = (x_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$ and $\Gamma \subset \Gamma_F(\Delta^{(\alpha)}, p, q)$.

Theorem 2.8 *The space* $\Gamma_F(\Delta^{(\alpha)}, p, q)$ *is solid.*

Proof. Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k},$$

since $F = (f_k)$ is non-decreasing. As $y = (y_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$, then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}y_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Hence

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\Delta^{(\alpha)}x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Therefore $x = (x_k) \in \Gamma_F(\Delta^{(\alpha)}, p, q)$.

Theorem 2.9 *The space* $\Gamma_F(\Delta^{(\alpha)}, p, q)$ *is monotone.*

Proof. We omit the proof as it is trivial.

3. Conclusions

We introduced some generalized entire sequence spaces and analytic sequence spaces defined by fractional difference operator and sequence of modulus functions. We also studied some topological properties and proved several inclusion relations between these spaces.

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