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# IDENTITIES AND RELATIONS INVOLVING PARAMETRIC TYPE BERNOULLI POLYNOMIALS

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ABSTRACT. The main purpose of this paper is to give explicit relations and identities for the parametric type Bernoulli polynomials. Further, we give some relations for the generalized Bernoulli polynomials.

Keywords: Bernoulli polynomials and numbers, Apostol-Bernoulli polynomials and numbers, Generalized Bernoulli numbers and polynomials, the parametric type Bernoulli polynomials.

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## 1. INTRODUCTION

The classical Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating functions:

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt}, \ |t| < 2\pi. \tag{1}$$

For $x = 0$, $B_n^{(\alpha)}(0) = B_n^{(\alpha)}$ are called the Bernoulli numbers.

The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{N}_0$ are defined by following generating function in ([3]-[16])

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt} , \tag{2}$$
$$(\lambda \in \mathbb{C}, \ |t| < 2\pi \text{ when } \lambda = 1, \ |t| < |\log \lambda| \text{ when } \lambda \neq 1).$$

Bernoulli polynomials and the generalized Apostol-Bernoulli polynomials have been studied by many authors ([1]-[18]).

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Natalini et al. [13] defined a new class of generalized Bernoulli polynomials $B_n^{[m-1]}(x)$, $m \geq 1$ by

$$\sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!} = \frac{t^m}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} e^{xt}. \tag{3}$$

When $x = 0$, $B_n^{[m-1]}(0) := B_n^{[m-1]}$ are called the new type Bernoulli numbers.

For $m = 1$, $B_n^{[0]}(x) := B_n(x)$ is the classical Bernoulli polynomials.

Kurt [6] defined the new type generalized Bernoulli polynomials $B_n^{[m-1,\alpha]}(x)$ of order $\alpha \in \mathbb{N}$, the following generating function:

$$\sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x) \frac{t^n}{n!} = \left( \frac{t^m}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} \right)^{\alpha} e^{xt}, \ \left(|t| < 2\pi, \ 1^{\alpha} := 1, \ h \in \mathbb{N}\right). \tag{4}$$

Srivastava et. al. in ([16], [17]) defined two parametric kind of special cases of the Apostol-Bernoulli polynomials $B_n^{(c,\alpha)}(x;\lambda)$ and $B_n^{(s,\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{N}$ as:

$$\sum_{n=0}^{\infty} B_n^{(c,\alpha)}(x,y;\lambda) \frac{t^n}{n!} = \left( \frac{t}{\lambda e^t - 1} \right)^{\alpha} e^{xt} \cos(yt) \tag{5}$$

and

$$\sum_{n=0}^{\infty} B_n^{(s,\alpha)}(x,y;\lambda) \frac{t^n}{n!} = \left( \frac{t}{\lambda e^t - 1} \right)^{\alpha} e^{xt} \sin(yt). \tag{6}$$

For $x, \ y \in \mathbb{R}$, the Taylor-Maclauren expansions of the two functions $e^{xt} \cos(yt)$ and $e^{xt} \sin(yt)$ are given by [11], respectively,

$$e^{xt} \cos(yt) = \sum_{n=0}^{\infty} C_n(x,y) \frac{t^n}{n!} \tag{7}$$

and

$$e^{xt} \sin(yt) = \sum_{n=0}^{\infty} S_n(x,y) \frac{t^n}{n!}. \tag{8}$$

From (7) and (8), we get the following relations, respectively,

$$C_n(x,y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}$$

and

$$S_n(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}.$$

## 2. Some Explicit Relation Related To Parametric Type Bernoulli Polynomials

In this section, we define the parametric type Bernoulli numbers and polynomials. We give some relations and identities for these polynomials. We define the generalized parametric cosine-Bernoulli polynomials ${}_{C}B_n^{[m-1,\alpha]}(x,y)$ of order $\alpha$ and the generalized parametric sine-Bernoulli polynomials ${}_{S}B_n^{[m-1,\alpha]}(x,y)$ of order $\alpha$ as, respectively:

$$\sum_{n=0}^{\infty} {}_{C}B_n^{[m-1,\alpha]}(x,y)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt}\cos(yt) \tag{9}$$

and

$$\sum_{n=0}^{\infty} {}_{S}B_n^{[m-1,\alpha]}(x,y)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt}\sin(yt). \tag{10}$$

For $m=\alpha=1$ and $y=0$ in (9) reduces the classical Bernoulli polynomials.

It easy to see that if we set $m=\alpha=1$ in (9) and (10), we arrive at

$$\sum_{n=0}^{\infty} {}_{C}B_n^{[0]}(x,y)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}\cos(yt)$$

and

$$\sum_{n=0}^{\infty} {}_{S}B_n^{[0]}(x,y)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}\sin(yt).$$

**Theorem 2.1.** *The following relations hold true:*

$$ {}_{C}B_n^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{[m-1,\alpha]}C_k(x,y), \tag{11}$$

$$ {}_{S}B_n^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{[m-1,\alpha]}S_k(x,y), \tag{12}$$

$$ {}_{C}B_n^{[m-1,\alpha]}(x+u,y) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{[m-1,\alpha]}(x,y)u^k \tag{13}$$

*and*

$$ {}_{S}B_n^{[m-1,\alpha]}(x+u,y) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{[m-1,\alpha]}(x,y)u^k. \tag{14}$$

*Proof.* Using equation (9), we write

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{C}B_n^{[m-1,\alpha]}(x,y)\frac{t^n}{n!} &= \left(\frac{t^m}{e^t - \sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt}\cos(yt)\\ &= \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}\frac{t^n}{n!}\sum_{n=0}^{\infty} C_n(x,y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{[m-1,\alpha]}C_k(x,y)\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ in the both sides of the above equation, we arrive at (11).

Proof of (12), (13) and (14) are similar to that of (11), so it is omitted. □

**Theorem 2.2.** *The following relations hold true:*

$$C_n\left(x_1+x_2,y_1+y_2\right)=\sum_{k=0}^{n}\binom{n}{k}\left\{C_{n-k}(x_1,y_1)C_k(x_2,y_2)-S_{n-k}(x_1,y_1)S_k(x_2,y_2)\right\} \quad (15)$$

*and*

$$S_n\left(x_1+x_2,y_1+y_2\right)=\sum_{k=0}^{n}\binom{n}{k}\left\{S_{n-k}(x_1,y_1)C_k(x_2,y_2)+C_{n-k}(x_1,y_1)S_k(x_2,y_2)\right\}. \quad (16)$$

*Proof.* By (7), we write

$$\begin{aligned}\sum_{n=0}^{\infty}C_n(x_1+x_2,y_1+y_2)\frac{t^n}{n!}&=e^{(x_1+x_2)t}\cos((y_1+y_2)\,t)\\&=e^{x_1t}\cos(y_1t)e^{x_2t}\cos(y_2t)-e^{x_1t}\sin(y_1t)e^{x_2t}\sin(y_2t)\end{aligned}$$

$$\begin{aligned}&=\sum_{m=0}^{\infty}C_m(x_1,y_1)\frac{t^m}{m!}\sum_{k=0}^{\infty}C_k(x_2,y_2)\frac{t^k}{k!}-\sum_{m=0}^{\infty}S_m(x_1,y_1)\frac{t^m}{m!}\sum_{k=0}^{\infty}S_k(x_2,y_2)\frac{t^k}{k!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left\{C_{n-k}(x_1,y_1)C_k(x_2,y_2)-S_{n-k}(x_1,y_1)S_k(x_2,y_2)\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we get (15).
Similarly, by (8), we get (16). □

For $x_1=x_2=x$ and $y_1=y_2=y$, we get respectively,

$$C_n\left(2x,2y\right)=\sum_{k=0}^{n}\binom{n}{k}\left\{C_{n-k}(x,y)C_k(x,y)-S_{n-k}(x,y)S_k(x,y)\right\}$$

and

$$S_n\left(2x,2y\right)=\sum_{k=0}^{n}\binom{n}{k}\left\{S_{n-k}(x,y)C_k(x,y)+C_{n-k}(x,y)S_k(x,y)\right\}.$$

**Theorem 2.3.** *The following summation formulas hold true:*

$${}_CB_n^{[m-1]}\left(x+1,y\right)-\ {}_CB_n^{[m-1]}\left(x,y\right)=n\sum_{k=0}^{n-1}\binom{n-1}{k}\ {}_CB_k^{[m-1]}\left(x,y\right)B_{n-1-k}^{(-1)} \quad (17)$$

*and*

$${}_SB_n^{[m-1]}\left(x+1,y\right)-\ {}_SB_n^{[m-1]}\left(x,y\right)=n\sum_{k=0}^{n-1}\binom{n-1}{k}\ {}_SB_k^{[m-1]}\left(x,y\right)B_{n-1-k}^{(-1)}. \quad (18)$$

*Proof.* For $\alpha=1$, using (9), we write as

$$\begin{aligned}\sum_{n=0}^{\infty}\ {}_CB_n^{[m-1]}\left(x+1,y\right)\frac{t^n}{n!}-\sum_{n=0}^{\infty}\ {}_CB_n^{[m-1]}\left(x,y\right)\frac{t^n}{n!}&=t\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)e^{xt}\cos(yt)\frac{(e^t-1)}{t}\\&=t\sum_{n=0}^{\infty}\ {}_CB_n^{[m-1]}\left(x,y\right)\frac{t^n}{n!}\sum_{l=0}^{\infty}\ B_l^{(-1)}\frac{t^l}{l!}\end{aligned}$$

$$= \sum_{n=0}^{\infty} n \sum_{k=0}^{n-1} \binom{n-1}{k} {}_{C}B_{k}^{[m-1]}(x,y)\, B_{n-1-k}^{(-1)} \frac{t^{n}}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ in the both sides of the above equation, we arrive at (17).

Proof of (18) is similar to that of (17), so it is omitted. □

**Theorem 2.4.** *The following relations hold true:*

$$ {}_{C}B_{n}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} {}_{C}B_{n-2k}^{[m-1,\alpha]}(x,0)\,(-1)^{k} y^{2k} \tag{19}$$

*and*

$$ {}_{S}B_{n}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2k+1} {}_{S}B_{n-2k-1}^{[m-1,\alpha]}(x,0)\,(-1)^{k} y^{2k+1}. \tag{20}$$

*Proof.* From (9), we write

$$\sum_{n=0}^{\infty} {}_{C}B_{n}^{[m-1,\alpha]}(x,y)\frac{t^{n}}{n!} = \left( \frac{t^{m}}{e^{t} - \sum\limits_{h=0}^{m-1} \frac{t^{h}}{h!}} \right)^{\alpha} e^{xt}\cos(yt)$$

$$= \sum_{m=0}^{\infty} {}_{C}B_{m}^{[m-1.\alpha]}(x,0)\frac{t^{m}}{m!} \sum_{k=0}^{\infty} (-1)^{k} y^{2k} \frac{t^{2k}}{(2k)!}$$

$$= \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} {}_{C}B_{n-2k}^{[m-1,\alpha]}(x,0)\,(-1)^{k} y^{2k+1} \right] \frac{t^{n}}{n!}.$$

Comparing the coefficients $\frac{t^n}{n!}$, we have (19).

From (10), we write

$$\sum_{n=0}^{\infty} {}_{S}B_{n}^{[m-1,\alpha]}(x,y)\frac{t^{n}}{n!} = \left( \frac{t^{m}}{e^{t} - \sum\limits_{h=0}^{m-1} \frac{t^{h}}{h!}} \right)^{\alpha} e^{xt}\sin(yt)$$

$$= \sum_{l=0}^{\infty} {}_{S}B_{l}^{[m-1.\alpha]}(x,0)\frac{t^{l}}{l!} \sum_{k=0}^{\infty} (-1)^{k} y^{2k+1} \frac{t^{2k+1}}{(2k+1)!}.$$

By using Cauchy product and comparing the coefficients $\frac{t^n}{n!}$, we have (20). □

**Theorem 2.5.** *The generalized parametric Bernoulli polynomials satisfies the following equations, respectively;*

$$ {}_{C}B_{n}^{[m-1,\alpha]}(x_1+x_2, y_1+y_2) = \sum_{k=0}^{n} \binom{n}{k} \left\{ {}_{C}B_{n-k}^{[m-1,\alpha]}(x_1,y_1)\, C_k(x_2,y_2) - {}_{S}B_{n-k}^{[m-1,\alpha]}(x_1,y_1)\, S_k(x_2,y_2) \right\} \tag{21}$$

*and*

$$ {}_{S}B_{n}^{[m-1,\alpha]}(x_1+x_2, y_1+y_2) = \sum_{k=0}^{n} \binom{n}{k} \left\{ {}_{S}B_{n-k}^{[m-1,\alpha]}(x_1,y_1)\, C_k(x_2,y_2) - {}_{C}B_{n-k}^{[m-1,\alpha]}(x_1,y_1)\, S_k(x_2,y_2) \right\}. \tag{22}$$

*Proof.* From (5), (6) and (9), we write

$$\sum_{n=0}^{\infty} {}_{C}B_n^{[m-1,\alpha]}(x_1+x_2,y_1+y_2)\frac{t^n}{n!} = \left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{(x_1+x_2)t}\cos((y_1+y_2)t)$$

$$= \left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{x_1t}\cos(y_1t)e^{x_2t}\cos(y_2t) - \left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{x_1t}\sin(y_1t)e^{x_2t}\cos(y_2t)$$

$$= \sum_{j=0}^{\infty} {}_{C}B_j^{[m-1,\alpha]}(x_1,y_1)\frac{t^j}{j!}\sum_{k=0}^{\infty} C_k(x_2,y_2)\frac{t^k}{k!} - \sum_{j=0}^{\infty} {}_{S}B_j^{[m-1,\alpha]}(x_1,y_1)\frac{t^j}{j!}\sum_{k=0}^{\infty} S_k(x_2,y_2)\frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left\{ {}_{C}B_{n-k}^{[m-1,\alpha]}(x_1,y_1)C_k(x_2,y_2) - {}_{S}B_{n-k}^{[m-1,\alpha]}(x_1,y_1)S_k(x_2,y_2)\right\}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have (21).

Similarly, from (5), (6) and (10), we have (22). □

Setting $x_1 = x_2 = x$ and $y_1 = y_2 = y$ in equation (21) and (22), we have the following equations, respectively

$${}_{C}B_n^{[m-1,\alpha]}(2x,2y) = \sum_{k=0}^{n}\binom{n}{k}\left\{ {}_{C}B_{n-k}^{[m-1,\alpha]}(x,y)C_k(x,y) - {}_{S}B_{n-k}^{[m-1,\alpha]}(x,y)S_k(x,y)\right\}$$

and

$${}_{S}B_n^{[m-1,\alpha]}(2x,2y) = \sum_{k=0}^{n}\binom{n}{k}\left\{ {}_{S}B_{n-k}^{[m-1,\alpha]}(x,y)C_k(x,y) - {}_{C}B_{n-k}^{[m-1,\alpha]}(x,y)S_k(x,y)\right\}.$$

## 3. Some Explicit Relations For ${}_{C}B_n^{[m-1]}(x,y)$ and ${}_{S}B_n^{[m-1]}(x,y)$

In this section, we provide the following symmetry identities for the parametric the generalization of the Bernoulli polynomials, respectively, ${}_{C}B_n^{[m-1]}(x,y)$ and ${}_{S}B_n^{[m-1]}(x,y)$.

**Theorem 3.1.** *The following symmetry equations for the parametric generalized Bernoulli polynomials, ${}_{C}B_n^{[m-1]}(x,y)$ holds true:*

$$\sum_{p=0}^{n}\binom{n}{p}a^{n-p}b^p\ {}_{C}B_p^{[m-1]}(0,ay)B_{n-p}^{[m-1]}\left(\frac{b}{a}x\right)$$
$$= \sum_{p=0}^{n}\binom{n}{p}b^{n-p}a^p\ {}_{C}B_p^{[m-1]}(0,by)B_{n-p}^{[m-1]}\left(\frac{a}{b}x\right) \tag{23}$$

*where $a$, $b \in \mathbb{Z}^+$.*

*Proof.* By using (3) and (9), we write as

$$h(t) = \frac{(at)^m}{e^{at}-\sum\limits_{h=0}^{m-1}\frac{(at)^h}{h!}}\frac{(bt)^m}{e^{bt}-\sum\limits_{h=0}^{m-1}\frac{(bt)^h}{h!}}e^{bxt}\cos(byt)$$

$$= \frac{(at)^m}{e^{at} - \sum\limits_{h=0}^{m-1} \frac{(at)^h}{h!}} e^{\left(\frac{bx}{a}\right)at} \frac{(bt)^m}{e^{bt} - \sum\limits_{h=0}^{m-1} \frac{(bt)^h}{h!}} \cos(byt)$$

$$= \sum_{n=0}^{\infty} B_n^{[m-1]} \left(\frac{bx}{a}\right) \frac{(at)^n}{n!} \sum_{n=0}^{\infty} {}_C B_n^{[m-1]}(0, by) \frac{(bt)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{n} \binom{n}{p} a^{n-p} b^p \; {}_C B_p^{[m-1]}(0, by) B_{n-p}^{[m-1]} \left(\frac{b}{a}x\right) \frac{t^n}{n!}.$$

Let us consider

$$h(t) = \frac{(bt)^m}{e^{bt} - \sum\limits_{h=0}^{m-1} \frac{(bt)^h}{h!}} \frac{(at)^m}{e^{at} - \sum\limits_{h=0}^{m-1} \frac{(at)^h}{h!}} e^{axt} \cos(ayt)$$

$$= \frac{(bt)^m}{e^{bt} - \sum\limits_{h=0}^{m-1} \frac{(bt)^h}{h!}} e^{\left(\frac{ax}{b}\right)bt} \frac{(at)^m}{e^{at} - \sum\limits_{h=0}^{m-1} \frac{(at)^h}{h!}} \cos(ayt)$$

$$= \sum_{n=0}^{\infty} B_n^{[m-1]} \left(\frac{ax}{b}\right) \frac{(bt)^n}{n!} \sum_{n=0}^{\infty} {}_C B_n^{[m-1]}(0, ay) \frac{(at)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{n} \binom{n}{p} b^{n-p} a^p \; {}_C B_p^{[m-1]}(0, ay) B_{n-p}^{[m-1]} \left(\frac{a}{b}x\right) \frac{t^n}{n!}.$$

Comparing the right-hand side of the last two equations, we get (23). □

**Corollary 3.1.** *The following relation holds true:*

$$\sum_{p=0}^{n} \binom{n}{p} a^{n-p} b^p \; {}_S B_p^{[m-1]}(0, ay) B_{n-p}^{[m-1]} \left(\frac{b}{a}x\right)$$

$$= \sum_{p=0}^{n} \binom{n}{p} b^{n-p} a^p \; {}_S B_p^{[m-1]}(0, by) B_{n-p}^{[m-1]} \left(\frac{a}{b}x\right).$$

**Theorem 3.2.** *The parametric generalized Bernoulli polynomials,* ${}_C B_n^{[m-1]}(x, y)$ *satisfies the following relation:*

$$\sum_{n=0}^{m} \binom{m}{n} \sum_{s=0}^{a-1} \sum_{r=0}^{b-1} {}_C B_m^{[m-1]}(bu + \frac{b}{a}s, by) B_{n-m}^{[m-1]} \left(ax + \frac{a}{b}r\right) a^m b^{n-m}$$

$$= \sum_{n=0}^{m} \binom{m}{n} \sum_{s=0}^{a-1} \sum_{r=0}^{b-1} {}_C B_m^{[m-1]}(ax + \frac{a}{b}r, ay) B_{n-m}^{[m-1]} \left(bu + \frac{b}{a}s\right) b^m a^{n-m}.$$

*Proof.* Let us consider

$$g(t) = \frac{(at)^m}{e^{at} - \sum\limits_{h=0}^{m-1} \frac{(at)^h}{h!}} e^{abut} \frac{\left(e^{abt} - 1\right)^2}{\left(e^{at} - 1\right)\left(e^{bt} - 1\right)} \frac{(bt)^m}{e^{bt} - \sum\limits_{h=0}^{m-1} \frac{(bt)^h}{h!}} e^{abxt} \cos(abyt)$$

$$= \frac{(at)^m}{e^{at} - \sum\limits_{h=0}^{m-1} \frac{(at)^h}{h!}} e^{abut} \cos(abyt) \frac{\left(e^{abt} - 1\right)}{\left(e^{bt} - 1\right)} \frac{(bt)^m}{e^{bt} - \sum\limits_{h=0}^{m-1} \frac{(bt)^h}{h!}} e^{abxt} \frac{\left(e^{abt} - 1\right)}{\left(e^{at} - 1\right)}$$

$$=\sum_{s=0}^{b-1}\sum_{n=0}^{\infty}\ {}_{C}B_n^{[m-1]}(bu+\frac{b}{a}s,by)\frac{(at)^n}{n!}\sum_{r=0}^{a-1}\sum_{p=0}^{\infty}B_p^{[m-1]}\left(ax+\frac{a}{b}r\right)\frac{(bt)^p}{p!}$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}a^p b^{n-p}\sum_{s=0}^{a-1}\sum_{r=0}^{b-1}\ {}_{C}B_p^{[m-1]}(bu+\frac{b}{a}s,by)B_{n-p}^{[m-1]}\left(ax+\frac{a}{b}r\right)\frac{t^n}{n!}.$$

Since, $g(t)$ is symmetric in $a$ and $b$, therefore above expression can also be expressed as

$$=\sum_{n=0}^{\infty}\sum_{n=0}^{m}\binom{m}{n}\sum_{s=0}^{a-1}\sum_{r=0}^{b-1}\ {}_{C}B_m^{[m-1]}(ax+\frac{a}{b}r,ay)B_{n-m}^{[m-1]}\left(bu+\frac{b}{a}s\right)b^m a^{n-m}\frac{t^n}{n!}.$$

Comparing the coefficients of the right-hand side of the last two equations, we obtain it. □

**Corollary 3.2.** *The following equation holds true:*

$$\sum_{n=0}^{m}\binom{m}{n}\sum_{s=0}^{a-1}\sum_{r=0}^{b-1}\ {}_{S}B_m^{[m-1]}(bx+\frac{b}{a}r,by)B_{n-m}^{[m-1]}\left(au+\frac{a}{b}r\right)a^m b^{n-m}$$

$$=\ \sum_{n=0}^{m}\binom{m}{n}\sum_{s=0}^{b-1}\sum_{r=0}^{a-1}\ {}_{S}B_m^{[m-1]}(ax+\frac{a}{b}r,ay)B_{n-m}^{[m-1]}\left(bu+\frac{b}{a}r\right)b^m a^{n-m}.$$

## 4. Conclusion

Many researchers ([1]-[18]) have studied and investigated intensively the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the generalized Bernoulli and Euler polynomials and numbers and Apostol-Bernoulli and Apostol-Euler polynomials and numbers. Bretti et al. [1] and Kurt [6] considered generalized Bernoulli polynomials $B_n^{[m-1]}(x)$. They gave some recurrence relations. Srivastava et al. [15] investigated parametric kind of the Fubini-type polynomials and they proved some recurrence relations.

In this work, we define the new two parametric Bernoulli polynomials, respectively, ${}_{C}B_n^{[m-1]}(x,y)$ and ${}_{S}B_n^{[m-1]}(x,y)$. We give some relations between these polynomials and prove some theorems for these polynomilas. Furthermore, we prove the symmetry identities for these polynomials.

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