TWMS J. App. Eng. Math. V.10, N.1, 2020, pp. 241-250

BISHOP'S PROPERTY (β) AND WEIGHTED CONDITIONAL TYPE OPERATORS IN k-QUASI CLASS $\mathcal{A}_n^*$

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ABSTRACT. An operator T is said to be k-quasi class $\mathcal{A}_n^*$ operator if $T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k \geq 0$, for some positive integers n and k. In this paper, we prove that the k-quasi class $\mathcal{A}_n^*$ operators have Bishop's property (β). Then, we give a necessary and sufficient condition for $T \otimes S$ to be a k-quasi class $\mathcal{A}_n^*$ operator, whenever T and S are both non-zero operators. Moreover, it is shown that the Riesz idempotent for a non-zero isolated point λ_0 of a k-quasi class $\mathcal{A}_n^*$ operator T say $\mathcal{R}_i$, is self-adjoint and $ran(\mathcal{R}_i) = ker(T-\lambda_0) = ker(T-\lambda_0)^*$. Finally, as an application in the last section, a necessary and sufficient condition is given in such a way that the weighted conditional type operators on $L^2(\Sigma)$, defined by $T_{w,u}(f) := wE(uf)$, belong to k-quasi- $\mathcal{A}_n^*$ class.

Keywords: Weighted translation, pre-frame, conditional expectation, measurable function.

AMS Subject Classification: Primary: 42C99; Secondary: 46C10.

1. Introduction

Let $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on an infinite dimensional complex Hilbert space $\mathcal{H}$. We shall write $ker(T)$ and $ran(T)$ for the null space and range of T, respectively. The spectrum of an operator $T \in B(\mathcal{H})$ is denoted by $\sigma(T)$. The operator T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. An operator $T \in B(\mathcal{H})$ is said to be

• normaloid, if $\|T^n\| = \|T\|^n$ for $n \in \mathbb{N}$ (equivalently, $\|T\| = r(T)$, the spectral radius of T);

• n-paranormal, if $\|Tx\|^{n+1} \leq \|T^{n+1}x\|\|x\|^n$ (If $n = 1$, then n-paranormal operators coincide with paranormal operators);

• n-$*$-paranormal, if $\|T^*x\|^{n+1} \leq \|T^{n+1}x\|\|x\|^n$ (If $n = 1$, then n-$*$-paranormal operators coincide with $*$-paranormal operators);

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§ Manuscript received: July 23, 2018; accepted: January 22, 2019.

• k-quasi-class $\mathcal{A}$, if $T^{k^*}\left(|T^2| - |T|^2\right)T^k \geq 0$ for $k \in \mathbb{N}$ (If $k = 0$ and $k = 1$, then k-quasi-class $\mathcal{A}$ operators coincide with class $\mathcal{A}$ operators and quasi-class $\mathcal{A}$ operators respectively, T^0 is the identity operator);
• k-quasi-$*$-class $\mathcal{A}$, if $T^{k^*}\left(|T^2| - |T^*|^2\right)T^k \geq 0$ for $k \in \mathbb{N}$ (If $k = 0$ and $k = 1$, then k-quasi-$*$-class $\mathcal{A}$ operators coincide with $*$-class $\mathcal{A}$ operators and quasi-$*$-class $\mathcal{A}$ operators respectively);
• p-hyponormal, if $|T|^{2p} - |T^*|^{2p} > 0$ for $0 < p < 1$ (If $p = 1$ then p-hyponormal operators coincide with hyponormal operators).
• An operator T is said to be k-quasi class $\mathcal{A}_n^*$ operator if

$$T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k \geq 0,$$

for some positive integers n and k. Note that if $n = 1$ and $k = 0$, then k-quasi class $\mathcal{A}_n^*$ operators coincide with k-quasi-$*$-class $\mathcal{A}$ operators and class $\mathcal{A}_n^*$ operators respectively. Moreover, if $T \in B(\mathcal{H})$ is an k-quasi class $\mathcal{A}_n^*$ operator and M is its invariant subspace, then the restriction of T i.e., $T_{|M}$ is also k-quasi class $\mathcal{A}_n^*$ operator([10, Theorem 2.4]).
• An operator T has Bishop's property (β) at $\lambda \in \mathbb{C}$, if for every open neighborhood G for λ of complex plane $\mathbb{C}$ and for every analytic function $f_n(z)$ on G such that $(T-z)f_n(z) \to 0$ uniformly on each compact subset of G, we have $f_n(z) \to 0$ uniformly on each compact subset of G. When T has Bishop's property (β) at each $\lambda \in \mathbb{C}$, then simply we say that T has property (β).

Let $\mathcal{K}$ be a complex Hilbert space and $\mathcal{H} \otimes \mathcal{K}$ the tensor product of $\mathcal{H}, \mathcal{K}$; i.e., the completion of the algebraic tensor product of $\mathcal{H}, \mathcal{K}$ with the inner product $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle\langle y_1, y_2\rangle$ for each $x_1, x_2 \in \mathcal{H}$ and $y_1, y_2 \in \mathcal{K}$. Let $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$. $T \otimes S \in B(\mathcal{H} \otimes \mathcal{K})$ denotes the tensor product of T and S defined by $(T \otimes S)(x \otimes y) = Tx \otimes Sy$ for each $x \in \mathcal{H}$ and $y \in \mathcal{K}$.
The operation of taking tensor product $T \otimes S$ preserves many properties of $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$, but by no means all of them. For example, the normaloid property is invariant under the tensor product, while the spectraloid property is not [16]. $T \otimes S$ is normal if and only if T and S are normal [9, 18]. However, there exist paranormal operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ such that $T \otimes S$ is not paranormal [1]. Duggal [3] showed that for nonzero $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$, $T \otimes S$ is p-hyponormal if and only if T, S are p-hyponormal. This result was then extended to $*$-class $\mathcal{A}$ operator [4], quasi-class $\mathcal{A}$ operators [11] and k-quasi-$*$-class $\mathcal{A}$ operators [7].

Let $T \in B(\mathcal{H})$. Pick an isolated point λ_0 in $\sigma(T)$. Then there exists a positive number $r > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r\} \cap \sigma(T) = \{\lambda_0\}$.
Let γ be the boundary of $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r\}$. Then

$$\mathcal{R}_i = \frac{1}{2\pi i}\int_\gamma (\lambda - T)^{-1} d\lambda$$

is called the Riesz idempotent of T for λ_0. It is well known that ([13]).

$$\mathcal{R}_i^2 = \mathcal{R}_i,\ \mathcal{R}_i T = T\mathcal{R}_i,\ \sigma(T_{|ran(\mathcal{R}_i)}) = \{\lambda_0\}\ and\ \ ker(T - \lambda_0 I) \subseteq ran(\mathcal{R}_i).$$

We know that the Riesz idempotent $\mathcal{R}_i$ is not an orthogonal projection and a necessary and sufficient condition for $\mathcal{R}_i$ to be orthogonal is that $\mathcal{R}_i$ is self-adjoint. In [17], Stampfli showed that if T satisfies the growth condition G_1, then the Riesz idempotent $\mathcal{R}_i$ for an isolated spectral point λ_0 of a hyponormal operator T is self-adjoint and $ran(\mathcal{R}_i) = ker(T - \lambda_0)$. Stampfli's result was then extended to p-hyponormal operators by Cho and

Tanahashi [2]. Moreover, incase $\lambda_0 \neq 0$ the Stampfli's result was extended to class $\mathcal{A}$ operators by Uchiyama and Tanahashi [21], to quasi class $\mathcal{A}$ operators by Jeon and Kim [12], to k-quasi-class $\mathcal{A}$ operators by Tanahashi, Jeon, Kim, Uchiyama [19], to paranormal operators by Uchiyama [20] and to k-quasi -$*$- class $\mathcal{A}$ operators by Mecheri [14].
This paper comprises of four sections. In Section 2, we study Bishop's property (β) for k-quasi class $\mathcal{A}_n^*$ operators. In Section 3, for non-zero operators T and S, a necessary and sufficient condition is given on which $T \otimes S$ is a k-quasi class $\mathcal{A}_n^*$ operator. In Section 4, it is proved that a corresponding Riesz idempotent of a k-quasi class $\mathcal{A}_n^*$ operator, is self-adjoint and $ran(\mathcal{R}_i) = ker(T-\lambda_0) = ker(T-\lambda_0)^*$. Finally in the last section i.e., Section 5, we will study $\mathcal{A}_n^*$ and k-quasi- $\mathcal{A}_n^*$ classes of the weighted conditional type operators on $L^2(\Sigma)$ defined by $T_{w,u}(f) := wE(uf)$.

2. Bishop's property (β) for k-quasi class $\mathcal{A}_n^*$ operators

In this section, we study the Bishop's property (β) for k-quasi class $\mathcal{A}_n^*$ operators. First, it may be worth reminding the reader some important results. If T is a class $\mathcal{A}_n^*$ operator, then T is a n-$*$-paranormal operator ([10, Theorem 2.5]). Furthermore, each n-$*$-paranormal operators satisfy property (β)([4, Proposition 2.4]).

Theorem 2.1. [10, Theorem 2.3] *Let $T \in B(\mathcal{H})$ be a k-quasi class $\mathcal{A}_n^*$ operator. T^k does not have a dense range and T has the following representation*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathcal{H} = \overline{ranT^k} \oplus kerT^{*^k}.$$

Then, T_1 is of class $\mathcal{A}_n^*$, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Theorem 2.2. [10, Theorem 3.1] *If $T \in B(\mathcal{H})$ is an k-quasi class $\mathcal{A}_n^*$ operator and $(T-\lambda)x = 0$, then $(T-\lambda)^*x = 0$ for all $\lambda \neq 0$.*

The following theorem is a structural result.

Theorem 2.3. *Let $T \in B(\mathcal{H})$ be k-quasi class $\mathcal{A}_n^*$. Then T has Bishop's property (β).*

Proof. Let $\lambda \in \mathbb{C}$ and G be an open neighborhood for λ of complex plane $\mathbb{C}$ and $f_n(z)$ be analytic on G. Suppose that $(T-z)f_n(z) \to 0$ uniformly on each compact subset of G. Then, using the representation of Theorem 2.1, we have

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_{n_1}(z) \\ f_{n_2}(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_{n_1}(z) + T_2 f_{n_2}(z) \\ (T_3 - z)f_{n_2}(z) \end{pmatrix} \longrightarrow 0.$$

Since T_3 is nilpotent, T_3 has Bishop's property (β). Indeed,
$(T_3 - z)f_n(z) \to 0$, and hence $T_3^{k-1}(T_3 - z)f_n(z) \to 0$, which implies in turn that $(T_3^k - T_3^{K-1}z)f_n(z) \to 0$. But we know that $T_3^k = 0$, so $(T_3^{K-1}z)f_n(z) \to 0$. Hence $f_{n_2}(z) \to 0$ uniformly on every compact subset of G. Then $(T_1 - z)f_{n_1}(z) \to 0$. Since T_1 is of class $\mathcal{A}_n^*$, T_1 has Bishop's property (β) by [10, Theorem 2.5] and [4, Proposition 2.4]. Hence, $f_{n_1}(z) \to 0$ uniformly on every compact subset of G. Thus, T has Bishop's property (β). □

3. Tensor product of k-quasi class $\mathcal{A}_n^*$ operators

In the following, we extend the result of Gao and Li [7] to k-quasi-class $\mathcal{A}_n^*$ operator T. We start with the following result.

Theorem 3.1. *Let $T \in B(\mathcal{H})$ be a k-quasi class $\mathcal{A}_n^*$ operator for a positive integer k. Then the bellow assertions hold.*
(1) $\|T^{n+1+m}x\|^{\frac{2}{n+1}}\|T^m x\|^{2(1-\frac{1}{n+1})} \geq \|T^*T^m x\|^2$ for all $x \in \mathcal{H}$ and all positive integers $m \geq k$.
(2) If $T^m = 0$ for some positive integer $m \geq k$, then $T^k = 0$.

Proof. Since

$$k - quasi\ class\, \mathcal{A}_n^* \subseteq (k+1) - quasi\ class\, \mathcal{A}_n^*,$$

we just need to prove the case $m = k$. Choose $x \in \mathcal{H}$ arbitrarily and then observe that

$$\langle T^{*k}|T^*|^2 T^k x, x\rangle = \langle TT^*T^k x, T^k x\rangle = \|T^*T^k x\|^2$$

and

$$\langle T^{*k}|T^{n+1}|^{\frac{2}{n+1}} T^k x, x\rangle = \langle (T^{*(n+1)}T^{n+1})^{\frac{1}{n+1}} T^k x, T^k x\rangle.$$

Now, by the Hölder-McCarthy inequality we have

$$\langle (T^{*(n+1)}T^{n+1})^{\frac{1}{n+1}} T^k x, T^k x\rangle \leq \|T^{n+1+k}x\|^{\frac{2}{n+1}}\|T^k x\|^{2(1-\frac{1}{n+1})}.$$

But T is a k-quasi class $\mathcal{A}_n^*$ operator and hence we get that

$$\|T^{n+1+k}x\|^{\frac{2}{n+1}}\|T^k x\|^{2(1-\frac{1}{n+1})} \geq \|T^*T^k x\|^2.$$

(2) If $m = k$, it is clear that $T^k = 0$. If $T^{k+1} = 0$, then $T^{n+1+k} = 0$. Therefore, by (1) we have $T^*T^k = 0$. Now for each $x \in \mathcal{H}$ consider that

$$\|T^k x\| = \langle T^*T^k x, T^{k-1}x\rangle = 0.$$

Hence, $T^k = 0$. □

The following theorem gives a necessary and sufficient condition for $T \otimes S$ to be a k-quasi class $\mathcal{A}_n^*$ operator, whenever T and S are both non-zero operators.

Theorem 3.2. *Let $T, S \in B(\mathcal{H})$ be non-zero operators. Then $T \otimes S$ is k-quasi class $\mathcal{A}_n^*$ operator if and only if one of the following holds:*
a) T and S are k-quasi class $\mathcal{A}_n^*$ operators.
b) $T^k = 0$ or $S^k = 0$.

Proof. Suppose that (a) or (b) holds. We are going to show that $T \otimes S$ is k-quasi class $\mathcal{A}_n^*$ operator i.e.,

$$(T \otimes S)^{*k}\left(|(T \otimes S)^{n+1}|^{\frac{2}{n+1}} - |(T \otimes S)^*|^2\right)(T \otimes S)^k \geq 0.$$

It is worth noting that $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ and $|T \otimes S|^q = |T|^q \otimes |S|^q$, for each positive real number q. Hence by using these facts, the above statement equivalently can be recast as follows

$$T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k \otimes S^{*k}|S^{n+1}|^{\frac{2}{n+1}}S^k$$

$$+T^{*k}|T^*|^2T^k \otimes S^{*k}\left(|S^{n+1}|^{\frac{2}{n+1}} - |S^*|^2\right)S^k \geq 0.$$

But the operators $S^{*k}|S^{n+1}|^{\frac{2}{n+1}}S^k$ and $T^{*k}|T^*|^2T^k$ are positive. Now if (a) or (b) holds, then the above statement is evidently positive which means that $T \otimes S$ is k-quasi class $\mathcal{A}_n^*$ operator.

Conversely, suppose that $T \otimes S$ is k-quasi class $\mathcal{A}_n^*$ operator. Then for every $x, y \in \mathcal{H}$ we have,

$$\langle T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k x, x\rangle\langle S^{*k}|S^{n+1}|^{\frac{2}{n+1}}S^k y, y\rangle \tag{1}$$
$$+\langle T^{*k}|T^*|^2T^k x, x\rangle\langle S^{*k}\left(|S^{n+1}|^{\frac{2}{n+1}} - |S^*|^2\right)S^k y, y\rangle \geq 0.$$

It suffices to show that if (a) does not hold, then (b) holds. To the contrary, suppose that neither T^k nor S^k is the zero operator.
Without loss of generality, assume that T is not a k-quasi class $\mathcal{A}_n^*$ operator. Then there exists $x_0 \in \mathcal{H}$ such that

$$\langle T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k x_0, x_0\rangle = \alpha < 0,$$

and

$$\langle T^{*k}|T^*|^2T^k x_0, x_0\rangle = \beta > 0.$$

Hence, for all $y \in \mathcal{H}$ with the aid of (1) we have,

$$(\alpha+\beta)\langle S^{*k}|S^{n+1}|^{\frac{2}{n+1}}S^k y, y\rangle \geq \beta\langle S^{*k}|S^*|^2S^k y, y\rangle. \tag{2}$$

This yields that S is k-quasi class $\mathcal{A}_n^*$ operator. Subsequently, we have

$$\begin{aligned}\langle S^{*k}|S^*|^2S^k y, y\rangle &= \langle SS^*S^k y, S^k y\rangle\\ &= \langle S^*S^k y, S^*S^k y\rangle\\ &= \|S^*S^k y\|^2\end{aligned}$$

and

$$\begin{aligned}\langle S^{*k}|S^{n+1}|^{\frac{2}{n+1}}S^k y, y\rangle &= \langle (S^{*(n+1)}S^{n+1})^{\frac{1}{n+1}}S^k y, S^k y\rangle\\ &\leq \langle S^{*(n+1)}S^{n+1}S^k y, S^k y\rangle^{\frac{1}{n+1}}\|S^k y\|^{2(1-\frac{1}{n+1})}\\ &= \|S^{n+1+k}y\|^{\frac{2}{n+1}}\|S^k y\|^{2(1-\frac{1}{n+1})}.\end{aligned}$$

Eventually, for all $y \in \mathcal{H}$ by (2), it is found that

$$(\alpha+\beta)\|S^{n+1+k}y\|^{\frac{2}{n+1}}\|S^k y\|^{2(1-\frac{1}{n+1})} \geq \beta\|S^*S^k y\|^2. \tag{3}$$

Since S is k-quasi class $\mathcal{A}_n^*$ operator, by Theorem 2.1 we can write

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad on \quad \mathcal{H} = \overline{ranS^k} \oplus kerS^{*k},$$

where S_1 is a class $\mathcal{A}_n^*$ operator. Therefore, for each $\eta \in \overline{ranS^k}$ we may rewrite inequality (3) as follows

$$(\alpha+\beta)\|S_1^{n+1+k}\eta\|^{\frac{2}{n+1}}\|S_1^k\eta\|^{2(1-\frac{1}{n+1})} \geq \beta\|S_1^*S_1^k\eta\|^2. \tag{4}$$

As mentioned above, S_1 is a class $\mathcal{A}_n^*$ operator and then is n-$*$-paranormal operator([10, Theorem 2.5]). Therefore, S_1 is normaloid (see [22]). In this circumstance, from inequality (4) one can easily deduce that

$$(\alpha+\beta)\|S_1\|^2 = (\alpha+\beta)\|S_1^2\| \geq \beta\|S_1^*\|^2 = \beta\|S_1\|^2.$$

This inequality in turn implies that $S_1 = 0$. Then, it is deduced that $S^{k+1} = 0$, because of

$$S^{k+1} = S_1S^k y = 0 \;\; for \; all \; y \in \mathcal{H}.$$

Eventually, by Theorem 3.1, we obtain that $S^k = 0$ which is a contradiction. $\square$

4. **Riesz idempotent for an isolated point of the spectrum**

In this section, we will show that the Riesz idempotent $\mathcal{R}_i$ for an isolated spectral point λ_0 of a k-quasi class $\mathcal{A}_n^*$ operator T is self-adjoint and

$$ran(\mathcal{R}_i) = ker(T - \lambda_0) = ker(T - \lambda_0)^*.$$

For this, we need the following lemma as a useful tool.

Lemma 4.1. *Let $T \in B(\mathcal{H})$ be a k-quasi-class $\mathcal{A}_n^*$ operator. Then T is isoloid.*

Proof. Assume that $T \in B(\mathcal{H})$ is a k-quasi class $\mathcal{A}_n^*$ operator with the representation given in Theorem 2.1. Let z be an isolated point in $\sigma(T)$. Since $\sigma(T) = \sigma(T_1) \cup \{0\}$, then z is an isolated point in $\sigma(T_1)$ or $z = 0$. If z isolated point in $\sigma(T_1)$, then by [10, Theorem 2.5] and [22, Lemma 2.2], z is an eigenvalue of T_1. Now assume that $z = 0$ and $z \notin \sigma(T_1)$. Then, we may find a non-zero vector $x \in ker(T_3)$ such that $-T_1^{-1}T_2x \oplus x \in ker(T)$. This fact, shows that z is an eigenvalue of T and the proof is completed. □

Theorem 4.1. *Let $T \in B(\mathcal{H})$ be a k-quasi class $\mathcal{A}_n^*$ operator. If λ_0 be a non-zero isolated point of $\sigma(T)$ and $\mathcal{R}_i$ is the Riesz idempotent for λ_0, then*

$$ran(\mathcal{R}_i) = ker(T - \lambda_0) = ker(T - \lambda_0)^*.$$

In particular, $\mathcal{R}_i$ is self adjoint.

Proof. By Lemma 4.1, λ_0 is an eigenvalue of T. So $ran(\mathcal{R}_i) = ker(T - \lambda_0)$ and $ker(\mathcal{R}_i) = ran(T-\lambda_0)$(see [13]). Moreover, by Theorem 2.2 we know that $ker(T-\lambda_0) \subseteq ker(T-\lambda_0)^*$, so it suffices only to show that $ker(T-\lambda_0)^* \subseteq ker(T-\lambda_0)$. For this, one can easily to check that $ker(T-\lambda_0)$ is a reducing subspace of T. But by [10, Theorem 2.4], the restriction of k-quasi class $\mathcal{A}_n^*$ operator to its reducing subspaces is also a k-quasi class $\mathcal{A}_n^*$ operator. Hence, the operator T can be written as follows: $T = \lambda_0 \oplus T_1$ on $\mathcal{H} = ker(T-\lambda_0) \oplus (ker(T-\lambda_0))^\perp$, where T_1 is k-quasi class $\mathcal{A}_n^*$ operator with $ker(T_1 - \lambda_0) = 0$. Since

$$\lambda_0 \in \sigma(T) = \{\lambda_0\} \cup \sigma(T_1)$$

is isolated, only two cases occur: $\lambda_0 \notin \sigma(T_1)$, or λ_0 is an isolated point of $\sigma(T_1)$ and this contradicts the fact $ker(T_1 - \lambda_0) = 0$. Since T_1 is invertible as an operator on $(ker(T - \lambda_0))^\perp$, we have $ker(T - \lambda_0) = ker(T - \lambda_0)^*$. Since $ker(T - \lambda_0) \subseteq ker(T - \lambda_0)^*$, we have $ker(T - \lambda_0) \perp ran(T - \lambda_0)$, and hence $ran(\mathcal{R}_i) \perp ker(\mathcal{R}_i)$. That is, $\mathcal{R}_i$ is self-adjoint. □

5. **The weighted conditional type operators satisfying k-quasi class $\mathcal{A}_n^*$**

In this section, we study $\mathcal{A}_n^*$ and k-quasi- $\mathcal{A}_n^*$ classes of the weighted conditional type operators. A necessary and sufficient condition is given on which the weighted conditional type operators belong to k-quasi class $\mathcal{A}_n^*$. Let (X, Σ, μ) be a complete σ-finite measure space and let $\mathcal{A}$ be a σ- subalgebra of Σ such that $(X, \mathcal{A}, \mu)$ is also σ-finite. The space of complex-valued Σ-measurable functions on X is $L^0(\Sigma)$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. For any non-negative $f \in L^0(\Sigma)$, corresponds a measure $\upsilon_f(B) = \int_B f d\mu$ for all $B \in \mathcal{A}$, which is absolutely continuous with respect to μ. Then by the Radon-Nikodym theorem, there exists a unique non-negative $\mathcal{A}$-measurable function $E(f)$ such that

$$\int_B E(f) d\mu = \int_B f d\mu \quad for\ all\ B \in \mathcal{A}.$$

Hence we obtain an operator E from $L^2(\Sigma)$ onto $L^2(\mathcal{A})$ which is called conditional expectation operator associated with respect to $\mathcal{A}$. The role of this operator is important in

this note and we list here some of its useful properties:
(i) If $g \in L^0(\mathcal{A})$, then $E(gf) = gE(f)$.
(ii) If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
(iii)$|E(f)|^p \leq E(|f|^p)$.
(iv) $E(|f|^2) = |E(f)|^2$ if and only if $f \in L^0(\mathcal{A})$.
For more details see [15].
For measurable weight functions $w : X \to \mathbb{C}$ and $u : X \to \mathbb{C}$ the weighted conditional type operator $T_{w,u} : L^2(\Sigma) \to L^2(\Sigma)$ is defined by $T_{w,u}(f) := wE(uf)$. This operator is bounded if and only if $(E(|w|^2)^{\frac{1}{2}}(E(|u|^2)^{\frac{1}{2}} \in L^\infty(\mathcal{A})$ and in this case its norm is given by $\|T_{w,u}\| = |(E(|w|^2)^{\frac{1}{2}}(E(|u|^2)^{\frac{1}{2}}\|_\infty$ ([6]). In case, $w = 1$ the operator $T_{w,u}$ has been widely discussed in [8]. Moreover, some classes of $T_{w,u}$ such as class $\mathcal{A}$, $*$-$\mathcal{A}$ class, quasi-$*$-$\mathcal{A}$ class and its spectra have been studied in [5, 6]. In the following theorem we give a necessary and a sufficient condition separately on which $T_{w,u}$ is to be $\mathcal{A}_n^*$-class operator.

Theorem 5.1. *Let operator $T_{w,u}$ be a bounded on $L^2(\Sigma)$. Then*
(*a*) *If for each $f \in L^2(\Sigma)$*

$$|E(uf)|^2|E(uw)|^{\frac{2n}{n+1}}\left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}}\chi_{\sigma(E(|u|^2))} \geq (E(|u|^2))|E(\overline{w}f)|^2,$$

then $T_{w,u}$ is $\mathcal{A}_n^*$ operator.
(*b*) *If $T_{w,u}$ is $\mathcal{A}_n^*$ operator, then*

$$|E(u)|^2|E(uw)|^{\frac{2n}{n+1}}\left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}}\chi_{\sigma(E(|u|^2))} \geq E(|u|^2)|E(\overline{w})|^2.$$

Proof. For each $f \in L^2(\Sigma)$, a routine computation shows that

$$|T_{w,u}^{n+1}|^{\frac{2}{n+1}}(f) = |E(uw)|^{\frac{2n}{n+1}}\left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}}\chi_{\sigma(E(|u|^2))}\overline{u}E(uf),$$

and

$$|T_{w,u}^*|^2(f) = E(|u|^2)wE(\overline{w}f).$$

Therefore,

$$\begin{aligned}
&\langle |T_{w,u}^{n+1}|^{\frac{2}{n+1}}(f) - |T_{w,u}^*|^2(f), f\rangle \\
&= \int_X |E(uw)|^{\frac{2n}{n+1}}\left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}}\chi_{\sigma(E(|u|^2))}\overline{uf}E(uf) \\
&- E(|u|^2)w\overline{f}E(\overline{w}f)d\mu \\
&= \int_X |E(uw)|^{\frac{2n}{n+1}}\left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}}\chi_{\sigma(E(|u|^2))}|E(uf)|^2 \\
&- E(|u|^2)|E(\overline{w}f)|^2d\mu.
\end{aligned}$$

Now, if for each $f \in L^2(\Sigma)$ we assume that

$$|E(uf)|^2|E(uw)|^{\frac{2n}{n+1}}\left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}}\chi_{\sigma(E(|u|^2))} \geq (E(|u|^2))|E(\overline{w}f)|^2,$$

then $T_{w,u}$ is easily $\mathcal{A}_n^*$ operator.

Conversely, suppose that $T_{w,u}$ is $\mathcal{A}_n^*$ operator, then for every $f \in L^2(\Sigma)$ we have

$$\int_X |E(uw)|^{\frac{2n}{n+1}} \left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}} \chi_{\sigma(E(|u|^2))}|E(uf)|^2 -$$

$$E(|u|^2)|E(\overline{w}f)|^2 d\mu \geq 0.$$

Pick $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A, we have

$$\int_A |E(uw)|^{\frac{2n}{n+1}} \left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}} \chi_{\sigma(E(|u|^2))}|E(u)|^2 -$$

$$E(|u|^2)|E(\overline{w})|^2 d\mu \geq 0.$$

Since $A \in \mathcal{A}$ is arbitrarily chosen, we get that

$$|E(u)|^2|E(uw)|^{\frac{2n}{n+1}} \left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}} \chi_{\sigma(E(|u|^2))} \geq (E(|u|^2))|E(\overline{w})|^2.$$

□

Theorem 5.2. *A bounded operator $T_{w,u}$ on $L^2(\Sigma)$ is a k-quasi class $\mathcal{A}_n^*$ operator if and only if*

$$|E(uw)|^{2k+2k(n+1)} \geq (E|u|^2)^{2n+1}(E|w|^2)^{2k(n+1)-1}.$$

Proof. There is no difficulty to check that for each $f \in L^2(\Sigma)$,

$$T_{w,u}^{*k}|T_{w,u}^{n+1}|^{\frac{2}{n+1}}T_{w,u}^k(f) =$$
$$|E(uw)|^{\frac{2n+2k(n+1)}{n+1}} \left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}} \chi_{\sigma(E(|u|^2))}\overline{u}E(uf)$$

and

$$T_{w,u}^{*k}|T_{w,u}^*|^2T_{w,u}^k(f) = E(|u|^2)\left(E(|w|^2)\right)^{2k}\overline{u}E(uf).$$

So, for all $f \in L^2(\Sigma)$ we obtain that,

$$\begin{aligned}
&\langle T_{w,u}^{*k}|T_{w,u}^{n+1}|^{\frac{2}{n+1}}T_{w,u}^k(f) - T_{w,u}^{*k}|T_{w,u}^*|^2T_{w,u}^k(f), f\rangle \\
&= \int_X |E(uw)|^{\frac{2n}{n+1}}|E(uw)|^{2k} \left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}} \chi_{\sigma(E(|u|^2))}\overline{uf}E(uf) \\
&- E(|u|^2)\left(E(|w|^2)\right)^{2k}\overline{uf}E(uf)d\mu \\
&= \int_X \Bigg(|E(uw)|^{\frac{2n+2k(n+1)}{n+1}} \left(\frac{E(|w|^2)}{(E(|u|^2))^n}\right)^{\frac{1}{n+1}} \chi_{\sigma(E(|u|^2))} \\
&-E(|u|^2)(E(|w|^2))^{2k}\Bigg)|E(uf)|^2 d\mu.
\end{aligned}$$

Hence, if we assume that

$$|E(uw)|^{2n+2k(n+1)} \geq (E|u|^2)^{2n+1}(E|w|^2)^{2k(n+1)-1},$$

then it is easily seen $T_{w,u}$ is a k-quasi class $\mathcal{A}_n^*$ operator.

Conversely, suppose that $T_{w,u}$ is a k-quasi class $\mathcal{A}_n^*$ operator. Then for all $f \in L^2(\Sigma)$ we have

$$\int_X \left(|E(uw)|^{\frac{2n+2k(n+1)}{n+1}} \left(\frac{E(|w|^2)}{(E(|u|^2))^n} \right)^{\frac{1}{n+1}} \chi_{\sigma(E(|u|^2))} - E(|u|^2) \left(E(|w|^2) \right)^{2k} \right) |E(uf)|^2 d\mu \geq 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A, we have

$$\int_A \left(|E(uw)|^{\frac{2n+2k(n+1)}{n+1}} \left(\frac{E(|w|^2)}{(E(|u|^2))^n} \right)^{\frac{1}{n+1}} \chi_{\sigma(E(|u|^2))} - E(|u|^2) \left(E(|w|^2) \right)^{2k} \right) |E(u)|^2 d\mu \geq 0.$$

Since $A \in \mathcal{A}$ is chosen arbitrarily, then

$$|E(uw)|^{\frac{2n+2k(n+1)}{n+1}} \left(\frac{E(|w|^2)}{(E(|u|^2))^n} \right)^{\frac{1}{n+1}} \chi_{\sigma(E(|u|^2))} \geq E(|u|^2) \left(E(|w|^2) \right)^{2k})$$

and the proof is completed. $\square$

Remark 5.1. *Note that if the inequality*

$$|E(uw)|^{2k+2k(n+1)} \geq (E|u|^2)^{2n+1} (E|w|^2)^{2k(n+1)-1}$$

holds, then by Theorem 5.2 and Theorem 2.3, $T_{w,u}$ has Bishop's property (β) on $L^2(\Sigma)$.

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