TWMS J. App. and Eng. Math. V.10, N.2, 2020, pp. 400-410

ON A CLASS OF $p(x)$-KIRHHOFF TYPE PROBLEMS WITH ROBIN BOUNDARY CONDITIONS AND INDEFINITE WEIGHTS

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Abstract. In this paper, we consider a class of $p(x)$-Kirhhoff type problems with Robin boundary conditions and indefinite weights. Under some suitable conditions on the nonlinearities, we establish the existence of at least one non-trivial weak solution for the problem by using the minimum principle and the Ekeland variational principle.

Keywords: $p(x)$-Kirhhoff type problems; Robin boundary condition; Singular weights; Variational methods

AMS Subject Classification: 35J70, 35J60, 35D05

1. Introduction

In this paper, we are interested in the existence of weak solutions for the following $p(x)$-Kirhhoff type problem with Robin boundary condition

$$\begin{cases} -M\left(L_{\beta(x)}(u)\right)\Delta_{p(x)}u = \lambda V_1(x)|u|^{q_1(x)-2}u - \mu V_2(x)|u|^{q_2(x)-2}u, \quad x\in\Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial\nu} + \beta(x)|u|^{p(x)-2}u = 0, \quad x\in\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N\geq 2$, $\frac{\partial u}{\partial\nu}$ is the outer normal derivative of u with respect to $\partial\Omega$, $L_{\beta(x)}(u) := \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma$, $1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < N$, $\beta\in L^\infty(\partial\Omega)$, $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$, λ,μ are two real parameters, V_1, V_2 are functions in some generalized Sobolev spaces. Throughout this paper, we assume that $M : \mathbb{R}_0^+ := [0,+\infty) \to \mathbb{R}$ is a continuous function, there exist $m_2 \geq m_1 > 0$ and $\alpha > 1$ such that the following condition hold:

(M_0) $m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}$ for all $t\in\mathbb{R}_0^+$.

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§ Manuscript received: July 30, 2018; accepted: Novembet 7, 2018.
TWMS Journal of Applied and Engineering Mathematics, Vol.10, No.2;
This research is supported by Quang Binh University Foundation for Science and Technology Development; (Grant No: CS. 09.2018).

Problem (1) was firstly proposed by Kirchhoff in 1883 as a model given by the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

where ρ is the mass density, P_0 is the initial tension, h represents the area of the cross-section, E is the Young modulus of the material and L is the length of the string, see [26]. Problem (2) is often called a nonlocal problem because it contains an integral over Ω. This causes some mathematical difficulties which make the study of such a problem particularly interesting. The nonlocal problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [15].

Kirchhoff type problems have been studied in many papers in the last decades, we refer to some recent works [3, 6, 8, 12, 13, 17, 16, 19, 20, 29] in which some interesting results on the problems with Dirichlet or Neumann boundary conditions have been obtained. Relatively speaking, Kirchhoff type problems with Robin boundary conditions have rarely been considered. Robin boundary conditions are a weighted combination of Dirichlet and Neuman boundary conditions and it is also called impedance boundary conditions, from their application in electromagnetic problems or convective boundary conditions from their application in heat transfer problems. Moreover, Robin conditions are commonly used in solving Sturm-Liouville problems which appear in many contexts in sciences and engineering, see [21]. We know that the $p(x)$-Laplacian operator where $p(.)$ is a continuous function possesses more complicated properties than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. The study of various mathematical problems with variable exponent are interesting in applications arising in the study of calculus of variations, partial differential equations [1, 18, 22], as well as in the modelling of electrorheological fluids [28], the analysis of Non-Newtonian fluids [30], fluid flow in porous media [4], magnetostatics [11], image restoration [9], and capillarity phenomena [7]. For this reason, ordinary differential and partial differential equations with nonstandard growth conditions have received specific attention in recent years.

In this paper, we are motivated by the results introduced in [2, 3, 10, 12, 14, 25], we study the existence of solutions for problem (1) with Robin boundary condition. In [3, 12], the authors considered a class of $p(x)$-Kirhhoff type problems with Dirichlet boundary conditions and positive weight functions. Using variational methods, they obtained some existence and multiplicity results for the problem with the singular or non-singular conditions imposed on the Kirhhoff function. In [10, 14, 25], the authors studied the existence of solutions for some local problems with indefinite weight functions. In a recent paper [2], Allaoui et al. have considered $p(x)$-Kirhhoff type problems with Robin boundary conditions of the form

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma\right)\Delta_{p(x)}u = f(x,u), & x \in \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 2$, $\frac{\partial u}{\partial \nu}$ is the outer normal derivative of u with respect to $\partial\Omega$, $M : \mathbb{R}_0^+ \to \mathbb{R}$ is a non-singular function, that is, $\inf_{t\in\mathbb{R}_0^+} M(t) > 0$ and $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the Ambrosetti-Rabinowitz type condition, see condition (f_1) of [3]. Using the mountain pass theorem and Fountain theorem, the authors obtained some existence and multiplicity results for problem (3). The purpose of this paper is to consider Robin problem (1) in the case when the Kirhhoff function M is singular at zero and the nonlinear term f involves indefinite weight functions. Therefore, our main results introduced here are natural extensions from the papers mentioned above. Finally, we point out that if $M(t) \equiv 1$ and $\mu \equiv 0$ then problem (1) has been studied in a very recent paper due to Kefi [25].

2. Preliminaries

In order to state and prove the results of the paper in the next section, we recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the books [22, 28] and the papers [21, 24, 27]. Set

$$C_+(\overline{\Omega}) := \{h;\ h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \text{ and } h^- = \inf_{x\in\overline{\Omega}} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u : \text{a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} := \inf\left\{\lambda > 0;\ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < +\infty$ and continuous functions are dense if $p^+ < +\infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < +\infty$ and p_1, p_2 are variable exponents so that $p_1(x) \le p_2(x)$ a.e. $x \in \Omega$ then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequalities

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)} \tag{4}$$

hold true. Moreover, if $p_1, p_2, p_3 : \overline{\Omega} \to (1, +\infty)$ are Lipschitz continuous functions such that $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_2(x)} = 1$, then for any $u \in L^{p_1(x)}(\Omega)$, $v \in L^{p_2(x)}(\Omega)$ and $w \in L^{p_3(x)}(\Omega)$ the following inequality holds

$$\left|\int_\Omega uvw\,dx\right| \le \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\right)|u|_{p_1(x)}|u|_{p_2(x)}|u|_{p_3(x)}. \tag{5}$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$ then the following relations hold

$$|u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{6}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} \tag{7}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow \ \rho_{p(x)}(u_n - u) \to 0. \tag{8}$$

If $p \in C_+(\overline{\Omega})$ the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, consisting of functions $u \in L^{p(x)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $[L^{p(x)}(\Omega)]^N$, endowed with the norm

$$\|u\| := \inf\left\{\lambda > 0;\ \int_\Omega \left[\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} + \left|\frac{u(x)}{\lambda}\right|^{p(x)}\right] dx \leq 1\right\}$$

or

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)},$$

is a separable and reflexive Banach space. The space of smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$, but if the exponent $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is,

$$|p(x) - p(y)| \leq -\frac{M}{\log(|x-y|)}, \quad \forall x, y \in \Omega, \quad |x - y| \leq \frac{1}{2},$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$. The space $\left(W^{1,p(x)}(\Omega), \|.\|\right)$ is a separable and Banach space. We note that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $\overline{\Omega}$ then the embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) > N$. If $s \in C_+(\partial\Omega)$ and $s(x) < p_*(x)$ for all $\partial\Omega$ then the trace embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\partial\Omega)$$

is compact and continuous, where $p_*(x) = \frac{(N-1)p(x)}{N-p(x)}$ if $p(x) < N$ or $p_*(x) = +\infty$ if $p(x) > N$. Moreover, for any $u \in W^{1,p(x)}(\Omega)$, let us define

$$\|u\|_\partial := |\nabla u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\partial\Omega)},$$

then $\|u\|_\partial$ is a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to the norm $\|u\|$, see [21, Theorem 2.1].

Now, let us introduce a norm which will be used later. Let $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$, and for any $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_{\beta(x)} := \inf\left\{\lambda > 0;\ \int_\Omega \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx + \int_{\partial\Omega} \beta(x) \left|\frac{u(x)}{\lambda}\right|^{p(x)} d\sigma \ \leq 1\right\},$$

where $d\sigma_x$ is the measure on the boundary $\partial\Omega$. Then $\|u\|_{\beta(x)}$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|.\|$ and $\|.\|_\partial$. Let

$$I_{\beta(x)}(u) = \int_\Omega |\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}\, d\sigma,$$

we have

$$\|u\|_{\beta(x)}^{p^-} \leq I_{\beta(x)}(u) \leq \|u\|_{\beta(x)}^{p^+} \tag{9}$$

provided $\|u\|_{\beta(x)} > 1$ while

$$\|u\|_{\beta(x)}^{p^+} \leq I_{\beta(x)}(u) \leq \|u\|_{\beta(x)}^{p^-} \tag{10}$$

provided $\|u\|_{\beta(x)} < 1$ and

$$\|u_n - u\|_{\beta(x)} \to 0 \Leftrightarrow I_{\beta(x)}(u_n - u) \to 0. \tag{11}$$

Proposition 2.1 (see [24]). *For $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$, let us define the functional $L_{\beta(x)} : W^{1,p(x)}(\Omega) \to \mathbb{R}$ by*

$$L_{\beta(x)}(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)} \, d\sigma \tag{12}$$

for all $u \in W^{1,p(x)}(\Omega)$. Then $L_{\beta(x)} \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ and its derivative is given by

$$L'_{\beta(x)}(u)(v) = \int_\Omega |\nabla u|^{p(x)-2} uv \, dx + \int_{\partial\Omega} \beta(x) |u|^{p(x)-2} uv \, d\sigma. \tag{13}$$

Moreover, we have the following assertions

(i) $L'_{\beta(x)} : W^{1,p(x)}(\Omega) \to W^{-1,p(x)}(\Omega)$ *is a continuous, bounded and strictly monotone operator;*

(ii) $L'_{\beta(x)} : W^{1,p(x)}(\Omega) \to W^{-1,p(x)}(\Omega)$ *is a mapping of type $(S)^+$, i.e. if $\{u_n\}$ converges weakly to u in $W^{1,p(x)}(\Omega)$ and $\limsup_{n\to\infty} L'_{\beta(x)}(u_n)(u_n - u) \leq 0$, then $\{u_n\}$ converges strongly to u in $W^{1,p(x)}(\Omega)$.*

3. Main results

In this section, we will state and prove the main results of the paper. Let us denote by X the Sobolev space with variable exponent $W^{1,p(x)}(\Omega)$ and denote by c_i positive constant whose value may change from line to line. In the sequel, we impose the following condition:

(H) $1 < q_1(x) < q_2(x) < p(x) < N < \alpha N < \min\{s_1(x), s_2(x)\}$ for all $x \in \overline{\Omega}$, where $q_1, q_2, s_1, s_2 \in C(\overline{\Omega})$, $V_1 \in L^{\frac{s_1(x)}{\alpha}}(\Omega)$ such that $V_1(x) > 0$ in $\Omega_0 \subset\subset \Omega$ with $|\Omega_0| > 0$ and $V_2 \in L^{\frac{s_2(x)}{\alpha}}(\Omega)$ such that $V_2(x) \geq 0$ in Ω.

Remark 3.1. *From (H) and (6), (7), it is clear that for all $u \in X$,*

$$\begin{aligned} \left| \int_\Omega \frac{V_i(x)}{q_i(x)} |u|^{q_i(x)} \, dx \right| &\leq \frac{1}{q_i^-} |V|_{\frac{s_i(x)}{\alpha}} ||u|^{q_i(x)}|_{\frac{s_i(x)}{s_i(x)-\alpha}} \\ &= \begin{cases} \frac{1}{q_i^-} |V|_{\frac{s_i(x)}{\alpha}} |u|^{q_i^-}_{\frac{s_i(x)q_i(x)}{s(x)-\alpha}} & \text{if } |u|_{\frac{s_i(x)q_i(x)}{s(x)-\alpha}} \leq 1, \\ \frac{1}{q_i^-} |V|_{\frac{s_i(x)}{\alpha}} |u|^{q_i^+}_{\frac{s_i(x)q_i(x)}{s(x)-\alpha}} & \text{if } |u|_{\frac{s_i(x)q_i(x)}{s(x)-\alpha}} \geq 1, \quad i = 1, 2. \end{cases} \end{aligned}$$

We set $h_i(x) = \frac{s_i(x)q_i(x)}{s_i(x)-\alpha}$ and $k_i(x) = \frac{s_i(x)q_i(x)}{s_i(x)-\alpha q_i(x)}$, $i = 1, 2$. By condition (H), it follows that

$$Ns_i(x) [q_i(x) - p(x)] < 0 < p(x) [s_i(x)q_i(x) - \alpha N]$$

for all $x \in \overline{\Omega}$ and thus, $h_i(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$, $i = 1, 2$. Similarly, we also have

$$Ns_i(x) [q_i(x) - p(x)] < 0 < p(x)q_i(x) [s_i(x) - \alpha N]$$

for all $x \in \overline{\Omega}$, so $k_i(x) < p^*(x)$ for all $x \in \overline{\Omega}$, $i = 1, 2$. Therefore, the embeddings $X \hookrightarrow L^{h_i(x)}(\Omega)$ and $X \hookrightarrow L^{k_i(x)}(\Omega)$ are continuous and compact.

Definition 3.1. *We say that $u \in X$ is a weak solution of problem (1) if*

$$M\left(L_{\beta(x)}(u)\right)\left(\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\,d\sigma\right) - \lambda\int_\Omega V_1(x)|u|^{q_1(x)-2}uv\,dx$$
$$+\mu\int_\Omega V_2(x)|u|^{q_2(x)-2}uv\,dx = 0$$

for all $v \in X$.

Theorem 3.1. *Assume that the conditions (M_0) and (H) hold. Then for all $\lambda > 0$ and $\mu > 0$, problem (1) has at least one non-trivial weak solution with negative energy.*

Theorem 3.2. *Assume that the conditions (M_0) and (H) hold. Then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ and all $\mu > 0$, problem (1) has at least one positive weak solution.*

In order to use variational methods, for each $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, let us define the functional $J_{\lambda,\mu} : X = W^{1,p(x)}(\Omega) \to \mathbb{R}$ corresponding to problem (1) by

$$J_{\lambda,\mu}(u) = \widehat{M}\left(L_{\beta(x)}(u)\right) - \lambda\int_\Omega \frac{V_1(x)}{q_1(x)}|u|^{q_1(x)}\,dx + \mu\int_\Omega \frac{V_2(x)}{q_2(x)}|u|^{q_2(x)}\,dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$ and

$$L_{\beta(x)}(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma.$$

By the conditions (M_0) and (H), some simple computations help us to show that $J_{\lambda,\mu} \in C^1(X, \mathbb{R})$ and its derivative is given by

$$J'_{\lambda,\mu}(u)(v) = M\left(L_{\beta(x)}(u)\right)\left(\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx + \int_{\partial\Omega}\beta(x)|u|^{p(x)-2}uv\,d\sigma\right)$$
$$-\lambda\int_\Omega V_1(x)|u|^{q_1(x)-2}uv\,dx + \mu\int_\Omega V_2(x)|u|^{q_2(x)-2}uv\,dx$$

for all $u, v \in X$.

Lemma 3.1. *Assume that the conditions (M_0) and (H) hold. Then for any $\lambda > 0$ and $\mu > 0$ the functional $J_{\lambda,\mu}$ is coercive on X.*

Proof. By condition (H), there exists $c_1 > 0$ such that

$$|u|_{h_i(x)} \le c_1\|u\|_{\beta(x)}, \quad \forall u \in X, \tag{14}$$

where $h_i(x) = \frac{s_i(x)q_i(x)}{s_i(x)-\alpha}$, $i = 1, 2$. Hence, by condition (M_0), the Hölder inequality and Remark 3.1, we deduce that

$$\begin{aligned}
J_{\lambda,\mu}(u) &= \widehat{M}\left(L_{\beta(x)}(u)\right) - \lambda\int_\Omega \frac{V_1(x)}{q_1(x)}|u|^{q_1(x)}\,dx + \mu\int_\Omega \frac{V_2(x)}{q_2(x)}|u|^{q_2(x)}\,dx\\
&\ge \frac{m_1}{\alpha(p^+)^\alpha}\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\beta(x)|u|^{p(x)}\,d\sigma\right)^\alpha - \frac{\lambda}{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}\left||u|^{q_1(x)}\right|_{\frac{s_1(x)}{s_1(x)-\alpha}}\\
&\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^-} - \frac{\lambda}{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}\min\left\{|u|_{h_i(x)}^{q_1^-}, |u|_{h_i(x)}^{q_1^+}\right\}\\
&\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^-} - \frac{\lambda}{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}\min\left\{c_1^{q_1^-}\|u\|_{\beta(x)}^{q_1^-}, c_1^{q_1^+}\|u\|_{\beta(x)}^{q_1^+}\right\}.
\end{aligned}$$

Since $q_1^+ < p^- < \alpha p^-$, we infer that $J_{\lambda,\mu}(u) \to +\infty$ as $\|u\|_{\beta(x)} \to +\infty$, which means that the functional $J_{\lambda,\mu}$ is coercive on X. $\square$

From Lemma 3.1 and the minimum principle, for any $\lambda > 0$ and $\mu > 0$, the functional $J_{\lambda,\mu}$ has a critical point and problem (1) has a weak solution. The following result shows that it is not trivial and so, Theorem 3.1 is proved.

Lemma 3.2. *Assume that the conditions (M_0) and (H) hold. Then for any $\lambda > 0$ and $\mu > 0$, there exists $u_0 \in X$ such that $u_0 \geq 0$, $u_0 \neq 0$ and $J_{\lambda,\mu}(tu_0) < 0$ for all $t > 0$ small enough.*

Proof. Set $q_{i,0}^- := \inf_{x\in\overline{\Omega}_0} q_i(x)$, $i = 1, 2$ and $p_0^- := \inf_{x\in\overline{\Omega}_0} p(x)$. Since $q_{1,0}^- < q_{2,0}^-$, let $\epsilon_0 > 0$ be such that $q_{1,0}^- + \epsilon_0 < q_{2,0}^-$. Since $q_1 \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_1 \subset\subset \Omega_0$ such that $|q_1(x) - q_{1,0}^-| < \epsilon_0$ for all $x \in \Omega_1$. Thus, $q_1(x) \leq q_{1,0}^- + \epsilon_0 < q_{2,0}^-$ for all $x \in \Omega_1$.

Let $u_0 \in C_0^\infty(\Omega)$ be such that $supp(u_0) \subset \Omega_1 \subset\subset \Omega_0$, $u_0 = 1$ in a subset $\Omega_1' \subset supp(u_0)$, $0 \leq u_0 \leq 1$ in Ω_1. Therefore, for any $t \in (0, 1)$ we have

$$
\begin{aligned}
J_{\lambda,\mu}(tu_0) &= \widehat{M}\left(L_{\beta(x)}(tu_0)\right) - \lambda \int_\Omega \frac{t^{q_1(x)}}{q_1(x)} V_1(x)|tu_0|^{q_1(x)}\,dx + \mu \int_\Omega \frac{t^{q_2(x)}}{q_2(x)} V_2(x)|tu_0|^{q_2(x)}\,dx \\
&\leq \frac{m_2 t^{\alpha p_0^-}}{\alpha (p_0^-)^\alpha} \left(\int_{\Omega_0} |\nabla u_0|^{p(x)}\,dx + \int_{\partial\Omega_0} \beta(x)|u_0|^{p(x)}\,d\sigma \right)^\alpha \\
&\quad - \lambda \int_{\Omega_1} \frac{t^{q_1(x)}}{q_1(x)} V_1(x)|u_0|^{q_1(x)}\,dx + \mu \int_{\Omega_0} \frac{t^{q_2(x)}}{q_2(x)} V_2(x)|u_0|^{q_2(x)}\,dx \\
&\leq \frac{m_2 t^{\alpha p_0^-}}{\alpha (p_0^-)^\alpha} \left(\int_{\Omega_0} |\nabla u_0|^{p(x)}\,dx + \int_{\partial\Omega_0} \beta(x)|u_0|^{p(x)}\,d\sigma \right)^\alpha \\
&\quad - \frac{\lambda t^{q_{1,0}^- + \epsilon_0}}{q_{1,0}^-} \int_{\Omega_1} V_1(x)|u_0|^{q_1(x)}\,dx + \frac{\mu t^{q_{2,0}^-}}{q_{2,0}^-} \int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx \\
&\leq t^{q_{2,0}^-} \frac{m_2}{\alpha (p_0^-)^\alpha} \left(\int_{\Omega_0} |\nabla u_0|^{p(x)}\,dx + \int_{\partial\Omega_0} \beta(x)|u_0|^{p(x)}\,d\sigma \right)^\alpha \\
&\quad + \frac{\mu t^{q_{2,0}^-}}{q_{2,0}^-} \int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx - \frac{\lambda t^{q_{1,0}^- + \epsilon_0}}{q_{1,0}^-} \int_{\Omega_1} V_1(x)|u_0|^{q_1(x)}\,dx.
\end{aligned}
$$

It follows that $J_{\lambda,\mu}(tu_0) < 0$ for all $0 < t < \delta^{\frac{1}{q_{2,0}^- - q_{1,0}^- - \epsilon_0}}$ with $0 < \delta < \min\{1, \delta_0\}$ and

$$
\delta_0 := \frac{\lambda \int_{\Omega_1} V_1(x)|u_0|^{q_1(x)}\,dx}{q_{1,0}^+ \left[\frac{m_2}{\alpha (p_0^+)^\alpha} \left(\int_{\Omega_0} |\nabla u_0|^{p(x)}\,dx + \int_{\partial\Omega_0} \beta(x)|u_0|^{p(x)}\,d\sigma \right)^\alpha + \frac{\mu}{q_{2,0}^-} \int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx \right]}.
$$

Finally, we point out that

$$
\frac{m_2}{\alpha (p_0^-)^\alpha} \left(\int_{\Omega_0} |\nabla u_0|^{p(x)}\,dx + \int_{\partial\Omega_0} \beta(x)|u_0|^{p(x)}\,d\sigma \right)^\alpha + \frac{\mu}{q_{2,0}^-} \int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx > 0.
$$

In fact, if it is not true then

$$
\int_{\Omega_0} |\nabla u_0|^{p(x)}\,dx + \int_{\partial\Omega_0} \beta(x)|u_0|^{p(x)}\,d\sigma = 0,
$$

which gives $\|u_0\|_{\beta(x)} = 0$, hence $u_0 = 0$ in Ω_0. This is a contradiction and thus the proof of Lemma 3.2 is now complete. ☐

Lemma 3.3. *Assume that the conditions (M_0) and (H) hold. Then for all $\rho \in (0, 1)$ there exist $\lambda^* > 0$ and a constant $a > 0$ such that for all $u \in X$ with $\|u\|_{\beta(x)} = \rho$ we have $J_{\lambda,\mu}(u) \geq a$ for any $\lambda \in (0, \lambda^*)$.*

Proof. Let us assume that $\|u\|_{\beta(x)} < \min\left\{1, \frac{1}{c_1}\right\}$, where c_1 is given by (14). It follows that $|u|_{h_i(x)} < 1$, where $h_i(x) = \frac{s_i(x)q_i(x)}{s_i(x)-\alpha}$, $i = 1, 2$. Using relations (4), (14), the condition (M_0) and Remark 3.1, we deduce that for any $u \in X$ with $\|u\|_{\beta(x)} = \rho \in (0,1)$ the following inequalities hold true

$$
\begin{aligned}
J_{\lambda,\mu}(u) &= \widehat{M}\left(L_{\beta(x)}(u)\right) - \lambda \int_\Omega \frac{V_1(x)}{q_1(x)}|u|^{q_1(x)}\,dx + \mu \int_\Omega \frac{V_2(x)}{q_2(x)}|u|^{q_2(x)}\,dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}\,d\sigma\right)^\alpha - \lambda \int_\Omega \frac{V_1(x)}{q_1(x)}|u|^{q_1(x)}\,dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^+} - \frac{\lambda}{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}\left||u|^{q_1(x)}\right|_{\frac{s_1(x)}{s_1(x)-\alpha}} \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^+} - \frac{\lambda}{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}|u|_{h_1(x)}^{q_1^-} \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^+} - \frac{\lambda}{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}} c_1^{q_1^-}\|u\|_{\beta(x)}^{q_1^-} \\
&= \rho^{q_1^-}\left(\frac{m_1}{\alpha(p^+)^\alpha}\rho^{\alpha p^+ - q_1^-} - \frac{\lambda}{q_1^-}c_1^{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}\right).
\end{aligned}
$$

This inequality shows that if we choose

$$
\lambda^* = \frac{m_1 q_1^-}{2\alpha(p^+)^\alpha c_1^{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}}\rho^{\alpha p^+ - q_1^-}, \tag{15}
$$

then for all $\lambda \in (0, \lambda^*)$ and for all $u \in X$ with $\|u\|_{\beta(x)} = \rho$, there exists $a > 0$ such that $J_{\lambda,\mu}(u) \geq a > 0$. The proof of Lemma 3.3 is complete. $\square$

Proof of Theorem 3.2 completed. Let $\lambda^* > 0$ be defined as in (15) and $\lambda \in (0, \lambda^*)$, $\mu > 0$. By Lemma 3.3 it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$, we have

$$
\inf_{\partial B_\rho(0)} J_{\lambda,\mu} > 0. \tag{16}
$$

On the other hand, by Lemma 3.2, there exists $u_0 \in X$ such that $J_{\lambda,\mu}(tu_0) < 0$ for all $t > 0$ small enough. Moreover, by hypothesis (M_0) and the proof of Lemma 3.3 we deduce that for any $u \in B_\rho(0)$,

$$
J_{\lambda,\mu}(u) \geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^+} - \frac{\lambda}{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}} c_1^{q_1^-}\|u\|_{\beta(x)}^{q_1^-}.
$$

It follows that

$$
-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} J_{\lambda,\mu} < 0.
$$

Let $0 < \epsilon < \inf_{\partial B_\rho(0)} J_{\lambda,\mu} - \inf_{B_\rho(0)} J_{\lambda,\mu}$. Using the above information, the functional $J_{\lambda,\mu} : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$, is lower bounded on $\overline{B_\rho(0)}$ and $J_{\lambda,\mu} \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then by Ekeland's variational principle [23], there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$
\begin{cases}
\underline{c} \leq J_{\lambda,\mu}(u_\epsilon) \leq \underline{c} + \epsilon \\
0 < J_{\lambda,\mu}(u) - J_{\lambda,\mu}(u_\epsilon) + \epsilon\|u - u_\epsilon\|_{\beta(x)}, \quad u \neq u_\epsilon.
\end{cases}
$$

Since

$$
J_{\lambda,\mu}(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} J_{\lambda,\mu} + \epsilon \leq \inf_{B_\rho(0)} J_{\lambda,\mu} + \epsilon < \inf_{\partial B_\rho(0)} J_{\lambda,\mu},
$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $\overline{J}_{\lambda,\mu} : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$ by $\overline{J}_{\lambda,\mu}(u) = J_{\lambda,\mu}(u) + \epsilon\|u - u_\epsilon\|_{\beta(x)}$. It is clear that u_ϵ is a minimum point of $\overline{J}_{\lambda,\mu}$ and thus

$$\frac{\overline{J}_{\lambda,\mu}(u_\epsilon + t \cdot v) - \overline{J}_{\lambda,\mu}(u_\epsilon)}{t} \geq 0,$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_{\lambda,\mu}(u_\epsilon + t \cdot v) - J_{\lambda,\mu}(u_\epsilon)}{t} + \epsilon\|v\|_{\beta(x)} \geq 0.$$

Letting $t \to 0$ it follows that $J'_{\lambda,\mu}(u_\epsilon)(v) + \epsilon\|v\|_{\beta(x)} \geq 0$ and we infer that $\|J'_{\lambda,\mu}(u_\epsilon)\|_{\beta(x)} \leq \epsilon$.

We deduce that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J_{\lambda,\mu}(u_n) \longrightarrow \underline{c} < 0 \quad \text{and} \quad J'_{\lambda,\mu}(u_n) \longrightarrow 0_{X^*}. \tag{17}$$

It is clear that $\{u_n\}$ is bounded in X. Thus, there exists u in X such that, up to a subsequence, $\{u_n\}$ converges weakly to u in X. Since $k_i(x) = \frac{s_i(x)q_i(x)}{s_i(x)-\alpha q_i(x)} < p^*(x)$ for all $x \in \overline{\Omega}$, $i = 1, 2$, we deduce that there exist compact embeddings $X \hookrightarrow L^{k_i(x)}(\Omega)$, hence the sequence $\{u_n\}$ converges strongly to u in $L^{k_i(x)}(\Omega)$, $i = 1, 2$.

Using Hölder's inequality (5) we have

$$\begin{aligned}\int_\Omega V_1(x)|u_n|^{q_1(x)-2}u_n(u_n - u)\,dx &\leq |V_1|_{\frac{s_1(x)}{\alpha}}\,||u_n|^{q_1(x)-2}u_n(u_n - u)|_{h_1(x)} \\ &\leq |V_1|_{\frac{s_1(x)}{\alpha}}\,||u_n|^{q_1(x)-2}u_n|_{\frac{q_1(x)}{q_1(x)-1}}\,|u_n - u|_{k_1(x)}.\end{aligned}$$

Now if $||u_n|^{q_1(x)-2}u_n|_{\frac{q_1(x)}{q_1(x)-1}} > 1$ then we get $||u_n|^{q_1(x)-2}u_n|_{\frac{q_1(x)}{q_1(x)-1}} \leq |u_n|^{q_1^+}_{q_1(x)}$. The compact embedding $X \hookrightarrow L^{q_1(x)}(\Omega)$ helps us to show that

$$\lim_{n\to\infty}\int_\Omega V_1(x)|u_n|^{q_1(x)-2}u_n(u_n - u)\,dx = 0. \tag{18}$$

Similarly, we get

$$\lim_{n\to\infty}\int_\Omega V_2(x)|u_n|^{q_2(x)-2}u_n(u_n - u)\,dx = 0. \tag{19}$$

Moreover, by (17) we have

$$\lim_{n\to\infty} J'_{\lambda,\mu}(u_n)(u_n - u) = 0$$

or

$$\begin{aligned}&M\left(L_{\beta(x)}(u_n)\right)\left(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot (\nabla u_n - \nabla u)\,dx + \int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n - u)\,d\sigma\right) \\ &\quad -\lambda\int_\Omega V_1(x)|u_n|^{q_1(x)-2}u_n(u_n - u)\,dx + \mu\int_\Omega V_2(x)|u_n|^{q_2(x)-2}u_n(u_n - u)\,dx \to 0.\end{aligned}$$

Combining this with relations (17)-(19) it follows that

$$M\left(L_{\beta(x)}(u_n)\right)\left(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot (\nabla u_n - \nabla u)\,dx + \int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n - u)\,d\sigma\right) \to 0. \tag{20}$$

If $L_{\beta(x)}(u_n) \to 0$ as $n \to \infty$ then it follows from (10) that $u_n \to 0$ strongly in X and the proof is finished. If $L_{\beta(x)}(u) \to t_0 > 0$ then for n large enough, we have

$$M\left(L_{\beta(x)}(u_n)\right) \to M(t_0) \geq m_1 t_0^{\alpha-1} > 0,$$

so it follows that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot(\nabla u_n-\nabla u)\,dx+\int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n-u)\,d\sigma=0$$

or

$$\lim_{n\to\infty} L'_{\beta(x)}(u_n)(u_n-u)=0.$$

Combining this with Proposition 2.1, we deduce that $\{u_n\}$ converges strongly to u in X. Since $J_{\lambda,\mu}\in C^1(X,\mathbb{R})$, we conclude that

$$J'_{\lambda,\mu}(u_n)\to J'_{\lambda,\mu}(u),\ \text{as } n\to\infty. \tag{21}$$

Relations (17) and (21) show that $J'_{\lambda,\mu}(u)=0$ and thus u is a weak solution for problem (1). Moreover, by relation (17), it follows that $J_{\lambda,\mu}(u)<0$ and thus, u is a nontrivial weak solution for (1). The proof of Theorem 3.2 is complete. □

4. Conclusions

In this paper, we consider a class of Kirchhoff type problems with Robin boundary conditions in Sobolev spaces with variable exponents. In particular, the Kirhhoff function is allowed to be singular at zero and the nonlinear term involves indefinite weights. Using the minimum principle and the Ekeland variational principle, we establish some results on the existence of nontrivial weak solutions for such problems.

Acknowledgement. The author would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

References

[1] Acerbi E. and Mingione G., (2002), Regularity results for stationary electro-rheological fluids, Arch. Ration. Mech. Anal., 164, (3), pp. 213-259.
[2] Allaoui M., (2017), Existence results for a class of $p(x)$-Kirchhoff problems, Studia Sci. Math. Hungarica, 54, (3), pp. 316-331.
[3] Allaoui M. and Ourraoui A., (2016), Existence results for a class of $p(x)$-Kirhhoff problem with a singular weight, Mediterr. J. Math., 13, (2), pp. 677-686.
[4] Amaziane B., Pankratov L. and A. Piatnitski, (2009), Nonlinear flow through double porosity media in variable exponent Sobolev spaces, Nonlinear Anal. Real World Appl., 10, (4), pp. 2521-2530.
[5] Ambrosetti A., Rabinowitz, P. H., (1973), Dual variational methods in critical points theory and applications, J. Funct. Anal., 14, pp. 349-381.
[6] Avci M., Cekic B. and Mashiyev R.A., (2011), Existence and multiplicity of the solutions of the $p(x)$-Kirchhoff type equation via genus theory, Math. Methods Appl. Sci., 34, (14), pp. 1751-1759.
[7] Avci M., (2013), Ni-Serrin type equations arising from capillarity phenomena with non-standard growth, Bound. Value Probl., 2013: 55.
[8] Bisci G. M., Radulescu V. D., (2015), Applications of local linking to nonlocal Neumann problems, Commun. Contemp. Math., 17, (1), 1450001.
[9] Blomgren P., Chan T. F., Mulet P. and Wong C. K., (1997), Total variation image restoration: numerical methods and extensions, in Proceedings of the International Conference on Image Processing, 1997, IEEE, 3, pp. 384-387
[10] Bouslimi, M. and Kefi, K., (2013), Existence of solution for an indefinite weight quasilinear problem with variable exponent, Complex Var. Elliptic Equa., 58, pp. 1655-1666.
[11] Cekic B., Kalinin A.V., Mashiyev R. A. and M. Avci, (2012), $L^{p(x)}(\Omega)$-estimates of vector fields and some applications to magnetostatics problems, J. Math. Anal. Appl., 389, (2), pp. 838-851.
[12] Chung, N. T., (2013), Multiple solutions for a $p(x)$-Kirchhoff-type equation with sign-changing nonlinearities, Complex Var. Elliptic Equa., 58(12), pp. 1637-1646.
[13] Chung, N. T., (2013), Multiple solutions for a class of $p(x)$-Kirchhoff type problems with Neumann boundary conditions, Adv. Pure Appl. Math., 4, (2), pp. 165-177.

[14] Chung, N. T., (2018), Some remarks on a class of $p(x)$-Laplacian Robin eigenvalue problems, Mediterr. J. Math., 15, (4): 147.

[15] Chipot, M, and Lovat, B., (1997), Some remarks on nonlocal elliptic and parabolic problems, Nonlinear Anal. (TMA), 30, (7), pp. 4619-4627.

[16] Colasuonno, F. and Pucci, P., (2011), Multiplicity of solutions for $p(x)$-polyharmonic Kirchhoff equations, Nonlinear Anal. (TMA), 74, pp. 5962-5974.

[17] Correa, F. J. S. A. and Figueiredo, G. M., (2006), On an elliptic equation of p-Kirchhoff type via variational methods, Bull. Aust. Math. Soc., 74, pp. 263-277.

[18] Cruz-Uribe, D. V. and Fiorenza A., (2013), Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Springer, Basel.

[19] Dai, G., (2013), Three solutions for a nonlocal Dirichlet boundary value problem involving the $p(x)$-Laplacian, Appl. Anal., 92(1), pp. 191-210.

[20] Dai, G. and Hao, R., (2009), Existence of solutions for a $p(x)$-Kirchhoff-type equation, J. Math. Anal. Appl., 359, pp. 275-284.

[21] Deng, S. G., (2009), Positive solutions for Robin problem involving the $p(x)$-Laplacian, J. Math. Anal. Appl., 360, pp. 548-560.

[22] Diening, L., Harjulehto, P., Hästö P. and Ružička M., (2011), Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, Vol. 2017, Springer-Verlag, Heidelberg.

[23] Ekeland I., (1974), On the variational principle, J. Math. Anal. Appl., 47, pp. 324-353.

[24] Ge, B. and Zhou, Q. M., (2017), Multiple solutions for a Robin-type differential inclusion problem involving the $p(x)$-Laplacian, Math. Meth. Appl. Sci., 40, (18), (2017), pp. 6229-6238.

[25] Kefi, K., (2018), On the Robin problem with indefinite weight in Sobolev spaces with variable exponents, *Zeitschrift für Analysis und ihre Anwendugen (ZAA)*, 37, pp. 25-38.

[26] Kirchhoff, G., (1883), Mechanik, Teubner, Leipzig, Germany.

[27] Kováčik, O. and Rákosník, J., (1991), On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J., 41, pp. 592-618.

[28] Ružička, M., (2000), Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, Vol. 1748, Springer-Verlag, Berlin.

[29] Wang, L., Xie, K. and Zhang, B., (2018), Existence and multiplicity of solutions for critical Kirchhoff-type p-Laplacian problems, J. Math. Anal. Appl., 458, pp. 361-378.

[30] Zhikov. V. V., (1997), Meyer-type estimates for solving the nonlinear Stokes system, Differential Equa., 33, (1), pp. 108-115.

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