TWMS J. App. Eng. Math. V.10, N.3, 2020, pp. 606-615

SOME COMMON FIXED POINT RESULTS VIA b-SIMULATION FUNCTION

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Abstract. Inspired by Khojasteh et al. [Filomat 29(6), 2015], Karapinar [Filomat 30(8), 2016] and Demma et al. [IJMSA 11(1), 2016], we establish some common fixed point results for a pair of self-mappings defined on a complete b-metric space employing a simulation function. Our results generalize several core results of the existing literature, particularly, the results contained in aforementioned articles.

Keywords: Common fixed point, b-metric space, b-simulation function; simulation function.

AMS Subject Classification: 54H25, 47H10.

1. Introduction

Due to extensive and broad applications, fixed point theory remains the most utilized tool from nonlinear analysis. Banach contraction principle is the most classical and fundamental results of metric fixed point theory. This principle guarantees a unique fixed point for each contraction defined on a complete metric space.

Many researchers have generalized Banach contraction principle utilizing a relatively more general contractive conditions. Other generalizations of this principle can be obtained either by proving it in various types of relatively larger classes of spaces (e.g. 2-metric space [1], b-metric [2,3], D-metric [4], partial metric [5], G-metric [6], cone metric space [7] etc.) or by increasing number of involved mappings and even lately by employing admissible mappings (e.g, [8]).

Improving Banach principle revived a new impetus when the authors attempted to produce unified-fixed point results. With such quest, Popa [9] initiated the notion of implicit function, Wardowski [10] initiated a new type of contractions called F-contractions, Wei-Shih Du and Khojasteh [11] presented the notion of manageable function, Khojasteh et al. [12] initiated the idea of simulation function and De Hierro and Shahzad [13] initiated

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§ Manuscript received: July 25, 2017; accepted: May 15, 2018.

the idea of R-function as well as R-contractions.

By now, there exists an extensive literature on this theme and still vigorous research is going on. For this kind of work one can consult [14–18] and references therein.

In this article, inspired by all above generalizations of Banach contraction principle, we establish common fixed point results for a pair of self-mappings in a b-metric space employing b-simulation function.

2. Preliminaries

In order to prove our results, the following definitions, notions and auxiliary results are needed. In the sequel, X stands for a non-empty set, $\alpha : X \times X \to [0, \infty)$, (T, g) is a pair of self-mappings on X and I_X denotes the identity mapping on X.

Definition 2.1. *[19] An element $x \in X$ is called a coincidence point of the pair (T, g) if $Tx = gx$. The point $z \in X$ with $z = Tx = gx$ is called a point of coincidence. Moreover, z is called a common fixed point of (T, g) if $x = z$.*

We denote the set of all coincidence points of the pair (T, g) by $Coin(T, g)$.

Definition 2.2. *[20] The pair (T, g) is said to be weakly compatible if T and g commute at all $x \in Coin(T, g)$.*

Definition 2.3. *[21] The mapping T is called g-continuous at $x_0 \in X$, if for any sequence $\{x_n\} \subseteq X$,*

$$gx_n \to gx_0 \Rightarrow Tx_n \to Tx_0.$$

If T is g-continuous at every $x \in X$, then T is said to be g-continuous.

Definition 2.4. *[8] The mapping T is called α-admissible with respect to g if for all $x, y \in X$, we have*

$$\alpha(gx, gy) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 2.5. *[22] Two points $x, y \in X$ are said to be a (g, α)-comparable if $\alpha(gx, gy) \geq 1$ or $\alpha(gy, gx) \geq 1$.*

Definition 2.6. *[23] A subset Y of a metric space (X, d) is said to be a* precomplete *if every Cauchy sequence $\{x_n\}$ in Y converges to a point of X.*

Bakhtin [2] and Czerwik [3] introduced the concept of b-metric space as follows:

Definition 2.7. *[2,3] Let X be a non-empty set and $b \geq 1$ a fixed real number. A function $\sigma : X \times X \to [0, \infty)$ is known as b-metric if it satisfies the following properties for each $x, y, z \in X$:*

(1) $\sigma(x, y) = 0$ iff $x = y$;
(2) $\sigma(x, y) = \sigma(y, x)$;
(3) $\sigma(x, z) \leq b[\sigma(x, y) + \sigma(y, z)]$.

The pair (X, σ) is referred as b-metric space.

Remark 2.1. *Every metric space is a b-metric space but the converse need not be true in general. Further, for $b = 1$, the concept of b-metric space coincides with the metric space.*

Khojasteh et al. [12] introduced the class of simulation function which was later refined by Argoubi et al. [24] and De-Hierro et al. [25]. Karapinar [26] enlarged this class to cover α-admissible mappings. Later on, Demma et al. [27] defined a type of simulation functions in b-metric space as under:

Definition 2.8. *[27] Let (X,σ) be a b-metric space. A b-simulation function $\zeta : [0,\infty)\times[0,\infty)\to\mathbb{R}$ is a function satisfying the following:*

(ζ_1) $\zeta(t,s) < s - t$, for all $s,t > 0$;

(ζ_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \lim_{n\to\infty} t_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq b \lim_{n\to\infty} t_n < \infty,$$

then

$$\limsup_{n\to\infty} \zeta(bt_n, s_n) < 0.$$

Remark 2.2. *Condition (ζ_2) in Definition 2.8 and (ζ_4) of [18, Definition 3.1] are independent.*

Some examples of simulation function are as under:

Example 2.1. *$\zeta(t,s) = ks - t$, for all $t,s\in[0,\infty)$, where $k\in[0,1)$.*

Example 2.2. *$\zeta(t,s) = \psi(s) - t$, for all $t,s\in[0,\infty)$, where $\psi : [0,\infty)\to[0,\infty)$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.*

Example 2.3. *$\zeta(t,s) = \psi(s) - t$, for all $t,s\in[0,\infty)$, where $\psi : [0,\infty)\to[0,\infty)$ is a non-decreasing function satisfying $\sum_{n=1}^{\infty}\psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.*

Example 2.4. *$\zeta(t,s) = s - \phi(s) - t$, for all $t,s\in[0,\infty)$, where $\phi : [0,\infty)\to[0,\infty)$ is a continuous function such that $\phi(t) = 0$ iff $t = 0$.*

3. Main Results

Inspired by [28], we introduce the following definition which is needed in our subsequent discussions.

Definition 3.1. *The mapping T is said to be a triangular α-orbital admissible with respect to g if*

(i) $\alpha(gx,Tx)\geq 1 \Rightarrow \alpha(Tx,T^2x)\geq 1$;

(ii) $\alpha(gx,gy)\geq 1$ and $\alpha(gy,Ty)\geq 1 \Rightarrow \alpha(gx,Ty)\geq 1$.

The following is our main result.

Lemma 3.1. *Let (X,σ) be a b-metric space and $T,g : X\to X$ which satisfy the following conditions:*

(a) the mapping T is triangular α-orbital admissible with respect to g;

(b) there exists $x_0\in X$, such that $\alpha(gx_0,Tx_0)\geq 1$;

(c) there exists a b-simulation function ζ such that

$$\zeta(b\alpha(gx,gy)\sigma(Tx,Ty),\sigma(gx,gy))\geq 0,\ \text{for all } x,y\in X. \tag{1}$$

Then the sequence $\{gx_n\}$, realized as $Tx_n = gx_{n+1}$, for all $n\in\mathbb{N}\cup\{0\}$, is a bounded sequence.

Proof. Let $x_0\in X$ as in (b) and generate a sequence $\{gx_n\}$ defined by $Tx_n = gx_{n+1}$, for all $n\in\mathbb{N}\cup\{0\}$. If there exists n_0 such that $gx_{n_0} = gx_{n_0+1}$, then x_{n_0} is a coincidence point of the pair (T,g). Otherwise $gx_n\neq gx_{n+1}$ for all $n\geq 0$, i.e. $\sigma(gx_n,gx_{n+1}) > 0$ for all $n\in\mathbb{N}\cup\{0\}$.
Using condition (a) and (c), we deduce that

$$\alpha(gx_n,gx_m)\geq 1,\ \text{for all } m,n\in\mathbb{N}\cup\{0\} \text{ and } m > n. \tag{2}$$

Since T satisfies (1), we have

$$\begin{aligned} 0 &\leq \zeta(b\alpha(gx_{n-1}, gx_n)\sigma(Tx_{n-1}, Tx_n), \sigma(gx_{n-1}, gx_n)) \\ &= \zeta(b\alpha(gx_{n-1}, gx_n)\sigma(gx_n, gx_{n+1}), \sigma(gx_{n-1}, gx_n)) \\ &< \sigma(gx_{n-1}, gx_n) - b\alpha(gx_{n-1}, gx_n)\sigma(gx_n, gx_{n+1}), \end{aligned}$$

so that

$$b\alpha(gx_{n-1}, gx_n)\sigma(gx_n, gx_{n+1}) < \sigma(gx_{n-1}, gx_n). \tag{3}$$

Since

$$\sigma(gx_n, gx_{n+1}) \leq \alpha(gx_{n-1}, gx_n)\sigma(gx_n, gx_{n+1}), \tag{4}$$

therefore, $\{\sigma(gx_n, gx_{n+1})\}$ is a strictly decreasing sequence of positive real numbers, so that it converges to some $r \geq 0$.
We assert that $r = 0$. Suppose on contrary, $r > 0$, then letting $n \to \infty$ in (3), we get

$$br \leq br \lim_{n\to\infty} \alpha(gx_{n-1}, gx_n) \leq r. \tag{5}$$

If $b > 1$, then equation (5) is possible only if $r = 0$. For $b = 1$, we have

$$\lim_{n\to\infty} \alpha(gx_{n-1}, gx_n) = 1.$$

Applying (ζ_2) with $t_n = \alpha(gx_{n-1}, gx_n)\sigma(gx_n, gx_{n+1})$ and $s_n = \sigma(gx_{n-1}, gx_n)$, we obtain

$$0 \leq \limsup_{n\to\infty} \zeta(b\alpha(gx_{n-1}, gx_n)\sigma(gx_n, gx_{n+1}), \sigma(gx_{n-1}, gx_n)) < 0$$

a contradiction so that, in all,

$$\lim_{n\to\infty} \sigma(gx_n, gx_{n+1}) = 0. \tag{6}$$

Now, we prove that $\{gx_n\}$ is a bounded sequence. Suppose, on contrary, it is not. Then there exists a subsequence $\{gx_{n_k}\}$ (of $\{gx_n\}$) such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the smallest integer such that

$$\sigma(gx_{n_{k+1}}, gx_{n_k}) > 1$$

and

$$\sigma((gx_m, gx_{n_k}) \leq 1, \ for\ n_k \leq m \leq n_{k+1} - 1.$$

Now, by the triangle inequality, we obtain

$$\begin{aligned} 1 < \sigma(gx_{n_{k+1}}, gx_{n_k}) &\leq b\sigma(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + b\sigma(gx_{n_{k+1}-1}, gx_{n_k}) \\ &\leq b\sigma(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + b, \text{ for all } k \in \mathbb{N}. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (6), we get

$$1 \leq \lim_{k\to\infty} \sigma(gx_{n_{k+1}}, gx_{n_k}) \leq b.$$

Again, from (1), we deduce that

$$b\alpha(gx_{n_k-1}, gx_{n_{k+1}-1})\sigma(gx_{n_k}, gx_{n_{k+1}}) < \sigma(gx_{n_k-1}, gx_{n_{k+1}-1}).$$

Using above inequality and triangle inequality, we obtain

$$\begin{aligned} b < b\alpha(gx_{n_k-1}, gx_{n_{k+1}-1})\sigma(gx_{n_k}, gx_{n_{k+1}}) &< \sigma(gx_{n_k-1}, gx_{n_{k+1}-1}) \\ &\leq b\sigma(gx_{n_k-1}, gx_{n_k}) + b\sigma(gx_{n_k}, gx_{n_{k+1}-1}) \\ &\leq b\sigma(gx_{n_k-1}, gx_{n_k}) + b. \end{aligned}$$

Letting $k \to \infty$ in the preceding inequality and applying (6), we get

$$b \leq b \lim_{k\to\infty} \alpha(gx_{n_k-1}, gx_{n_{k+1}-1})\sigma(gx_{n_k}, gx_{n_{k+1}}) \leq b$$

so that

$$\lim_{k\to\infty} \alpha(gx_{n_k-1}, gx_{n_{k+1}-1})\sigma(gx_{n_k}, gx_{n_{k+1}}) = 1$$

and

$$\lim_{k\to\infty} \sigma(gx_{n_k-1}, gx_{n_{k+1}-1}) = b.$$

Then, by condition (ζ_2), we have

$$0 \leq \limsup_{n\to\infty} \zeta(b\alpha(gx_{n_k-1}, gx_{n_{k+1}-1})\sigma(gx_{n_k}, gx_{n_{k+1}}), \sigma(gx_{n_k-1}, gx_{n_{k+1}-1}) < 0,$$

which is a contradiction. Thus our claim is established and hence $\{gx_n\}$ is bounded. □

Using Lemma 3.1, we prove our main result which runs as follows:

Theorem 3.1. *Let (X, σ) be a b-metric space and $T, g : X \to X$. Suppose the following conditions hold:*

(a) T is triangular α-orbital admissible with respect to g;
(b) there exists $x_0 \in X$, such that $\alpha(gx_0, Tx_0) \geq 1$;
(c) there exists $Y \subseteq X$ such that Y is precomplete in $g(X)$ and $T(X) \subseteq Y \subseteq g(X)$;
(d) there exists a b-simulation function ζ satisfying (1);
(e) T is g-continuous.

Then there exists $u \in X$ such that $Tu = gu$. Furthermore, if T satisfies the following:

(f) for every $x, y \in Coin(T, g)$, there exists $z \in X$ such that z is (g, α)-comparable to x and y;
(g) T and g are weakly compatible,

then the pair (T, g) has a unique common fixed point.

Proof. Choose $x_0 \in X$ such as in (b) and define a sequence $\{gx_n\}$ satisfying $Tx_n = gx_{n+1}$ for all $n \geq 0$. By lemma 3.1, we obtain that $\{gx_n\}$ is a bounded sequence. Next, we show that $\{gx_n\}$ is a Cauchy sequence. To accomplish this let us consider $C_n = \sup\{\sigma(gx_i, gx_j) : i, j \geq n\}$; $n \in \mathbb{N}$. Then, $C_n < \infty$ for every $n \in \mathbb{N}$. As $\{gx_n\}$ is a bounded sequence, therefore, in view of the definition of $\{C_n\}$, it is a bounded below as well as positive decreasing. Hence,

$$\lim_{n\to\infty} C_n = C.$$

where $C \geq 0$. Now, we assert that $C = 0$. Suppose, on contrary, that $C > 0$. By the definition of C_n, for every $k \in \mathbb{N}$, there exists $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ with

$$C_k - \frac{1}{k} < \sigma(gx_{m_k}, gx_{n_k}) \leq C_k.$$

Letting $k \to \infty$, one gets

$$\lim_{k\to\infty} \sigma(gx_{m_k}, gx_{n_k}) = C. \tag{7}$$

Again, from (1) and the definition of C_n, we deduce that

$$b\alpha(gx_{n_k-1}, gx_{m_k-1})\sigma(gx_{m_k}, gx_{n_k}) \leq C_{k-1}.$$

Letting $k \to \infty$ in the above inequality and using equation (7), we get

$$bC \leq bC \lim_{k\to\infty} \alpha(gx_{n_k-1}, gx_{m_k-1}) \leq C. \tag{8}$$

For $b > 1$, equation (8) is possible only if $C = 0$ and for $b = 1$, we obtain

$$\lim_{k\to\infty} \alpha(gx_{n_k-1}, gx_{m_k-1}) = 1.$$

On using (ζ_2), we have

$$0 \leq \limsup_{n\to\infty} \zeta(b\alpha(gx_{n_k-1}, gx_{m_k-1})\sigma(gx_{n_k}, gx_{m_k}), \sigma(gx_{n_k-1}, gx_{m_k-1}) < 0,$$

which is a contradiction so that

$$\lim_{k\to\infty} \sigma(gx_{m_k}, gx_{n_k}) = 0. \tag{9}$$

Thus, we have shown that $\{gx_n\}$ is a Cauchy sequence in Y. The precompleteness of Y in $g(X)$ ensures that there exists some $x \in X$ such that

$$\lim_{n\to\infty} gx_n = gx.$$

Now, owing to the g-continuity of T, we have

$$\lim_{n\to\infty} Tx_n = Tx.$$

As $Tx_n = gx_{n+1}$, the uniqueness of the limit implies that

$$Tx = gx. \tag{10}$$

Next, assume that $x, y \in Coin(T, g)$, then by condition (f), there exists $z_0 \in X$ such that $\alpha(gz_0, gx) \geq 1$ or $\alpha(gx, gz_0) \geq 1$. In case $\alpha(gz_0, gx) \geq 1$, consider the sequence $\{gz_n\}$ based at z_0 by $Tz_n = gz_{n+1}$. Since T is triangular α-orbital admissible with respect to g, inductively, we find $\alpha(gz_n, gx) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Now by equation (1), we derive

$$b\alpha(gz_n, gx)\sigma(gz_{n+1}, gx) < \sigma(gz_n, gx), \tag{11}$$

which shows that $\{\sigma(gz_n, gx)\}$ is a strictly decreasing sequence of positive real numbers. Hence, there exists some $L \geq 0$ such that

$$\lim_{n\to\infty} \sigma(gz_n, gx) = L.$$

Suppose $L > 0$. On taking limit in (11), we get

$$bL \leq bL \lim_{n\to\infty} \alpha(gz_n, gx) \leq L.$$

Again, proceeding on the lines as before, for $b > 1$ one can show that $L = 0$ while for $b = 1$, using (ζ_2) with $t_n = \alpha(gz_n, gx)\sigma(gz_{n+1}, gx)$ and $s_n = \sigma(gz_n, gx)$, we have

$$0 \leq \limsup_{n\to\infty} \zeta(b\alpha(gz_n, gx)\sigma(gz_{n+1}, gx), \sigma(gz_n, gx) < 0$$

a contradiction. Thus, in all, we have $L = 0$, i.e., $\lim_{n\to\infty} gz_n = gx$. In the same way, we can also prove $\lim_{n\to\infty} gz_n = gy$. Owing to the uniqueness of the limit, $gx = gy$, i.e., (T, g) has a unique point of coincidence.

Now we have to establish the uniqueness of a common fixed point. Let $x \in Coin(T, g)$, then there exists some $z \in X$, such that $Tx = gx = z$. By condition (g), $gz = g(Tx) = T(gx) = Tz$, then due to uniqueness of the point of coincidence

$$Tz = gz = z.$$

Hence, z is a common fixed point of the pair (T, g). Now, appealing to the uniqueness of the point of coincidence, z is unique. This completes the proof. □

Next, we demonstrate Theorem 3.1 by the following example:

Example 3.1. *Let* $X = [0, 1]$ *and* $\sigma : X \times X \to [0, \infty)$ *be defined by*

$$\sigma(x, y) = (x - y)^2, \ \forall x, y \in X.$$

Then (X, σ) *is a complete b-metric space with b=2. Define* $T, g : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$ *as,*

$$Tx = \frac{x}{4(1 + x)}, \ gx = \frac{x}{2}, \forall \ x, y \in X$$

and

$$\alpha(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Then, for $\zeta(t, s) = ks - t, \ \forall t, s \in [0, \infty)$ *and* $k \in [\frac{1}{32}, 1)$*, all the conditions of Theorem 3.1 are satisfied. Observe that,* (1) *is obvious for* $gx = gy$*. Next, for* $gx \neq gy$*:*

$$\begin{aligned} \zeta(2\alpha(gx, gy)\sigma(Tx, Ty), \sigma(gx, gy) &= k\sigma(gx, gy) - 2\alpha(gx, gy)\sigma(Tx, Ty) \\ &= \frac{k}{4}(x - y)^2 - 2(1)\frac{(x - y)^2}{(4(1 + x)(1 + y))^2} \\ &\geq \frac{k}{4}(x - y)^2 - 2\left(\frac{1}{16}\right)\frac{(x - y)^2}{16} \\ &= \left(\frac{k}{4} - \frac{1}{128}\right)(x - y)^2 \\ &\geq 0, \ \text{for } k \in \left[\frac{1}{32}, 1\right). \end{aligned}$$

Now, appealing to Theorem 3.1, the pair (T, g) *has a unique common fixed point (namely* $x = 0$*).*

Another version of Theorem 3.1 is the following.

Theorem 3.2. *The conclusion of Theorem 3.1 remains true if we replace the assumption (e) by the following:*

- (e') *if* $\{gx_n\}$ *is a sequence in* X *such that* $\alpha(gx_n, gx_{n+1}) \geq 1$ *for all* n *and* $gx_n \to gu \in g(X)$ *as* $n \to \infty$*, then there exists a subsequence* $\{gx_{n_k}\}$ *of* $\{gx_n\}$ *such that* $\alpha(gx_{n_k}, gu) \geq 1$ *for all* k*.*

Proof. The proof runs on the lines of the proof of Theorem 3.1 up to (10). From equation (2) and condition (e'), there exists a subsequence $\{gx_{n_k}\}$ (of $\{gx_n\}$) such that $\alpha(gx_{n_k}, gu) \geq 1$ for all k. Since T satisfies (1), therefore

$$\begin{aligned} 0 &\leq \zeta(b\alpha(gx_{n_k}, gu)\sigma(Tx_{n_k}, Tu), \sigma(gx_{n_k}, gu)) \\ &= \zeta(b\alpha(gx_{n_k}, gu)\sigma(gx_{n_k+1}, Tu), \sigma(gx_{n_k}, gu)) \\ &< \sigma(gx_{n_k}, gu) - b\alpha(gx_{n_k}, gu)\sigma(gx_{n_k+1}, Tu), \end{aligned}$$

which implies that

$$\sigma(gx_{n_k+1}, Tu) \leq b\alpha(gx_{n_k}, gu)\sigma(gx_{n_k+1}, Tu) < \sigma(gx_{n_k}, gu). \tag{12}$$

On letting $k \to \infty$ in (12), we obtain $\sigma(gu, Tu) = 0$, i.e. $gu = Tu$. The remaining part of the proof runs as in Theorem 3.1. □

The following consequences exhibit that our results are general enough to deduce several results of the existing literature.

Corollary 3.1. *(Banach [29] type) Let (X, σ) be a b-metric space, $\alpha : X \times X \to [0, \infty)$ and $T, g : X \to X$ which satisfy the following:*

$$b\alpha(gx, gy)\sigma(Tx, Ty) \leq k\sigma(gx, gy),$$

for all $x, y \in X$, where $k \in [0, 1)$. If the pair (T, g) satisfies all the conditions of Theorem 3.1 (resp. 3.2), then the pair (T, g) has a unique common fixed point.

Proof. In view of Example 2.1 and Theorem 3.1 (resp. Theorem 3.2), the result follows. □

Corollary 3.2. *(Boyd-Wong [30] type) Let (X, σ) be a b-metric space, $\alpha : X \times X \to [0, \infty)$ and $T, g : X \to X$ such that there exists an upper semi-continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(t) < t$ for all $t > 0$, $\psi(0) = 0$ and*

$$b\alpha(gx, gy)\sigma(Tx, Ty) \leq \psi(\sigma(gx, gy)) \; for \; all \; x, y \in X(\quad wherin \quad gx \neq gy).$$

If the pair (T, g) satisfies all the conditions of Theorem 3.1 (resp. Theorem 3.2), then (T, g) has a unique common fixed point.

Proof. The result follows from Example 2.2 and Theorem 3.1 (resp. 3.2). □

The following result is a generalized version of the main result of Samet et al. [8].

Corollary 3.3. *[8] Let (X, σ) be a b-metric space, $\alpha : X \times X \to [0, \infty)$ and $T, g : X \to X$ which satisfy the following:*

$$b\alpha(gx, gy)\sigma(Tx, Ty) \leq \psi(\sigma(gx, gy)),$$

for all $x \in X$ where $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$. If all the conditions of Theorem 3.1 (resp. Theorem 3.2) are satisfied, then the pair (T, g) has a unique common fixed point.

Proof. The result follows in view of the Example 2.3 and Theorem 3.1 (resp. Theorem 3.2). □

Next, we describe Rhoades type result.

Corollary 3.4. *[31] Let (X, σ) be a b-metric space, $\alpha : X \times X \to [0, \infty)$ and $T, g : X \to X$ which satisfy the following:*

$$b\alpha(gx, gy)\sigma(Tx, Ty) \leq \sigma(gx, gy) - \phi(\sigma(gx, gy)),$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\phi(x) = 0$ iff $x = 0$. If all the conditions of Theorem 3.1 (resp. Theorem 3.2) are satisfied, then T and g have a unique common fixed point.

Proof. The result follows in view of Example 2.4 and Theorem 3.1 (resp. Theorem 3.2). □

On setting $g = I_X$, $Y = X$ and take $\alpha = 1$ in Theorem 3.1, we get the following:

Corollary 3.5. *Theorem 5.3 of [27] and Theorem 2.3 of [32] follow immediately from Theorem 3.1.*

Choosing $g = I_X$, $Y = X$ and $b = 1$ in Theorems 3.1 and 3.2, we get the following:

Corollary 3.6. *Theorem 1.7 of [26] follows from Theorems 3.1 and 3.2 immediately.*

Similarly, one can deduce a fixed point (or a common fixed point) result corresponding to every b-simulation functions. Naturally, all the above results remain true in metric spaces.

Acknowledgement: The authors are very thankful to all the reviewers for their valuable suggestions towards the improvement of this paper. The second author is thankful to UGC for awarding fellowship.

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