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A GEOMETRICAL STUDY OF WANG-CHEN SYSTEM IN VIEW OF KCC THEORY

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Abstract. The present paper discuss the stability of Wang-Chen system from the approach of KCC theory. we show that the system is Jacobi unstable for any value of parameter where as it is linear stable for some choosen parameter. We also show the dynamics of deviation vector near the equilibrium point.

Keywords: Finsler Space; geodesics; KCC-theory; Jacobi stability; Wang-Chen system.
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1. Introduction

In 1994, Sprott has proposed 19 distinct simple examples of chaotic flows with quadratic non-linearities and these are distinct in the sense that there is no transformation of one to another [1]. In the list Sprott E system is described by

$$\frac{dx}{dt} = yz,\ \frac{dy}{dt} = x^2 - y,\ \frac{dz}{dt} = 1 - 4x. \tag{1}$$

There are five terms in which two are nonlinear. This system has a degenerate equilibrium point in the sense that eigenvalues at the equilibrium point are $\lambda_1 = -1,\ \lambda_{2,3} = \pm 0.5i$, that is, one real number and conjugate pair of pure imaginary number. The equilibrium point $(0.25, 0.625, 0)$ is unstable. In the sprott E system, a modification has been made by adding a control parameter in the first equation by Wang and Chen [2]. The Wang-Chen system can be described as

$$\begin{aligned}\frac{dx}{dt} &= yz + a,\\ \frac{dy}{dt} &= x^2 - y,\\ \frac{dz}{dt} &= 1 - 4x,\end{aligned} \tag{2}$$

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where a is the control parameter. They have obtain the eigenvalues for some chosen values of parameter a and have also shown that the system has single stable node-focus equilibrium point and preserves its chaotic dynamics. Through some simple coordinate transformation on the Wang-Chen system, the different number of equilibria will be obtained [3]. The horseshoe chaos in the Wang-Chen system is shown by Huan *et al.* by using computer aided approach [4]. Wei and Wang constructed an extended form of Wang-Chen system and its modified function projection on synchronisation was described in [5]. The coexistense of point, stable limit cycle and starange attractor in Wang-Chen system demonstrated by Sprott *et al.* [6].

There are various method to discuss the stability of dynamical system. Some are well established and some going to their development stage. For example the linear stability and lyapunov stabilty methods are well established. Now, we study the dynamical system through geometro-dynamical approach will be introduced independently by Kosambi [7], Cartan [8] and Chern [9], known as KCC theory. In the KCC theory it is considered that second order differential equation and geodesic equation are topologically equivalent in the Finsler space. The KCC theory is a differential geometric theory of the variational equation for the deviation equation of the whole trajectories to nearby ones [10].

2. KCC-Theory and Jacobi Stability

The terminology and basic concepts are reffered to [[7]-[20]]. Let $(x^1, x^2, ...x^n) = (x)$, $(\frac{dx^1}{dt}, \frac{dx^2}{dt}, ..., \frac{dx^n}{dt}) = (\frac{dx}{dt}) = y$ and t be $(2n+1)$ co-ordinates in an open connected subset Ω of the $(2n+1)$-dimensional euclidean space $R^n \times R^n \times R^1$. Suppose that system of Second Order Differential Equation (SODE) is of the form

$$\frac{d^2x^i}{dt^2} + 2g^i(x^j, y^j, t) = 0,\ i, j = 1, 2, ..., n\,. \tag{3}$$

where each function g^i is C^∞ in a neighbourhood of initial points $((x)_0, (y)_0, t_0) \in \Omega$.

The intrinsic geometric properties of (3) under a non-singular coordinate transformations of the type

$$\begin{aligned} &\overline{x}^i = f^i(x^1, x^2, ...x^n),\ i = 1, 2, ..., n\,. \\ &\bar{t} = t, \end{aligned} \tag{4}$$

were described by the five KCC- differential invariants, named after D. Kosambi[7], E. Cartan[8] and S. S. Chern[9]. Let us define KCC-covariant derivative of a contravariant vector field $\xi^i(x)$ on Ω by [11, 12]

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N^i_j\xi^j\ , \tag{5}$$

where $N^i_j = \frac{\partial g^i}{\partial y^j}$, is the coefficients of the non-linear connection and we have used the Einstein summation convention throughout.
By putting $\xi^i = y^i$ and using (3), the above equation becomes

$$\frac{Dy^i}{dt} = N^i_j y^j - 2g^i = -\epsilon^i, \tag{6}$$

where ϵ^i is the contravariant vector field, called as first KCC- invariant, represents an 'external force' [12].

Now, we consider trajectories variation $x^i(t)$ of system (3) into nearby ones according to

$$\overline{x}^i(t) = x^i(t) + \xi^i(t)\eta\ , \tag{7}$$

where η denotes a parameter, with $|\eta|$ small and $\xi^i(t)$ are the components of contravariants vector defined along the path $x^i = x^i(t)$. Since $\overline{x}$ and x are solution of (3), it is not difficult to see that $\eta\ddot{\xi} + (\overline{g} - g) = 0$, where $\overline{g} - g := g(t, x + \eta\xi, \dot{x} + \eta\dot{\xi}) - g(t, x, \dot{x})$. In the view of $\overline{g} - g$ as a function of η, applying the mean value theorem and taking the limit as $\eta \to 0$, we get [11, 19, 20]

$$\frac{d^2\xi^i}{dt^2} + 2N^i_j\frac{d\xi^j}{dt} + 2\frac{\partial g^i}{\partial x^j}\xi^j = 0 \ , \tag{8}$$

Using the KCC-covariant differential (5), the above equation (8) becomes

$$\frac{D^2\xi^i}{dt^2} = P^i_j\xi^j \ , \tag{9}$$

where

$$P^i_j = -2\frac{\partial g^i}{\partial x^j} - 2g^l g^i_{jl} + y^l\frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j + \frac{\partial N^i_j}{\partial t} \ . \tag{10}$$

Here $g^i_{jl} = \frac{\partial N^i_j}{\partial y^l}$ is the Berwald connection [11, 12]. The tensor P^i_j is second KCC-invariant or *'deviation tensor'* of (3). The third, fourth and fifth invariants of the system (3) are [11, 12]

$$P^i_{jk} = \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right), \ P^i_{jkl} = \frac{\partial P^i_{jk}}{\partial y^l}, \ D^i_{jkl} = \frac{\partial g^i_{jk}}{\partial y^l}. \tag{11}$$

The third, fourth and fifth invariants are called the torsion tensor, Riemann-curvature tensor and Douglas curvature tensor respectively. Alternatively, we give another definition for the third and fourth invariants as [12]

$$B^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j} \ , \tag{12}$$

and

$$B^i_{jkl} = \frac{\partial B^i_{kl}}{\partial y^i} \ , \tag{13}$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j}. \tag{14}$$

2.1. **Jacobi stability of dynamical system:** Suppose that the trajectories $x^i = x^i(t)$ of (3) as curves in the Euclidean space $(R^n, \langle\cdot\,,\cdot\rangle)$, where $\langle\cdot\,,\cdot\rangle$ is the canonical inner product of the R^n. We assume that the deviation vector ξ satisfies the initial conditions

$$\xi(0) = O, \ \dot{\xi}(0) = W \neq O,$$

where $O \in R^n$ is the null vector. Let us consider an adapted inner product $\langle\langle\cdot\,,\cdot\rangle\rangle$ to the deviation tensor ξ by

$$\langle\langle X, Y\rangle\rangle := \frac{1}{\langle W, W\rangle}\cdot\langle X, Y\rangle,$$

for any vectors $X, Y \in R^n$. Obviously, $\|W\|^2 := \langle\langle W, W\rangle\rangle = 1$. Then, for $t \approx 0^+$, the trajectories of (3) are [19, 20, 21]

- bunching together if and only if the real part of the eigenvalues of deviation vector $P^i_j(0)$ are strictly negative.
- dispersing if and only if the real part of eigenvalues of deviation vector $P^i_j(0)$ are strictly positive.

Now, we define the Jacobi stability for SODE based on above cosideration [19, 20]. In a small vicinity of t_0 this type of stability refers to the focusing tendency of trajectories of (3) with respect to the variation (7) that satisfy the conditions

$$\|x^i(t_0) - \overline{x}^i(t_0)\| = 0, \ \|\dot{x}^i(t_0) - \dot{\overline{x}}^i(t_0)\| \neq 0.$$

Definition 2.1. *The trajectory of (3) are called Jacobi stable at $(x(t_0), \dot{x}(t_0))$ if and only if real parts of the eigenvalues of the deviation tensor $P^i_j|_{t_0}$ are strictly negative, and Jacobi unstable, otherwise.*

A basic result of the KCC theory is the following [10]:

Two systems of the form (3) on Ω can be locally transfered, relative to equation (4), one into another, if and only if the five KCC-invariants $\epsilon^i, P^i_j, \ P^i_{jk}, \ P^i_{jkl}, \ D^i_{jkl}$ are equivalent tensor. In particular, there exist coordinates $(\overline{x})$ for which the $G^i(\overline{x}, \overline{y}, t)$ vanish if and only if all KCC-invariants are zero.

In two dimensional space the matrix form of the deviation tensor can be written as

$$P^i_j = \begin{pmatrix} P^1_1 & P^1_2 \\ P^2_1 & P^2_2 \end{pmatrix}, \tag{15}$$

with eigenvalues as

$$\lambda_{\pm} = \frac{1}{2}\left[P^1_1 + P^2_2 \pm \sqrt{(P^1_1 - P^2_2)^2 + 4P^1_2 P^2_1}\right]. \tag{16}$$

The eigenvalues are the solution of the equation

$$\lambda^2 - (P^1_1 + P^2_2)\lambda + (P^1_1 P^2_2 - P^1_2 P^2_1) = 0. \tag{17}$$

3. Jacobi Stability of the Wang-Chen System

The present section is devoted to the dynamical properties of Wang-Chen system by differential geometric (KCC) approach. The nonlinear connection, Berwald connection and deviation tensor are obtained.

Differentiating second equation of (2) with respect to t, we get

$$\frac{d^2y}{dt^2} = 2x\frac{dx}{dt} - \frac{dy}{dt}.$$

Substitute the value of $\frac{dx}{dt} = yz + a$ from first equation of (2) and $x = \frac{1}{4}(1 - \frac{dz}{dt})$ from third equation of (2), then above equation becomes

$$\frac{d^2y}{dt^2} - \frac{1}{2}(yz + a) + \frac{1}{2}(yz + a)\frac{dz}{dt} + \frac{dy}{dt} = 0. \tag{18}$$

Again, from the third equation of (2) write $x = \frac{1}{4}(1 - \frac{dz}{dt})$ and differentiating with respect to t, we get $\frac{dx}{dt} = -\frac{1}{4}\frac{d^2z}{dt^2}$. Substitute the value of $\frac{dx}{dt}$ in first equation of (2), we get

$$\frac{d^2z}{dt^2} + 4yz + 4a = 0. \tag{19}$$

Let us introduce the new notation as $y = x^1, \frac{dy}{dt} = y^1, z = x^2, \frac{dz}{dt} = y^2$, the equations (18) and (19) yields the form

$$\begin{aligned} &\frac{d^2x^1}{dt^2} - \frac{1}{2}(x^1x^2 + a) + \frac{1}{2}(x^1x^2 + a)y^2 + y^1 = 0. \\ &\frac{d^2x^2}{dt^2} + 4x^1x^2 + 4a = 0. \end{aligned} \tag{20}$$

3.1. **The Nonlinear connection, Berwald connection and KCC invariants.** The second order differential formulation of Wang-Chen system is of the form

$$\frac{d^2x^i}{dt^2} + 2g^i(x^i, y^i) = 0, \ i = 1, 2, \tag{21}$$

where

$$\begin{aligned} g^1 &= -\frac{1}{4}\left[(x^1x^2 + a) - (x^1x^2 + a)y^2 - 2y^1\right], \\ g^2 &= 2(x^1x^2 + a). \end{aligned} \tag{22}$$

The components of nonlinear connection $N^i_j = \frac{\partial g^i}{\partial y^j}$ are

$$N^1_1 = \frac{1}{2}, \ N^1_2 = \frac{1}{4}(x^1x^2 + a), \ N^2_1 = 0, \ N^2_2 = 0.$$

The components of Berwald connection $G^i_{jl} = \frac{\partial N^i_j}{\partial y^l}$, vanish identically. The components of first KCC invariant, $\epsilon^i = 2g^i - N^i_j y^j$, are given as

$$\begin{aligned} \epsilon^1 &= -\frac{1}{2}(x^1x^2 + a) + \frac{1}{4}(x^1x^2 + a)y^2 + \frac{1}{2}y^1, \\ \epsilon^2 &= 4(x^1x^2 + a). \end{aligned} \tag{23}$$

The components of deviation tensor of Wang-Chen system obtained by equation (10) are as follows

$$\begin{aligned} P^1_1 &= \frac{1}{4} + \frac{1}{2}(1 - y^2)x^2, \\ P^1_2 &= \frac{1}{2}x^1 + \frac{1}{4}x^1y^2 + \frac{1}{4}x^2y^1 + \frac{1}{8}(x^1x^2 + a), \\ P^2_1 &= -4x^2, \quad P^2_2 = -4x^1. \end{aligned} \tag{24}$$

Time variation of deviation tensor components are represented in figure 1.

3.2. **Jacobi stability at the equilibrium point.** The system (2) has only one equilibrium $E(u, v, w) = (\frac{1}{4}, \frac{1}{16}, -16a)$. In respect to system of SODE (20), the equilibrium point is $E(x^1, x^2) = (\frac{1}{16}, -16a)$.
The components of first KCC invariant at the equilibrium point E are identically equal to zero, i.e.

$$\epsilon^1(E) = \epsilon^2(E) = 0,$$

The components of second KCC invariant (deviation tensor) are

$$P^1_1 = \frac{1}{4} - 8a, \ P^1_2 = \frac{1}{32}, \ P^2_1 = 64a, \ P^2_2 = -\frac{1}{4}.$$

The Jacobi matrix at the equilibrium point is

$$P = \begin{pmatrix} \frac{1}{4} - 8a & \frac{1}{32} \\ 64a & -\frac{1}{4} \end{pmatrix}.$$

Its characteristics equation is

$$\lambda^2 + 8a\lambda - \frac{1}{16} = 0, \tag{25}$$

thereore eigenvalues are $\lambda_1 = -4a - \frac{\sqrt{1+256a^2}}{4}$ and $\lambda_2 = -4a + \frac{\sqrt{1+256a^2}}{4}$, which shows that λ_1 is always negative and λ_2 is always positive irespective of the choice of the parameter a. Thus, we have:

Theorem 3.1. *The equilibrium point E is Jacobi unstable.*

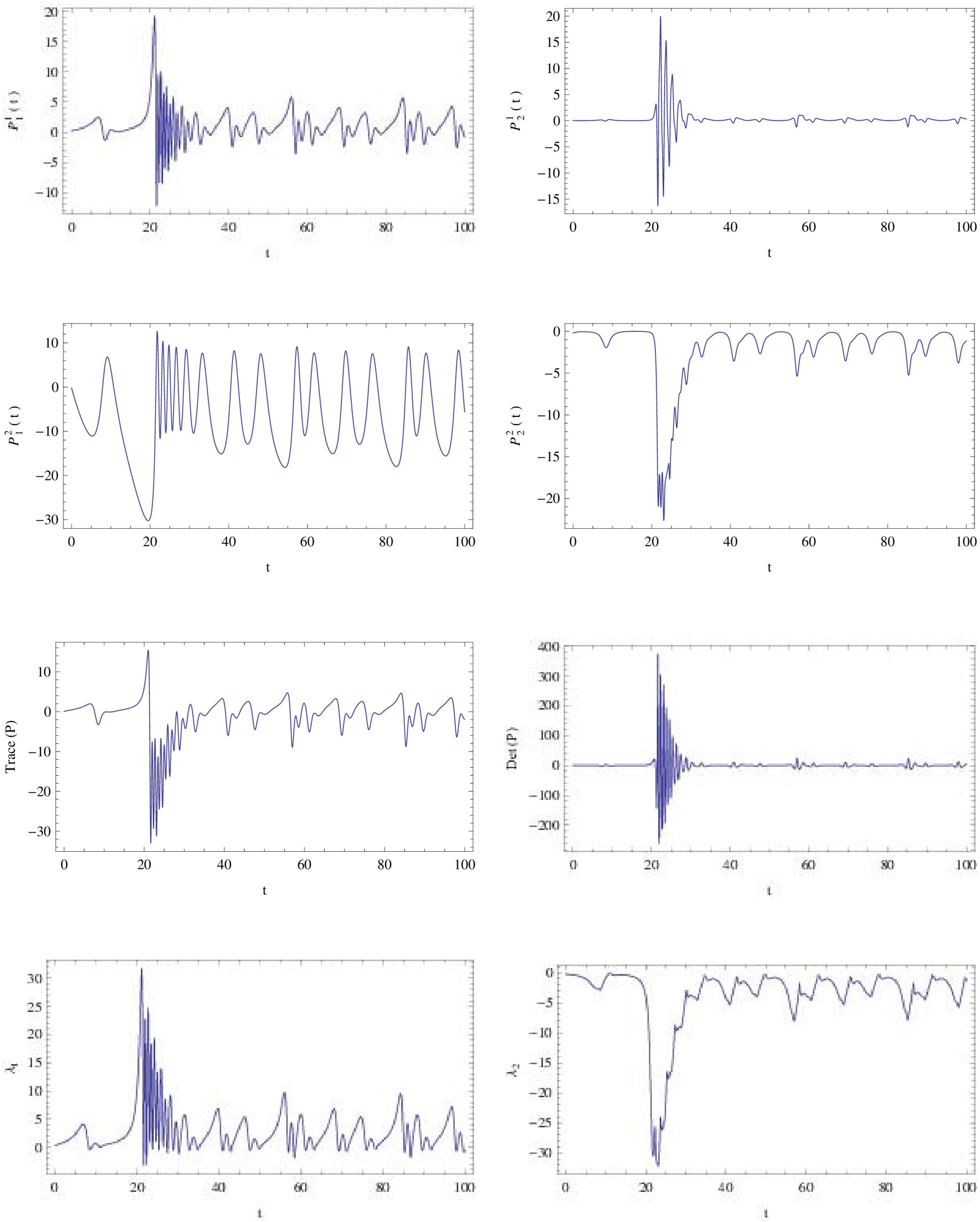

FIGURE 1. Time variation of deviation tensor component $P^1_1(t)$, $P^1_2(t)$, $P^2_1(t)$, $P^2_2(t)$, $trace(P(t))$, $det(P(t))$, eigenvalue λ_1 and eigenvalue λ_2 respectively, for parameters value $a = 0.006$. The initial conditions for the numerical integration are $x^1(0) = x^2(0) = x^3(0) = 0.1$.

4. Dynamics of deviation vector

In the view of equation (8), the trajectories behavior of dynamical system is described by the behaviour of deviation vector near the equilibrium point. In this case equations are

$$\frac{d^2\xi^1}{dt^2}+\frac{d\xi^1}{dt}+\frac{1}{2}(x^1x^2+a)\frac{d\xi^2}{dt}-\frac{1}{2}x^2(1-y^2)\xi^1-\frac{1}{2}x^1(1-y^2)\xi^2=0,$$
$$\frac{d^2\xi^2}{dt^2}+4x^2\xi^1+4x^1\xi^2=0,$$

The deviation vector is obtained as

$$\xi(t)=\sqrt{[\xi^1(t)]^2+[\xi^2(t)]^2}.$$

The instability exponents $\delta_i, i=1,2$ analogous to lyapunove exponents defined as [18]

$$\delta_i(E)=lim_{t\to\infty}\frac{1}{t}ln\frac{\xi^i(t)}{\xi_{i0}},\ \ i=1,2,\ \text{and}\ \ \delta(E)=lim_{t\to\infty}\frac{1}{t}ln\frac{\xi(t)}{\xi_{10}}.$$

4.1. **Dynamics of deviation vector near E.** The dynamics of deviation vector near the equilibrium point E is given by the differential equation,

$$\frac{d^2\xi^1}{dt^2}+\frac{d\xi^1}{dt}+8a\xi^1-\frac{1}{32}\xi^2=0,$$
$$\frac{d^2\xi^2}{dt^2}-64a\xi^1+\frac{1}{4}\xi^2=0,$$

Time variation of deviation vector and instability exponents are represented in the figure 2.

5. Comparison in Linear and Jacobi stability

By linearising the Wang-Chen system (2) at the equilibrium point E, the Jacobian matrix is

$$J(E)=\begin{pmatrix} 0 & -16a & -\frac{1}{16} \\ -\frac{1}{2} & -1 & 0 \\ -4 & 0 & -1 \end{pmatrix}. \tag{26}$$

Its characteristics equation is

$$\lambda^3+\lambda^2+(0.25+8a)\lambda+0.25=0. \tag{27}$$

In respect of system of equation (2) the equilibria and eigenvalue of Wang-Chen system are given in the following table as in the paper [2],

Table 1

Value of a	Equilibria	Eigenvalue	Linear Stability
-0.005	(0.25, 0.0625, 0.08)	$-1.03140,\ 0.01570\pm0.49208i$	Unstable
0.006	(0.25, 0.0625, -0.096)	$-0.96069,\ -0.01966\pm0.50975i$	Stable
0.022	(0.25, 0.0625, -0.352)	$-0.84580,\ -0.07710\pm0.53818i$	Stable
0.030	(0.25, 0.0625, -0.48)	$-0.78217,\ -0.10891\pm0.55476i$	Stable
0.050	(0.25, 0.0625, -0.8)	$-0.60746,\ -0.19627\pm0.61076i$	Stable

However, in view of theorem (3.1), the Wang-Chen system is Jacobi unstable for every parameter a.

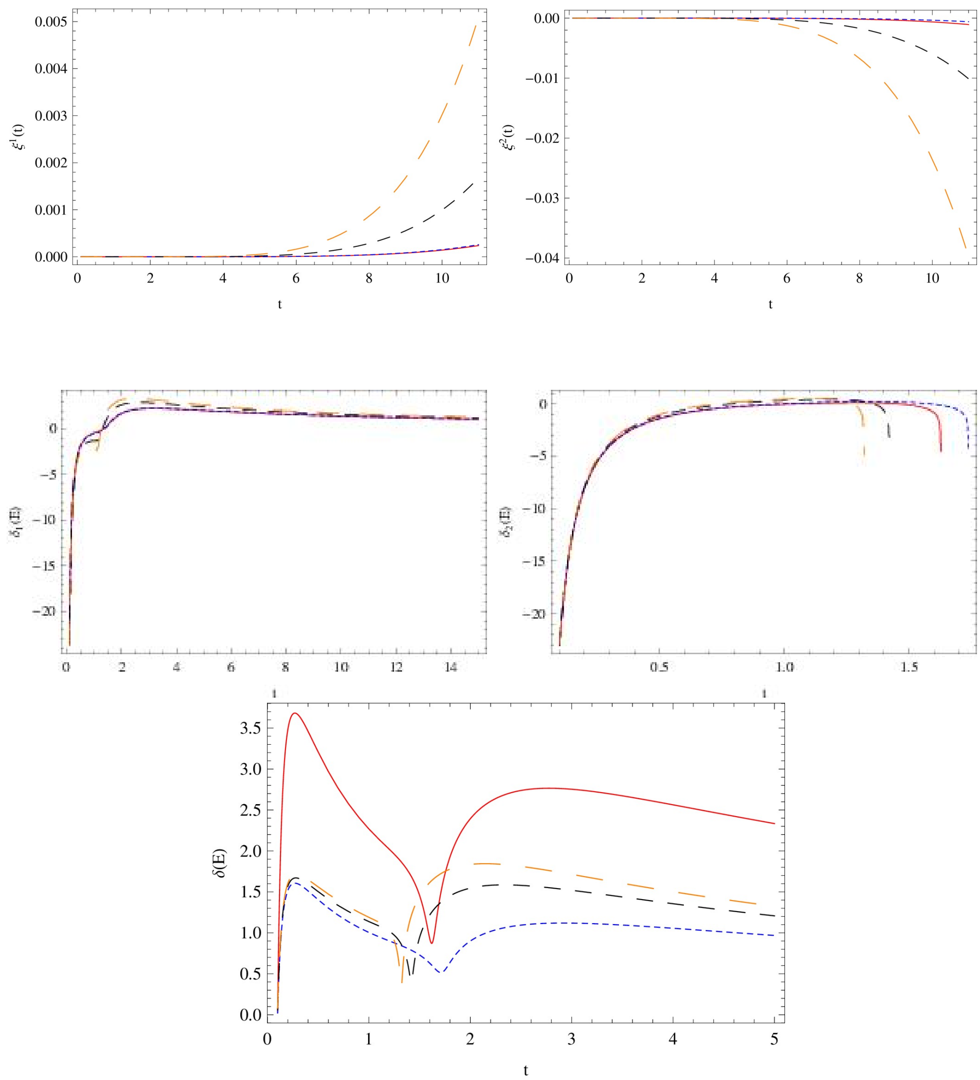

FIGURE 2. Time variation of deviation vector component $\xi^1(t)$ in above left fig., $\xi^2(t)$ in above right fig., $\delta_1(t)$ in middle left fig., $\delta_2(t)$ in middle right fig., $\delta(t)$ in below fig. respectively for parameter value $a = 0.006$ (solid, red), $a = 0.06$ (dashed, blue), $a = 0.6$ (long dashed, black), $a = 1$ (ultra long dashed, orange). The initial conditions for the numerical integration are $\xi^1(0) = \xi^2(0) = 0$ and $\dot{\xi}^1(0) = 10^{-10}, \dot{\xi}^2(0) = 10^{-9}$.

6. Conclusion

The Wang-Chen system has been investigsted from the view point of geometro-dynamical method. This theory is known as KCC theory. We converted the first order system to second order system by elimination of one equation. Then we appled the KCC theory and obtained the KCC invariant. The KCC invariants described the intrinsic geometric properties of Wang-Chen system. All the KCC invariant vanish except the first, second and third. The second KCC invariant (deviation tensor) gives the Jacobi stability of the system. At the equilibrium point the Wang-Chen system is Jacobi unstable for any values of parameter because one of the eigenvalue is positive. However, it is both linear stable and unstable for some choosen values of parameter as shown in the table 1 . The dynamics of deviation vector is represented near the equilibrium point. This show that the chaotic nature of the system.

References

[1] Sprott, J. C., (1994), Some simple chaotic flows, *Physical review E*, 50, pp. 647-650.

[2] Wang, X. and Chen, G., (2012), A chaotic system with only one equilibrium, Commun. Nonlinear Sci. Numer. Simulat., 17, pp. 1264-1272.

[3] Wang, X. and Chen, G., (2013), Constructing a chaotic system with any number of equilibria, Nonlin. Dyn., 71, pp. 429-436.

[4] Huan, S., Li, Q. and Yang, X. (2013), Hoeseshoes in a chaotic system with only one srable equilibrium, Int. J. Bifurcation Chaos, 23, 1350002.

[5] Wei, Z. and Wang, Z., (2013), Chaotic behavior and modified function projective synchronisation of a simple system with one stable equilibrium, Kybernetika, 49, pp. 359-374.

[6] Sprott, J. C., Wang, X. and Chen, G., (2013), Coexistence of point, periodic and strange attractors, Int. J. Bifurcation Chaos, 23, 1350093.

[7] Kosambi, D. D., (1933), Parallelism and path-space, Math. Zeitschrift, 37, pp. 608-818.

[8] Cartan, E., (1933), Observations sur le memoire precedent, Math. Zeitschrift, 37, pp. 619-622.

[9] Chern, S. S., (1939), Sur la geometrie dún systeme d'equations differentielles du second ordre, Bull. Sci. Math., 63, pp. 206-249.

[10] Antonelli, P. L., (2000), Equivalence problem for systems of second order ordinary differential equations, Encyclopedia of mathematics, Kluwer Acad. Publishers, Dordrecht.

[11] Antonelli, P. L., (2003), Handbook of Finsler geometry I, II, Kluwer Academic Publishers, Boston/ Dordrecht.

[12] Antonelli, P. L., Ingarden, R. S. and Matsumoto, M., (1993), The theories of sprays and Finsler spaces with application in physics and biology, Kluwer Acad. Publishers, Dordrecht/Boston/London.

[13] Boehmer. C. G., Harko, T. and Sabau, S. V., (2012), Jacobi stability analysis of Dynamical systems - Applications in Gravitation and Cosmology, Adv. Theor. Math. Phys., 16, pp. 1145-1196.

[14] Gupta, M. K. and Yadav, C. K., (2016), Jacobi stability analysis of Rikitake system, Int. J. Geom. Methods Mod. Phys., 13(07), 1650098.

[15] Gupta, M. K. and Yadav, C. K., (2017), Jacobi stability analysis of Rossler system, Int. J. Bifurcation Chaos, 27(04), 1750089.

[16] Gupta, M. K. and Yadav, C. K., (2017), Jacobi stability analysis of Modified Chua circuit system, Int. J. Geom. Methods Mod. Phys., 14(06), 1750056.

[17] Gupta, M. K. and Yadav, C. K., (2017) KCC theory and its application in a tumor growth model, Math Meth Appl Sci., 40, pp. 7470-7487.

[18] Harko, T., Ho, C. Y., Leung, C. S. and Yip, S., (2015), Jacobi stability analysis of Lorenz system, Int. J. Geom. Methods Mod. Phys., 12(07), 1550081.

[19] Sabau, S. V., (2005), Some remarks on jacobi stability, Nonlinear Analysis, 63, pp. e143-e153.

[20] Sabau, S. V., (2005), Systems biology and deviation curvature tensor, Nonlinear Analysis: Real World Applications, 6, pp. 563-587.

[21] Bao, D., Chern, S. S. and Shen, Z., 2000, An Introduction to Riemann-Finsler Geometry, Graduate Texts in Mathematics, vol. 200, Springer, NewYork.

[22] Pham, V. T. *et al.*, (2016), From Wang-Chen system with only one stable equilibrium to a new Chaotic system without equilibrium, Int. J. Bifurcation Chaos, 27, 1750097.

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