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STARLIKE SYMMETRICAL FUNCTIONS

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ABSTRACT. The objective of the present paper is to study subclass $\mathcal{S}^{(j,k)}(A,B)$ of analytic functions that is defined by using the class of Janowski functions combined with the (j,k)-symmetrical functions. This class generalizes various classes defined by different authors. Distortion theorem, argument theorem, covering theorem,and convolution condition are obtained. Finally we give analogous definition of neighborhood for the class $\mathcal{S}^{(j,k)}(A,B)$ and then investigate related neighborhood result for this new class.

Keywords Janowski functions, Subordination, Starlike functions, Neighborhood, k-fold symmetric function, (j,k)-symmetrical function.

AMS Subject Classification: 30C45.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}\ and\ |z| < 1\}$, and suppose $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions that are univalent in $\mathcal{U}$. Also, let Ω be the family of functions w, analytic in $\mathcal{U}$ and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$. If f and g are analytic in $\mathcal{U}$, we say that a function f is subordinate to a function g in $\mathcal{U}$, if there exists a function $w \in \Omega$ such that $f(z) = g(w(z))$, and we denote this by $f \prec g$. If g is univalent in $\mathcal{U}$ then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. The Hadamard product or convolution of two functions f and g in $\mathcal{A}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,\ z \in \mathcal{U}.$$

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Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\operatorname{Re} p(z) > 0$, $z \in \mathcal{U}$. The class of functions with positive real part $\mathcal{P}$ plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses like class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$.
A function $f \in \mathcal{S}$ defined on $\mathcal{U}$ is said to be *starlike* in $\mathcal{U}$ if and only if $\frac{zf'(z)}{f(z)}$ belongs to $\mathcal{P}$, also a function $f \in \mathcal{S}$ defined on $\mathcal{U}$ is said to be *convex* in $\mathcal{U}$ if and only if $\frac{(zf'(z))'}{f'(z)}$ belongs to $\mathcal{P}$. The class of starlike functions with respect to symmetric points have been defined by using the concept of class of functions with positive real part.

Definition 1.1. [8] *Let $\mathcal{P}[A, B]$, with $-1 \leq B < A \leq 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \dfrac{1 + Aw(z)}{1 + Bw(z)}$, $z \in \mathcal{U}$, where $w \in \Omega$.*

Note that $p \in \mathcal{P}[A, B]$ if and only if $p(z) \prec \dfrac{1 + Az}{1 + Bz}$. The class of *generalized Janowski type functions* $\mathcal{P}[A, B, \alpha]$ was introduced in [11] as follows:
For arbitrary fixed numbers A, B, α, with $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$,

$$p \in \mathcal{P}[A, B, \alpha] \Longleftrightarrow p(z) \prec \frac{1 + [(1 - \alpha)A + \alpha B]z}{1 + Bz}.$$

This implies

$$f \in \mathcal{S}^*[A, B, \alpha] \Longleftrightarrow \frac{zf'(z)}{f(z)} \in \mathcal{P}[A, B, \alpha].$$

In order to define a new class of Janowski symmetrical functions defined in the open unit disk $\mathcal{U}$, we first recall the notion of k-fold symmetric functions defined in k-fold symmetric domain, where k is any positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{D}$ if for every z in $\mathcal{D}$ we have

$$f(\varepsilon z) = \varepsilon f(z),\ z \in \mathcal{D},\ \varepsilon = e^{\frac{2\pi i}{k}}.$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$, and for $k = 2$ we get class of odd univalent functions. In 1995, Liczberski and Polubinski [10] constructed the theory of (j, k)-symmetrical functions for $(j = 0, 1, 2, \ldots, k - 1)$ and $(k = 2, 3, \ldots)$. If $\mathcal{D}$ is k-fold symmetric domain and j is any integer, then a function $f : \mathcal{D} \to \mathbb{C}$ is called (j, k)-symmetrical if $f(\varepsilon z) = \varepsilon^j f(z)$, for each $z \in \mathcal{D}$. We note that the (j, k)-symmetrical functions is a generalization of the notions of even, odd, and k-symmetrical functions

The theory of (j, k)-symmetrical functions has many interesting applications; for instance, in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings, see [10].

Denote the family of the classes of starlike with (j, k)-symmetrical functions by $\mathcal{S}^{(j,k)}$. We observe that $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are the classes of starlike with respect to even, odd and k-symmetric functions respectively. We have the following decomposition theorem:
Theorem 1 [10, Page 16] For every mapping $f : \mathcal{U} \to \mathbb{C}$, and a k-fold symmetric set $\mathcal{U}$, there exists exactly one sequence of (j, k)-symmetrical functions $f_{j,k}$ such that

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z),$$

where

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f\left(\varepsilon^{v}z\right),\ z\in\mathcal{U}. \tag{2}$$

Remark 1.1. *Equivalently,* (2) *may be written as*

$$f_{j,k}(z)=\sum_{n=1}^{\infty}\delta_{n,j}a_nz^n,\ \ a_1=1, \tag{3}$$

where

$$\delta_{n,j}=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(n-j)v}=\begin{cases}1, & n=lk+j;\\ 0, & n\neq lk+j;\end{cases}, \tag{4}$$

$$(l\in\mathbb{N},\ k=1,2,\dots,\ j=0,1,2,\dots,k-1)$$

Al Sarari and Latha [1] introduced and studied the classes $\mathcal{S}^{(j,k)}(A,B)$ and $\mathcal{K}^{(j,k)}(A,B)$ which are starlike and convex with respect to (j,k)-symmetric points. For more details about the classes with (j,k)-symmetrical functions see [3, 4, 2].

Definition 1.2. *A function $f\in\mathcal{A}$ is said to belongs to the class $\mathcal{S}^{(j,k)}(A,B)$, with $-1\leq B<A\leq 1$ if*

$$\frac{zf'(z)}{f_{j,k}(z)}\prec\frac{1+Az}{1+Bz},$$

where $f_{j,k}$ are defined by (2).

Remark 1.2. *Using the definition of the subordination, we can easily obtain that the equivalent condition for a function f belonging to the class $\mathcal{S}^{(j,k)}(A,B)$, with $-1\leq B<A\leq 1$, which is*

$$\left|\frac{zf'(z)}{f_{j,k}(z)}-1\right|<\left|A-B\frac{zf'(z)}{f_{j,k}(z)}\right|,\ z\in\mathcal{U}.$$

We note that special values of j,k,A and B yield the following classes:
For $j=1$ the class is studied by Kwon and Sim [9].
For $j=1,\ A=1$ *and* $B=-1$, $\mathcal{S}_k^*=\mathcal{S}_k^*(1,-1)$, the class is studied by Sakaguchi [13]
For $j=1,\ A=1-2\alpha(0\leq\alpha<1)$ *and* $B=-1$, $\mathcal{S}_k^*(\alpha)=\mathcal{S}_k^*(1-2\alpha,-1)(0\leq\alpha<1)$, the class of functions that are starlike of order α with respect to k-symmetric points.

We need to recall the following neighborhood concept introduced by Goodman [7] and generalized by Ruscheweyh [12]

Definition 1.3. *For any $f\in\mathcal{A}$, ρ-neighborhood of function f can be defined as:*

$$\mathcal{N}_\rho(f)=\left\{g\in\mathcal{A}:g(z)=z+\sum_{n=2}^{\infty}b_nz^n,\ \sum_{n=2}^{\infty}n|a_n-b_n|\leq\rho\right\}. \tag{5}$$

For $e(z)=z$, we can see that

$$\mathcal{N}_\rho(e)=\left\{g\in\mathcal{A}:g(z)=z+\sum_{n=2}^{\infty}b_nz^n,\ \sum_{n=2}^{\infty}n|b_n|\leq\rho\right\}. \tag{6}$$

Ruscheweyh [12] proved, among other results that for all $\eta\in\mathbb{C}$, with $|\mu|<\rho$,

$$\frac{f(z)+\eta z}{1+\eta}\in\mathcal{S}^*\Rightarrow\mathcal{N}_\rho(f)\subset\mathcal{S}^*.$$

In this paper, we investigate distortion theorem, argument theorem, covering theorem, convolution condition for our class $\mathcal{S}^{(j,k)}(A,B)$. Finally motivated by Definition 29. Further, we give analogous definition of neighborhood for the class $\mathcal{S}^{(j,k)}(A,B)$ and then investigate related neighborhood result for this new class.
The following lemmas are needed in our investigations.

Lemma 1.1. [11] *Let $f \in \mathcal{S}^*(A,B,\alpha)$, then $f(z)$ can be written in the form*

$$f(z) = \begin{cases} z\left(1+Bw(z)\right)^{\frac{(1-\alpha)(A-B)}{B}}, & \text{if } \ B \neq 0, \\ z\exp[(1-\alpha)Aw(z)], & \text{if } \ B = 0, \end{cases} \tag{7}$$

Lemma 1.2. [8] *Let $p \in \mathcal{P}[A,B]$, then*

$$\frac{1-A|z|}{1-B|z|} \leq |p(z)| \leq \frac{1+A|z|}{1+B|z|},\ z \in \mathcal{U}.$$

2. Main results

Lemma 2.1. *If $f \in \mathcal{S}^{(j,k)}(A,B)$, then*

$$f_{j,k}(z) = \begin{cases} z\left(1+Bw(z)\right)^{\frac{A-B}{B}}, & \text{if } \ B \neq 0, \\ z\exp[Aw(z)], & \text{if } \ B = 0, \end{cases} \tag{8}$$

where $f_{j,k}$ are defined by (2).

Proof. Supposing that $f \in \mathcal{S}^{(j,k)}(A,B)$, we can get

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz}. \tag{9}$$

Substituting z by $\varepsilon^v z$ in (9), it follows

$$\frac{\varepsilon^v z f'(\varepsilon^v z)}{f_{j,k}(\varepsilon^v z)} \prec \frac{1+A\varepsilon^v z}{1+B\varepsilon^v z} \prec \frac{1+Az}{1+Bz},$$

hence

$$\frac{\varepsilon^{v-vj} z f'(\varepsilon^v z)}{f'_{j,k}(z)} \prec \frac{1+Az}{1+Bz}, \tag{10}$$

Letting $v = 0,1,2,\ldots,k-1$ in (10) and using the fact that $\mathcal{P}[A,B]$ is a convex set, we get

$$\frac{z\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{v-vj} f'(\varepsilon^v z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz},$$

or equivalently

$$\frac{zf'_{j,k}(z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz}, \tag{11}$$

that is $f_{j,k} \in \mathcal{S}^*(A,B)$, and by Lemma 1.1 for $\alpha = 0$, we finally obtain our result.

Theorem 2.1. *If $f \in \mathcal{S}^{(j,k)}(A,B)$, then*

$$\left.\begin{matrix} \frac{1-Ar}{1-Br}(1-Br)^{\frac{A-B}{B}}, & \text{if } \ B \neq 0, \\ \left[1-Ar\right]\exp\left[-Ar\right], & \text{if } \ B = 0 \end{matrix}\right\} \leq |f'(z)| \leq \left\{\begin{matrix} \frac{1+Ar}{1+Br}(1+Br)^{\frac{A-B}{B}}, & \text{if } \ B \neq 0, \\ \left[1+Ar\right]\exp\left[Ar\right], & \text{if } \ B = 0, \end{matrix}\right.$$

where $|z| = r < 1$.

Proof. For an arbitrary function $f \in \mathcal{S}^{(j,k)}(A,B)$, according to Lemma 2.1 we need to distinguish the next two cases:

(i) If $B \neq 0$, then there exists a function $w \in \Omega$, such that $f_{j,k}(z) = z\left(1 + Bw(z)\right)^{\frac{A-B}{B}}$, and therefore

$$\frac{1-Ar}{1-Br}\left|1+Bw(z)\right|^{\frac{A-B}{B}} \leq |f'(z)| \leq \frac{1+Ar}{1+Br}\left|1+Bw(z)\right|^{\frac{A-B}{B}}, \ |z| \leq r < 1. \tag{12}$$

Since $w \in \Omega$, we have

$$1 - |B|r \leq |1 + Bw(z)| \leq 1 + |B|r, \ |z| \leq r < 1.$$

Case 1. If $B > 0$, using the fact that $-1 \leq B < A \leq 1$, we have

$$(1 - |B|r)^{\frac{A-B}{B}} \leq |1 + Bw(z)|^{\frac{A-B}{B}} \leq (1 + |B|r)^{\frac{A-B}{B}}, \ |z| \leq r < 1,$$

and from (12) we obtain

$$\frac{1-Ar}{1-Br}(1-|B|r)^{\frac{A-B}{B}} \leq |f'(z)| \leq \frac{1+Ar}{1+Br}(1+|B|r)^{\frac{A-B}{B}}, \ |z| \leq r < 1. \tag{13}$$

Case 2. If $B < 0$, from the fact that $-1 \leq B < A \leq 1$, we have

$$(1 - |B|r)^{\frac{A-B}{B}} \geq |1 + Bw(z)|^{\frac{A-B}{B}} \geq (1 + |B|r)^{\frac{A-B}{B}}, \ |z| \leq r < 1,$$

and from (12) we obtain

$$\frac{1-Ar}{1-Br}(1-|B|r)^{\frac{A-B}{B}} \geq |f'(z)| \geq \frac{1+Ar}{1+Br}(1+|B|r)^{\frac{A-B}{B}}, \ |z| \leq r < 1. \tag{14}$$

Now, combining the inequalities (13) and (14), we finally conclude that

$$\frac{1-Ar}{1-Br}(1-Br)^{\frac{A-B}{B}} \leq |f'(z)| \leq \frac{1+Ar}{1+Br}(1+Br)^{\frac{A-B}{B}}, \ |z| \leq r < 1. \tag{15}$$

(ii) If $B = 0$, there exists a function $w \in \Omega$, such that $f_{j,k}(z) = z\exp\left[Aw(z)\right]$, and therefore

$$[1 - Ar]\left|\exp\left[Aw(z)\right]\right| \leq |f'(z)| \leq [1 + Ar]\left|\exp\left[Aw(z)\right]\right|, \ |z| \leq r < 1. \tag{16}$$

Since $|\exp\left[Aw(z)\right]| = \exp\left[A\Re w(z)\right]$, $z \in \mathcal{U}$, using a similar computation as in the previous case, we deduce

$$\exp\left[-Ar\right] \leq \left|\exp\left[Aw(z)\right]\right| \leq \exp\left[Ar\right], \ |z| \leq r < 1.$$

Thus, (16) yields to

$$[1 - Ar]\exp\left[-Ar\right] \leq |f'(z)| \leq [1 + Ar]\exp\left[Ar\right], \ |z| \leq r < 1, \tag{17}$$

which completes the proof of our theorem. □

Theorem 2.2. *Let $f \in \mathcal{S}^{(j,k)}(A,B)$. Then*

$$|\arg f'(z)| \leq \begin{cases} \dfrac{(A-B)}{B}\arcsin(Br) + \arcsin\left(\dfrac{(A-B)}{1-ABr^2}\right), & \textit{if} \quad B \neq 0, \\ Ar + \arcsin(Ar), & \textit{if} \quad B = 0, \end{cases}$$

Proof. Suppose that $f \in \mathcal{S}^{(j,k)}(A,B)$, then

$$|\arg f'(z)| \leq \left|\arg \frac{f_{j,k}(z)}{z}\right| + |\arg p(z)|, \tag{18}$$

where $p \in P[A,B]$, using Lemma 2.1 for $B \neq 0$, we have

$$\frac{f_{j,k}(z)}{z} = (1 + Bw(z))^{\frac{(A-B)}{B}},$$

Case (i), $B > 0$.

$$\begin{aligned}
\left|(1 + Bw(z))^{\frac{A-B}{B}}\right| &= \left|\exp\left\{\frac{A-B}{B}\log(1 + Bw(z))\right\}\right| \\
&= \exp\left\{\frac{A-B}{B}\log|(1 + Bw(z))|\right\} \\
&= |(1 + Bw(z))|^{\frac{A-B}{B}} \\
&\leq (1 + Br)^{\frac{A-B}{B}}.
\end{aligned}$$

Case (ii) $B < 0$. Let $B = -C, C > 0$. Then

$$\begin{aligned}
\left|(1 + Bw(z))^{\frac{A-B}{B}}\right| &= \left|\left\{(1 - Cw(z))^{-1}\right\}^{\frac{A+C}{C}}\right| \\
&= \left|(1 - Cw(z))^{-1}\right|^{\frac{A+C}{C}} \\
&\leq \left(\frac{1}{1 - Cr}\right)^{\frac{A+C}{C}} \\
&= (1 + Br)^{\frac{A-B}{B}}.
\end{aligned}$$

Combining the cases (i) and (ii), we get

$$\left|\arg\left(\frac{f_{j,k}(z)}{z}\right)\right| \leq \frac{A-B}{B}|\arg(1 + Br)| \leq \frac{A-B}{B}\arcsin(Br). \tag{19}$$

For $B = 0$ it is clear

$$\left|\arg\left(\frac{f_{j,k}(z)}{z}\right)\right| \leq Ar. \tag{20}$$

Now for $p \in \mathcal{P}[A, B]$, we have

$$|\arg p(z)| \leq \arcsin\left(\frac{(A-B)r}{1 - ABr^2}\right). \tag{21}$$

For $B = 0$, directly we get

$$|\arg p(z)| \leq \arcsin(Ar). \tag{22}$$

From (19),(20),(21) and (22) we get the proof.

Theorem 2.3. *If* $f \in \mathcal{S}^{(j,k)}(A, B)$, *then*

$$|f(z)| \leq \begin{cases} \displaystyle\int_0^r \frac{1 + A\rho}{1 + B\rho}(1 + B\rho)^{\frac{A-B}{B}}\, d\rho, & \text{if } B \neq 0, \\[2ex] \displaystyle\int_0^r [1 + A\rho]\exp[A\rho]\, d\rho, & \text{if } B = 0, \end{cases}$$

where $|z| \leq r < 1$.

Proof. Integrating the function f' along the close segment connecting the origin with an arbitrary $z \in \mathcal{U}$, since a point on this segment is of the form $\zeta = \rho e^{i\theta}$, with $\rho \in [0, r]$, where $\theta = \arg z$ and $r = |z|$, we get

$$f(z) = \int_0^z f'(\zeta)d\zeta, \ z = re^{i\theta},$$

hence

$$|f(z)| = \left| \int_0^r f'\left(\rho e^{i\theta}\right) e^{i\theta} d\rho \right| \leq \int_0^r \left| f'\left(\rho e^{i\theta}\right) e^{i\theta} \right| d\rho.$$

Using this inequality and the right-hand side inequalities of Theorem 2.1, we discuss the following two cases:

(i) If $B \neq 0$, then

$$|f(z)| \leq \int_0^r \left| f'\left(\rho e^{i\theta}\right) e^{i\theta} \right| d\rho \leq \int_0^r \frac{1 + A\rho}{1 + B\rho}(1 + B\rho)^{\frac{(A-B)}{B}} d\rho,$$

that is

$$|f(z)| \leq \int_0^r \frac{1 + A\rho}{1 + B\rho}(1 + B\rho)^{\frac{(A-B)}{B}} d\rho, \ |z| \leq r < 1.$$

(ii) If $B = 0$, then

$$|f(z)| \leq \int_0^r \left| f'\left(\rho e^{i\theta}\right) e^{i\theta} \right| d\rho \leq \int_0^r [1 + |A|\rho] \exp[A\rho] d\rho,$$

that is

$$|f(z)| \leq \int_0^r [1 + A\rho] \exp[A\rho] d\rho, \ |z| \leq r < 1.$$

□

Theorem 2.4. *A function $f \in \mathcal{S}^{(j,k)}(A, B)$ if and only if*

$$\frac{1}{z}\left[f * \left\{\frac{z}{(1-z)^2}(1 + Be^{i\theta}) - q(z)(1 + Ae^{i\theta})\right\}\right] \neq 0 \tag{23}$$

where $-1 \leq B < A \leq 1, 0 \leq \theta < 2\pi$ and $q(z)$ is given by

$$q(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{(1-j)v} \frac{z}{1 - \varepsilon^v z},$$

.

Proof. Suppose that $f \in \mathcal{S}^{(j,k)}(A, B)$, then

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1 + Az}{1 + Bz},$$

if and only if

$$\frac{zf'(z)}{f_{j,k}(z)} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}}, \tag{24}$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$. The condition (24) can be written as

$$\frac{1}{z}[zf'(z)(1 + Be^{i\theta}) - f_{j,k}(z)(1 + Ae^{i\theta})] \neq 0. \tag{25}$$

Since

$$zf'(z) = f(z) * \frac{z}{(1-z)^2} \tag{26}$$

and

$$f_{j,k}(z) = f(z) * q(z), \tag{27}$$

where

$$q(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(1-j)v}\frac{z}{1-\varepsilon^{v}z}, \tag{28}$$

we have (23) from substituting (26) and (27) into (25).
To find some neighborhood results for the class $\mathcal{S}^{(j,k)}(A,B)$ analogous to those obtained by Ruscheweyh [12], we introduce the following concept of neighborhood.

Definition 2.1. *For $-1\le B<A\le 1$ and $\rho\ge 0$ we define the neighborhood of a function $f\in\mathcal{A}$ as*

$$\mathcal{N}^{j,k}_{A,B,\rho}(f)=\left\{g\in\mathcal{A}:g(z)=z+\sum_{n=2}^{\infty}b_nz^n,d(f,g)=\sum_{n=2}^{\infty}\frac{n-\delta_{n,j}+|Bn-A\delta_{n,j}|}{A-B}|b_n-a_n|\le\rho,\right\} \tag{29}$$

where $\delta_{n,j}$ is defined by (4).

Remark 2.1. *For parametric values $j=k=A=-B=1$, (29) reduces to (5).*

Theorem 2.5. *Let $f\in\mathcal{A}$, and for all complex number η, with $|\mu|<\rho$, if*

$$\frac{f(z)+\eta z}{1+\eta}\in\mathcal{S}^{(j,k)}(A,B). \tag{30}$$

Then

$$\mathcal{N}^{j,k}_{A,B,\rho}(f)\subset\mathcal{S}^{(j,k)}(A,B).$$

Proof. We assume that a function g defined by $g(z)=\sum_{n=2}^{\infty}b_nz^n$ is in the class $\mathcal{N}^{j,k}_{A,B,\rho}(f)$. In order to prove the theorem, we only need to prove that $g\in\mathcal{S}^{(j,k)}(A,B)$. We would prove this claim in next three steps.
We first note that Theorem 2.4 is equivalent to

$$f\in\mathcal{S}^{(j,k)}(A,B)\Leftrightarrow\frac{1}{z}[(f*t_\phi)(z)]\neq 0,\quad z\in\mathcal{U}, \tag{31}$$

where

$$t_\phi(z)=\frac{\frac{z}{(1-z)^2}(1+Be^{i\theta})-q(z)(1+Ae^{i\theta})}{(B-A)e^{i\theta}},$$

$0\le\theta<2\pi$ and $q(z)$ is given by (28). We can write $t_\phi(z)=z+\sum_{n=2}^{\infty}t_nz^n$, where

$$t_n=\frac{n-\delta_{n,j}+(Bn-A\delta_{n,j})e^{i\theta}}{(B-A)e^{i\theta}}. \tag{32}$$

Secondly we obtain that (30) is equivalent to

$$\left|\frac{f(z)*t_\phi(z)}{z}\right|\ge\rho, \tag{33}$$

because, if $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{A}$ and satisfy (30), then (31) is equivalent to

$$t_\phi\in\mathcal{S}^{(j,k)}(A,B,)\Leftrightarrow\frac{1}{z}\left[\frac{f(z)*t_\phi(z)}{1+\eta}\right]\neq 0,\qquad |\eta|<\rho.$$

Thirdly letting $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ we notice that

$$\begin{aligned}\left|\frac{g(z)*t_\phi(z)}{z}\right|&=\left|\frac{f(z)*t_\phi(z)}{z}+\frac{(g(z)-f(z))*t_\phi(z)}{z}\right|\\&\ge\rho-\left|\frac{(g(z)-f(z))*t_\phi(z)}{z}\right|,\quad\text{(by using (33))},\end{aligned}$$

$$= \rho - \left| \sum_{n=2}^{\infty} (b_n - a_n) t_n z^n \right|,$$

$$\geq |z| \sum_{n=2}^{\infty} \frac{n - \delta_{n,j} + (Bn - A\delta_{n,j})e^{i\theta}}{(B-A)e^{i\theta}} |b_n - a_n|$$

$$\geq \rho - \rho = 0, \qquad \text{by applying (32).}$$

This proves that

$$\frac{g(z) * t_\phi(z)}{z} \neq 0, \qquad z \in \mathcal{U}.$$

In view of our observations and by (31), it follows that $g \in \mathcal{S}^{(j,k)}(A, B)$. This completes the proof of the theorem.

When $j = k = A = -B = 1$ in the above theorem we get (6) proved by Ruscheweyh in [12].

Corollary 2.1. *Let $\mathcal{S}^*$ the class of starlike functions. Let $f \in \mathcal{A}$ and for all complex number η, with $|\mu| < \rho$, if*

$$\frac{f(z) + \eta z}{1 + \eta} \in \mathcal{S}^*. \tag{34}$$

Then $\mathcal{N}_\sigma(f) \subset \mathcal{S}^*$.

3. Competing interests

The author declares that there is no conflict of interests.

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