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LAGUERRE WAVELET SOLUTION OF BRATU AND DUFFING EQUATIONS

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Abstract. The aim of this study is to solve the Bratu and Duffing equations by using the Laguerre wavelet method. The solution of these nonlinear equations is approximated by Laguerre wavelets which are defined by well known Laguerre polynomials. One of the advantages of the proposed method is that it does not require the approximation of the nonlinear term like other numerical methods. The application of the method converts the nonlinear differential equation to a system of algebraic equations. The method is tested on four examples and the solutions are compared with the analytical and other numerical solutions and it is observed that the proposed method has a better accuracy.

Keywords: Laguerre wavelets, Bratu equation, Duffing equation.

AMS Subject Classification: 65L05, 65T60

1. Introduction

The Bratu equation is used to model many phenomena such as the fuel ignition of the thermal combustion, thermal reaction, radiative heat transfer, chemical reactor theory and the expansion of the universe. [1, 2]. Consequently, researchers pay attention towards analytical and numerical solutions of this equation. Wazwaz [3] studied the analytical solutions of Bratu-type equations using Adomian decomposition method. Restarted Adomian decomposition method is used to solve the same problem numerically by Vahidi and Hasanzade [4]. Al-Mazmumy et.al [5] used both the Adomian and restarted Adomian decomposition methods with new techniques. Other numerical methods used in the solution of Bratu equation can be listed as: the shooting method [1], the finite difference method [6], weighted residual method [7], decomposition technique [8], Legendre wavelets [9, 11, 12], Bernoulli-collocation method [13], Chebyshev wavelets [14, 15], B-spline method [16], Jacobi-Gauss collocation method [17], Laplace transform decomposition method [18], Variational iteration method [19, 20], perturbation-iteration algorithms [21], Green's functions [22, 23], Haar wavelets [24], the fifth order Runge-Kutta method [25], Homotopy analysis method [26], Homotopy perturbation method [27], Differential quadrature method [28], Nonstandard finite differences [29], Optimal Homotopy Asymptotic Method [30], sinc-Collocation method [31], Chebyshev pseudospectral method [32] and Taylor wavelets [33].

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The Duffing equation is also a well known nonlinear equation which is used to model some important practical phenomena such as orbit extraction, classical oscillator in chaotic systems, nonlinear vibration of beams and plates, and the prediction of diseases. Hence, Duffing equation has been widely investigated by many researchers using several numerical methods. Among them one can mention: the improved Taylor matrix method [35], generalized differential quadrature rule [36], shifted Chebyshev polynomials [37], Daftardar-Jafari method [34], Runge-Kutta-Fehlenberg algorithm [38], Laplace decomposition algorithm [39], Differential transform method [40], Homotopy method [41], Legendre wavelets [10, 11, 12], and restarted Adomian decomposition method [50].

Laguerre wavelet method has been used to solve many problems including fractional order delay differential equations [42, 43], one dimensional partial differential equations [44], Lane-Emden type differential equations [45, 46], nonlinear delay differential equations with damping [47], linear and nonlinear singular boundary value problems [48], and Benjamina-Bona-Mohany equations [49]. The aim of this study is using Laguerre wavelet method to solve the Bratu and Duffing equations:

2. Laguerre Polynomials and Some Properties

The Laguerre polynomials are m-th degree polynomials, which are known as the solutions of the differential equation, which

$$xy''(x) + (1-x)y'(x) + my(x) = 0, \;\; x \in [0, \infty)$$

which is also called the Laguerre differential equation [51, 52]. The Laguerre polynomial of degree m, usually denoted by $L_m(x)$, satisfies

$$\int_0^\infty e^{-x} L_m(x) L_n(x) dx = \delta_{mn},$$

where δ_{mn} is the Kronecker delta symbol. This leads the orthogonality of these polynomials with respect to the weight function $w(x) = e^{-x}$ over the interval $[0, \infty)$. With the aid of the multiplier e^{-x}, it is also possible to define the Laguerre polynomials as the eigenfunctions of the Sturm-Liouville equation

$$e^x (xe^{-x} y'(x))' + \mu_m y(x) = 0$$

corresponding to the eigenvalues $\mu_m = m$, $m = 0, 1, 2, \ldots$

The Laguerre polynomials are easily determined by the recurrence relation with $L_0(x) = 1$ and $L_1(x) = 1 - x$,

$$(m+2)L_{m+2}(x) = (2m+3-x)L_{m+1}(x) - (m+1)L_m(x), \;\; m = 0, 1, \ldots$$

or equivalently, by the closed form $\sum_{s=0}^{m} \frac{(-1)^s m!}{(m-s)!(s!)^2} x^s$.

3. Laguerre Wavelets and Some Results on Laguerre Wavelets

Wavelets constitute orthonormal set of functions, which are obtained by dilation and translation of a single function $\phi(t)$ as

$$\phi_{a,b}(t) = |a|^{-1/2} \phi\left(\frac{t-b}{a}\right), \;\; a, b \in \mathbf{R}, \;\; a \neq 0, \tag{1}$$

if the dilation parameter a and the translation parameter b are chosen by $a = 2^{-k}$ and $b = n2^{-k}$, for a nonnegative integer k [53, 54]. Here, $\phi(t)$ is the generating function which is called mother wavelet.

Legendre wavelets, Chebyshev wavelets, Laguerre wavelets and Haar wavelets are some examples of the wavelets which form a basis in $L^2(\mathbf{R})$ [53, 54]. These wavelets combine the basic properties of corresponding polynomials with a compact support and this gives them the advantage of being good at modeling localized features in applications [54]. Due to this and such advantages, Haar wavelet [24], Legendre wavelet [9, 11, 12], Chebyshev wavelet [14, 15] are frequently used to solve varieties of differential and integral equations.

If the dilation and translation parameters are chosen respectively as $a = 2^{-(k+1)}$ and $b = (2n+1)2^{-(k+1)}$ in (1), then the Laguerre wavelets $\phi_{nm} = \phi_{nm}(k, n, m, t)$ with four arguments can be defined on $[0, 1)$ for integers $k = 0, 1, 2, \ldots$ $n = 0, 1, 2, \ldots, 2^k - 1.$, using the Laguerre polynomials L_m, of order m as

$$\phi_{nm}(t) = \begin{cases} 2^{(k+1)/2} L_m(2^{k+1}t - 2n - 1), & \text{if } \frac{n}{2^k} < t < \frac{n+1}{2^k}, \\ 0, & otherwise \end{cases}$$

where $m = 0, 1, 2, \ldots, M$ and t is the normalized time. Here, the Laguerre wavelets are orthogonal with respect to the dilated and translated weight function $w_n(t) = e^{-(2^{k+1}t-2n-1)}$. Moreover, the Laguerre wavelets ϕ_{nm} form a wavelet basis in $L^2(\mathbf{R})$ so that a square integrable function $f(t)$, defined in $[0, 1]$ can be expanded by an infinite series of Laguerre wavelets

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{nm} \phi_{nm}(t), \tag{2}$$

where B_{nm} are Laguerre wavelet coefficients in the form $B_{nm} =< f(t), \phi_{nm}(t) >$ and $< \cdot, \cdot >$ is the inner product. If the series is truncated then,

$$f(t) \approx \sum_{k=0}^{2^k-1} \sum_{m=0}^{M} B_{nm} \phi_{nm}(t) \tag{3}$$

The convergence of the Laguerre wavelet expansion in (2) and error estimation of the truncated series (3) are analyzed by the given theorems.

Theorem 3.1. *The Laguerre wavelet series expansion* $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{nm} \phi_{nm}(t)$ *converges to* $f(t)$.

Proof. Let $L^2(\mathbf{R})$ denote the Hilbert space. Since the Laguerre wavelets form a wavelet basis in $L^2(\mathbf{R})$, any function $f(t)$ can be expanded by the series $f(t) = \sum_{j=0}^{M} B_{\rho j} \phi_{\rho j}(t)$ for a fixed ρ, where $B_{\rho j} =< f(t), \phi_{\rho j} >$.

In order to show the convergence of this series to $f(t)$, a partial sum S_n of the sequence $\{B_{\rho j} \phi_{\rho j}\}_{j=0}^{n}$, for $m < n < M$, is defined in the form $S_n = \sum_{j=1}^{n} B_{\rho j} \phi_{\rho j}(t)$. Here, the main aim is to show that S_n is a Cauchy sequence in Hilbert space.

$$< f(t), S_n >=< f(t), \sum_{j=1}^{n} B_{\rho j} \phi_{\rho j}(t) >= \sum_{j=1}^{n} \overline{B}_{\rho j} < f(t), \phi_{\rho j}(t) >= \sum_{j=1}^{n} |B_{\rho j}|^2.$$

On the other hand, it is clear that $S_n - S_m = \sum_{j=m+1}^{n} B_{\rho j} \phi_{\rho j}(t)$, for $m < n < M$. Then consider

$$\begin{aligned} ||S_n - S_m||^2 &= || \sum_{j=m+1}^{n} B_{\rho j} \phi_{\rho j}(t)||^2 =< \sum_{i=m+1}^{n} B_{\rho i} \phi_{\rho i}(t), \sum_{j=m+1}^{n} B_{\rho j} \phi_{\rho j}(t) > \\ &= \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} B_{\rho i} \overline{B}_{\rho j} < \phi_{\rho i}(t), \phi_{\rho j}(t) >= \sum_{j=m+1}^{n} |B_{\rho j}|^2. \end{aligned}$$

By Bessel's inequality, $\sum_{j=m+1}^{n}|B_{\rho j}|^2 \leq \sum_{j=0}^{n}|B_{\rho j}|^2 \leq ||f(t)||^2$ [51], this implies that $\sum_{j=m+1}^{n}|B_{\rho j}|^2$ is bounded and therefore, $||S_n - S_m||^2 = \sum_{j=m+1}^{n}|B_{\rho j}|^2$ is convergent as $m, n \to \infty$. Hence, S_n is Cauchy sequence in Hilbert space and therefore, it converges to a sum S, then

$$\begin{aligned} < S - f(t), \phi_{\rho i}(t) > &= < S, \phi_{\rho j}(t) > - < f(t), \phi_{\rho j}(t) > = < \lim_{n\to\infty} S_n, \phi_{\rho j}(t) > - B_{\rho j} \\ &= \lim_{n\to\infty} < S_n, \phi_{\rho i}(t) > - B_{\rho j} \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} B_{\rho j} < \phi_{\rho j}(t), \phi_{\rho i}(t) > - B_{\rho j} \\ &= \lim_{n\to\infty} (B_{\rho j} - B_{\rho j}) = 0. \end{aligned}$$

As a result, $< S - f(t), \phi_{\rho j}(t) >= 0$ which implies that $S = f(t)$ and therefore, the series $\sum_{j=1}^{\infty} B_{\rho j}\phi_{\rho j}(t)$ converges to $f(t)$. The justification of this theorem has been done for a fixed $n = \rho$. Similar explanations can be provided for also fixed m as $n \to \infty$. □

Lemma 3.1. *The Laguerre wavelet series expansion* (2) *of a continuous function* $f(t)$ *converges to* $f(t)$.

Proof. Assume that the infinite series of the Laguerre wavelet basis of the function $g(t)$ converges to the function $f(t)$ that is

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\phi_{nm}(t),$$

where $B_{nm} =< g(t), \phi_{nm} >$. Now $f(t)$ is multiplied by $\phi_{rs}(t)w_n(t)$ for fixed values of r and s then integration term by term gives the inner product

$$\begin{aligned} < f(t), \phi_{rs} >= \int_0^1 f(t)\phi_{rs}(t)w_n(t)dt &= \int_0^1 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\phi_{nm}(t)\phi_{rs}(t)w_n(t)dt \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\int_0^1 \phi_{nm}(t)\phi_{rs}(t)w_n(t)dt \\ &= B_{rs} =< g(t), \phi_{rs} > . \end{aligned}$$

This implies $f(t) = g(t)$, which is the desired result. □

These theorems stated above imply the convergence of the infinite series of the Laguerre wavelets $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\phi_{nm}(t)$ to a unique function $f(t)$. If the only finite terms of the approximation are considered then the error bound $|f(t) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(t)|$ can be determined by the next theorem.

Theorem 3.2. *Let* $f(t)$ *be* i*-times differentiable function on* $[0, 1]$, *then there exists a mean error bound for the approximation of Laguerre wavelets* $\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(t)$ *to* $f(t)$ *as follows*

$$||f(t) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(t)|| \leq \frac{1}{i!2^{ik}} \sup_{t\in[0,1]} |f^{(i)}(t)|$$

Proof. The justification of this theorem can be done similar to [46]. Assume that $f(t)$ is a i-times continuously differentiable function in $[0, 1]$. There exists an approximation of the Laguerre wavelets, say $f_{kM}(t)=\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}B_{nm}\phi_{nm}(t)$ to $f(t)$, the error bound for this approximation follows

$$||f(t)-f_{kM}(t)||\leq \frac{1}{i!2^{ik}}\sup_{t\in[0,1]}|f^{(i)}(t)|$$

To show this inequality, the interval $[0,1]$ is divided into subintervals $[\frac{n}{2^k},\frac{n+1}{2^k}]$ and $f(t)$ is approximated by $i-$th degree polynomial $f_{kM}(t)$ in these subintervals. Then

$$\begin{aligned}||f(t)-f_{kM}(t)||^2 &= \sum_{n=0}^{2^k-1}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} w_n(t)[f(t)-f_{kM}(t)]^2dt\\ &\leq \sum_{n=0}^{2^k-1}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} w_n(t)[f(t)-f^*(t)]^2dt\end{aligned}$$

where $f^*(t)$ is the $i-$th order interpolation of $f(t)$ on these subintervals with the maximum error bound

$$|f(t)-f^*(t)|\leq \frac{1}{i!2^{ik}}\sup_{t\in[0,1]}|f^{(i)}(t)|$$

Then

$$\begin{aligned}||f(t)-f_{kM}(t)||^2 &\leq \sum_{n=0}^{2^k-1}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} w_n(t)[\frac{1}{i!2^{ik}}\sup_{t\in[0,1]}|f^{(i)}(t)|]^2dt\\ &\leq \int_0^1 w_n(t)[\frac{1}{i!2^{ik}}\sup_{t\in[0,1]}|f^{(i)}(t)|]^2dt=||\frac{1}{i!2^{ik}}\sup_{t\in[0,1]}|f^{(i)}(t)|||^2\end{aligned}$$

Taking square roots of both sides gives the desired result. □

4. Applications of Laguerre Wavelets

In this section, the application of the method to the second order nonlinear differential equations of Bratu and Duffing type is discussed.

4.1. **Application to Bratu's Equation.** The boundary value problem (BVP) of the classical Bratu equation can be expressed as [3]

$$u''(x)+\lambda e^{u(x)}=0,\quad 0<x<1 \tag{4}$$

$$u(0)=u(1)=0. \tag{5}$$

Here λ is a constant. This problem has the exact solution $u(x)=-2\ln\Big[\dfrac{\cosh\left((\frac{x-1}{2})\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\Big]$, for $\lambda>0$, where θ is the solution of $\theta=\sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right)$.

In order to solve Bratu's problem, the solution of (4)-(5) is expressed as an expansion of the Laguerre wavelets of the form

$$u(x)=\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}B_{nm}\phi_{nm}(x) \tag{6}$$

where B_{nm} are unknown coefficients. Note that for the determination of $2^k(M+1)$ unknown coefficients, $2^k(M+1)$ algebraic equations are required and two of these equations are obtained by the boundary conditions in Eq.(5)

$$u(0) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(0) = 0 \tag{7}$$

$$u(1) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(1) = 0 \tag{8}$$

In order to obtain the other $2^k(M-1)$ equations, the differential equation (4) is expressed using (6) as follows

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi''_{nm}(x) + \lambda e^{\left(\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(x)\right)} = 0$$

and then x can be replaced by the first $2^k(M-1)$ roots, x_i, of very well-known shifted Chebyshev polynomials $T_{2^k(M+1)}$ as collocation points in the form

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi''_{nm}(x_i) + \lambda e^{\left(\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(x_i)\right)} = 0 \tag{9}$$

for $i = 1, 2, \ldots, 2^k(M-1)$. The system of algebraic equations (7), (8) and (9) with the same number of unknown coefficients $\{B_{nm}\}_{n=0,1,\ldots,2^k-1; m=0,1,\ldots,M}$ can be solved successfully by using MATLAB tools to find the solution (6) of the problem (4)-(5).

4.2. **Application to Duffing Equation.** The initial value problem (IVP) of Duffing type differential equation is given by the second order nonlinear differential equation and the initial conditions as [35]

$$u''(x) + pu'(x) + p_1u(x) + p_2u^3(x) = g(x), \tag{10}$$

$$u(0) = \alpha, \ u'(0) = \beta. \tag{11}$$

where p, p_1, p_2 and α, β are real constants and $g(x)$ is a given forcing function. Let $u(x)$ be the solution of (10)-(11), then $u(x)$ can be expanded by

$$u(x) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(x) \tag{12}$$

where B_{nm} are unknown coefficients. Substituting the expansion in Eq.(12) into the initial conditions (11) and into the differential equation (10) gives a system of nonlinear equations

$$\begin{aligned}
\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(0) &= \alpha \\
\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi'_{nm}(0) &= \beta, \\
\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi''_{nm}(x_i) &+ p\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi'_{nm}(x_i) \\
&+ p_1\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(x_i) + p_2\left[\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} B_{nm}\phi_{nm}(x_i)\right]^3 = g(x_i)
\end{aligned}$$

where x_i, $i = 1, 2, \ldots, 2^k(M-1)$ are the first $2^k(M-1)$ roots of the shifted Chebyshev polynomial $T_{2^k(M+1)}(x)$. Finally the obtained system is solved by MATLAB tools for the coefficients $\{B_{nm}\}_{n=0,1,\ldots,2^k-1;m=0,1,\ldots,M}$ to find the solution (12) of the IVP (10)-(11).

5. Numerical examples and Discussion

In this section, the proposed method is applied to different cases of Bratu and Duffing equations. Results are presented in terms of absolute errors calculated at each point $x_i \in [a, b]$, and compared with the analytical and other numerical solutions.

5.1. **Example 1: Bratu's problem.**

The solution of Bratu's problem (4)-(5) is investigated for the cases $\lambda = 1$ and $\lambda = 2$.

5.1.1. *Case 1:* Consider the Bratu's problem (4)-(5) when $\lambda = 1$

$$\begin{aligned}
&u''(x) + e^{u(x)} = 0, \quad 0 < x < 1 \\
&u(0) = u(1) = 0
\end{aligned}$$

This problem is solved by taking $M = 6$. Recall that M refers to the order of the approximation polynomial. Table 1 presents the comparison of the absolute errors obtained from the present method, Laplace transform decomposition method (LTDM) [18], decomposition method (DM) [8], perturbation iteration algorithm (PIA) [21], and variational iteration method (VIM) [20]. It can be seen that the present method has a similar accuracy with LTDM [18] and has a better accuracy than the other methods.

5.1.2. *Case 2:* As the second case, Bratu's problem is considered with $\lambda = 2$ as follows

$$\begin{aligned}
&u''(x) + 2e^{u(x)} = 0, \quad 0 < x < 1 \\
&u(0) = u(1) = 0
\end{aligned}$$

The problem is solved by taking $M = 6$. Table 2 presents the comparison of the absolute errors obtained from the present method, Laplace transform decomposition method (LTDM) [18], decomposition method (DM) [8], restarted Adomian decomposition method with Taylor series (RADM) [5], and variational iteration method (VIM) [20]. One can observe that the absolute errors are increased in all methods compared to the previous case. The accuracy of the present method is better than all of the numerical methods compared

TABLE 1. Comparison of the present method with other numerical methods

x_i	Present Method	VIM [20]	PIA $(1,1)$ [21]	LTDM [18]	DM [8]
0.1	2.09e-06	6.46e-05	1.10e-03	1.97e-06	2.68e-03
0.2	4.10e-06	1.20e-04	2.20e-03	3.93e-06	2.02e-03
0.3	6.15e-06	1.94e-04	2.80e-04	5.85e-06	1.52e-04
0.4	8.14e-06	2.64e-04	3.80e-04	7.70e-06	2.20e-03
0.5	9.90e-06	3.51e-04	4.00e-04	9.46e-06	3.01e-03
0.6	1.15e-05	4.76e-04	3.80e-04	1.11e-05	2.20e-03
0.7	1.34e-05	6.77e-04	2.80e-04	1.25e-05	1.52e-04
0.8	1.47e-05	1.01e-03	2.20e-03	1.34e-05	2.02e-03
0.9	1.25e-05	1.59e-03	1.10e-03	1.19e-05	2.68e-03

TABLE 2. Comparison of the present method with other numerical methods

x_i	Present Method	VIM [20]	RADM [5]	LTDM [18]	DM [8]
0.1	8.13e-05	3.65e-03	6.50e-05	2.12e-03	1.52e-02
0.2	1.59e-04	7.22e-03	1.30e-04	4.20e-03	1.46e-02
0.3	2.35e-04	1.39e-03	1.80e-04	6.18e-03	5.88e-03
0.4	3.05e-04	1.78e-02	2.10e-04	8.00e-03	3.24e-03
0.5	3.63e-04	2.10e-02	2.30e-04	9.59e-03	6.98e-03
0.6	4.12e-04	2.31e-02	2.10e-04	1.09e-02	3.24e-03
0.7	4.58e-04	2.36e-02	1.80e-04	1.19e-02	5.88e-03
0.8	4.83e-04	2.18e-02	1.30e-04	1.23e-02	1.46e-02
0.9	3.96e-04	1.68e-02	6.50e-05	1.08e-02	1.52e-02

in this case. Considering these two cases, it can be seen that the present method is more effective than the other methods.

5.2. **Example 2: The Damped/Undamped Duffing problem.**

The effect of the damping coefficient p is investigated in two cases.

5.2.1. *Case 1:* Let us consider the damping case of the Duffing equation, (when $p \neq 0$)

$$\begin{aligned} &u''(x) + 2u'(x) + u(x) + 8u^3(x) = e^{-3x}, \\ &u(0) = \tfrac{1}{2} \quad u'(0) = \tfrac{-1}{2} \end{aligned}$$

The exact solution of the problem is $u(x) = \frac{1}{2}e^{-x}$. The problem is solved by taking $M = 5$. The absolute errors at the points $x_i \in [0, 1]$ obtained from the present method, the Adomian decomposition method (ADM) and the restarted Adomian decomposition method (RADM) [50] are presented in Table 3. It can be seen that the difference in the absolute errors obtained from ADM and RADM are similar and furthermore the present method have a better accuracy than the other two methods. Thus, it is more effective than these methods in the solution of this case.

5.2.2. *Case 2:* Finally, let us consider the undamping case, ($p = 0$)

Table 3. Comparison of the present method with differential transform method (DTM) and restarted Adomian decomposition method (RADM)

x_i	Present Method	ADM [4]	RADM [4]
0.1	1.61e-09	4.53e-09	4.53e-09
0.2	1.08e-09	5.47e-07	5.47e-07
0.3	3.27e-09	8.85e-06	8.81e-06
0.4	5.13e-09	6.29e-05	6.24e-05
0.5	1.04e-09	2.86e-04	2.81e-04
0.6	7.99e-10	9.86e-04	9.57e-04
0.7	1.36e-08	2.80e-03	2.67e-03
0.8	5.08e-11	6.93e-03	6.48e-03
0.9	2.93e-07	1.54e-02	1.40e-02
1	1.58e-06	3.18e-02	2.80e-02

Table 4. Comparison of the present method with other numerical methods)

x_i	Present Method	DJM [34]	ITM [35]
0.1	5.07e-09	5.07e-09	4.62e-08
0.2	4.57e-09	4.47e-09	6.12e-07
0.3	1.17e-08	8.29e-09	4.28e-07
0.4	1.95e-08	1.57e-08	2.29e-07
0.5	1.15e-08	1.53e-07	4.23e-07
0.6	6.82e-09	2.32e-07	4.03e-07
0.7	4.49e-08	1.99e-06	3.32e-07
0.8	2.20e-08	1.76e-05	5.66e-07
0.9	7.06e-07	8.66e-05	8.87e-06
1	4.04e-06	3.26e-04	1.43e-05

$$u''(x) + 3u(x) - 2u^3(x) = \cos x \sin(2x),$$
$$u(0) = 0 \quad u'(0) = 1$$

The exact solution of the problem is $u(x) = \sin x$. The problem is solved by taking $M = 6$. The absolute errors obtained from the present method and other numerical methods such as Daftardar-Jafari method (DJM) [34] and improved Taylor matrix method (ITM) [35] are presented in Table 4. As one can observe that the present method has a better accuracy than the other two methods.

6. Conclusion

In this study, Laguerre wavelet method is applied to solve the Bratu and Duffing equations which are stiff ordinary differential equations. One of the advantages of the method is that it converts the problem of nonlinear differential equation to a system of algebraic equations, and hence simplifies the solution of the problems. The other advantage is that unlike many numerical methods, it does not require the approximation of the nonlinear term. Test problems show that the proposed method is simple to implement and more effective than some other numerical methods.

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