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SOME NEW APPROACHES ON PROPERTIES OF U-CROSS GRAM MATRIX

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Abstract. The operator equation $Uf = b$ can be transferred to matrix level by $Mc = d$, where M can be considered as a generalization of Gram matrix. This matrix obtained by the composition of the analysis and synthesis operators of two different sequences with inserting an operator U on a Hilbert space, called a U-cross Gram matrix. In this paper, we investigate the U-cross Gram operator $G_{U,\Phi,\Psi}$, associated to the sequences $\{\phi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ and sufficient and necessary conditions for boundedness, invertibility, compactness of this operator are determined depending on the associated sequences. We show that invertibility of $G_{U,\Phi,\Psi}$ is not possible when the associated sequences are frames but not Riesz Bases or at most one of them is a Riesz basis.

Keywords: U-cross Gram matrix, Riesz basis, Invertible operator, Hilbert space.

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1. Introduction

Frames as a generalization of orthonormal basis, are flexible tools which prepare non-independent representation of vectors in a vector space. The availability of frames has encouraged mathematicians to use them in a variety of areas throughout mathematics and engineering, such as solving equations [1, 3, 14], wireless communications [15] and image processing [5].
For some operator $U \in B(H)$ considering the operator equation $Uf = b$, the matrix M is obtained by $M_{k,\ell} = \langle Ue_k, e_\ell\rangle$, where $\{e_k\}$ is an orthonormal basis for H. The operator equations transferred to matrix level in the standard way by using orthonormal basis. But recently frames are used for this discretization [14], this means that by inserting the operator $U \in B(H)$, the matrix obtained by $M_{k,\ell} = \langle U\psi_k, \phi_\ell\rangle$, where $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_\ell\}_{\ell=1}^{\infty}$ are arbitrary frames. Therefore in this way [13], the matrix representation of operators considered as a generalization of Gram matrices. In this paper, we study the U-cross Gram operator and investigate the cases that this operator can be bounded, compact, invertible and injective or surjective. Also to illustrate the proposed concepts numerical examples are presented and the obtained results are discussed.

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Throughout this paper H is a separable Hilbert space and $B(H)$ is the set of all linear and bounded operators on H.
A countable family of elements $\{\psi_k\}_{k=1}^{\infty} \subset H$ is a frame for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{1}$$

The constants A and B are called the lower and upper frame bounds, respectively. We call $\{\psi_k\}_{k=1}^{\infty}$ a Bessel sequence with bound B, if we have only the second inequality in (1). We say that $\{\psi_k\}_{k=1}^{\infty}$ is a frame sequence for H if it is a frame for a $\overline{span}\{\psi_k\}_{k=1}^{\infty}$ [7].
Let $\{\psi_k\}_{k=1}^{\infty}$ be a sequence in H and suppose that $\sum_{k=1}^{\infty} c_k \psi_k$ is convergent for all $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$, Then

$$T : \ell^2(\mathbb{N}) \to H, \quad T(\{c_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} c_k \psi_k,$$

defines a bounded linear operator called the synthesis operator. The adjoint operator given by:

$$T^* : H \to \ell^2(\mathbb{N}); \quad T^* f = \{\langle f, \psi_k \rangle\}_{k=1}^{\infty},$$

is called the analysis operator. Composing T with its adjoint T^*, we obtain the frame operator

$$S : H \to H; \quad Sf = TT^* f = \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \psi_k.$$

If $\{\psi_k\}_{k=1}^{\infty}$ is a Bessel sequence, we can compose the synthesis operator T and its adjoint T^*; hereby we obtain the bounded operator

$$T^*T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}); \quad T^*T\{c_k\}_{k=1}^{\infty} = \left\{ \left\langle \sum_{\ell=1}^{\infty} c_\ell \psi_\ell, \psi_k \right\rangle \right\}_{k=1}^{\infty}.$$

Let $\{e_k\}_{k=1}^{\infty}$ be the canonical orthonormal basis for $\ell^2(\mathbb{N})$, the jk-th entry in the matrix representation of T^*T is as follows:

$$T^*T = \{\langle \psi_k, \psi_j \rangle\}_{j,k=1}^{\infty}.$$

The matrix $\{\langle \psi_k, \psi_j \rangle\}_{j,k=1}^{\infty}$ is called the matrix associated with $\{\psi_k\}_{k=1}^{\infty}$ or Gram matrix and it defines a bounded operator on $\ell^2(\mathbb{N})$ when $\{\psi_k\}_{k=1}^{\infty}$ is a Bessel sequence.

Lemma 1.1. [6] *Suppose that $\{\psi_k\}_{k=1}^{\infty} \subseteq H$. Then the following statements are equivalent:*
(1) $\{\psi_k\}_{k=1}^{\infty}$ is a Bessel sequence with bound B.
(2) The Gram matrix associated with $\{\psi_k\}_{k=1}^{\infty}$ defines a bounded operator on $\ell^2(\mathbb{N})$ with norm at most B.

Definition 1.1. [6] *A Riesz basis $\{\psi_k\}_{k=1}^{\infty}$ for H is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H and $U \in B(H)$ is a bounded bijective operator.*

Proposition 1.1. [6] *A sequence $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis for H if and only if it is an unconditional basis for H and*

$$0 < \inf \|\psi_k\| \leq \sup \|\psi_k\| < \infty.$$

Theorem 1.1. [6] *Suppose that $\{\psi_k\}_{k=1}^{\infty} \subseteq H$. Then the following conditions are equivalent:*
(1) $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis for H.

(2) $\{\psi_k\}_{k=1}^{\infty}$ *is complete and its Gram matrix* $\{\langle\psi_k,\psi_j\rangle\}_{j,k=1}^{\infty}$ *defines a bounded, invertible operator on* $\ell^2(\mathbb{N})$.

Definition 1.2. [8] *Suppose that* $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal basis for* H. *We say that* U *is a Hilbert-Schmidt operator for* H *if*

$$\|U\|_2 = \left(\sum_{k=1}^{\infty}\|Ue_k\|^2\right)^{\frac{1}{2}} < \infty.$$

2. U-Cross Gram matrix

In this section, we propose the concept of U-cross Gram matrix and by considering some proper conditions on its associated sequences, we obtain good properties on this matrix like boundedness, compactness, being injective and surjective. Also some practical examples are proposed to illustrate our results.

Definition 2.1. [13] *Suppose that* $\Psi = \{\psi_k\}_{k=1}^{\infty}$ *and* $\Phi = \{\phi_k\}_{k=1}^{\infty}$ *are Bessel sequences in a Hilbert space* H. *For* $U \in B(H)$, *the matrix* $G_{U,\Phi,\Psi}$ *given by*

$$(G_{U,\Phi,\Psi})_{j,k} = \langle U\psi_k, \phi_j\rangle, \quad k, j \in \mathbb{N},$$

is called the U-cross Gram matrix [4]. *If* $U = I_H$, *it is called cross Gram matrix and is denoted by* $G_{\Phi,\Psi}$ [2, 12, 9] *and if* $\Phi = \Psi$, $G_{\Psi,\Psi}$ *is called Gram matrix* [7, 10] *and is denoted by* G_{Ψ}.

If $\Psi = \{\psi_k\}_{k=1}^{\infty}$ and $\Phi = \{\phi_k\}_{k=1}^{\infty}$ are Bessel sequences in a Hilbert space H. For $U \in B(H)$, we can compose the synthesis operator of the sequence $\{\psi_k\}_{k=1}^{\infty}$, T_Ψ, and the analysis operator of the sequence $\{\phi_k\}_{k=1}^{\infty}$, T_Φ^*, to obtain a bounded operator on $\ell^2(\mathbb{N})$ given by

$$T_\Phi^* U T_\Psi : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}); \quad T_\Phi^* U T_\Psi \{c_k\}_{k=1}^{\infty} = \left\{\left\langle \sum_{\ell=1}^{\infty} c_\ell U(\psi_\ell), \phi_k \right\rangle\right\}_{k=1}^{\infty}.$$

The bounded operator, $G_{U,\Phi,\Psi} = T_\Phi^* U T_\Psi$, is called the U-cross Gram operator associated to $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$.
If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\ell^2(\mathbb{N})$, the jk-th entry in the matrix representation for $T_\Phi^* U T_\Psi$ is

$$\langle T_\Phi^* U T_\Psi e_k, e_j\rangle = \langle U T_\Psi e_k, T_\Phi e_j\rangle = \langle U\psi_k, \phi_j\rangle.$$

Therefore the matrix representation of $T_\Phi^* U T_\Psi$ is as follows:

$$T_\Phi^* U T_\Psi = \{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}.$$

Lemma 2.1. [13] *Let* $\Phi = \{\phi_k\}_{k=1}^{\infty}$ *and* $\Psi = \{\psi_k\}_{k=1}^{\infty}$ *be two Bessel sequences in* H *and* $U \in B(H)$. *Then the following assertions hold:*
(1) $G_{U,\Phi,\Psi} = T_\Phi^* U T_\Psi$. *In particular, the U-cross Gram matrix* $G_{U,\Phi,\Psi}$ *defines a bounded operator on* $\ell^2(\mathbb{N})$ *and* $\|G_{U,\Phi,\Psi}\| \leq \sqrt{B_\Phi B_\Psi}\|U\|$.
(2) $(G_{U,\Phi,\Psi})^* = G_{U^*,\Psi,\Phi}$.

The above lemma shows that if $\Phi = \{\phi_k\}_{k=1}^{\infty}$ and $\Psi = \{\psi_k\}_{k=1}^{\infty}$ are Bessel sequences and $U \in B(H)$, then the U-cross Gram matrix is a bounded operator on $\ell^2(\mathbb{N})$. But the following example shows that the inverse of the above assertion is not true.

Example 2.1. *Let* $\{e_k\}_{k=1}^{\infty}$ *be an orthonormal basis for* H. *Consider* $\{\phi_k\}_{k=1}^{\infty} = \{\frac{1}{k}e_k\}_{k=1}^{\infty}$, $\{\psi_k\}_{k=1}^{\infty} = \{ke_k\}_{k=1}^{\infty}$ *and* $U \in B(H)$; $Ux = \sum_{n=1}^{\infty} \frac{1}{n^2}\langle x, e_n\rangle e_n$. *Since*

$$\left(\sum_{k=1}^{\infty}\|Ue_k\|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{\infty}\frac{1}{k^4}\right)^{\frac{1}{2}} < \infty,$$

Therefore U *is a Hilbert-Schmidt operator and so is bounded. Also*

$$G_{U,\Phi,\Psi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{9} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{16} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{25} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is a well-defined and bounded operator but $\{\psi_k\}_{k=1}^{\infty}$ *is not a Bessel sequence.*
We can have this example when $U = I_H$. *In this case* $G_{\phi,\psi}$ *is the identity matrix.* $\{\phi_k\}_{k=1}^{\infty}$ *is a Bessel sequence, but* $\{\psi_k\}_{k=1}^{\infty}$ *is not a Bessel one.*

In Lemma 2.1 we see that if $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are two Bessel sequences, then the U-cross Gram operator is well-defined and bounded. In the following theorem we consider a strong condition on the sequence $\{\psi_k\}_{k=1}^{\infty}$ than being Bessel and obtain a strong property for the operator $G_{U,\Phi,\Psi}$ as compactness.

Theorem 2.1. *Suppose that* $\{\psi_k\}_{k=1}^{\infty}$ *and* $\{\phi_k\}_{k=1}^{\infty}$ *are sequences in* H, $\{\phi_k\}_{k=1}^{\infty}$ *is a Bessel sequence with bound* B' *and* $U \in B(H)$. *Assume that there exists* $M > 0$ *such that* $\sum_{k=1}^{\infty}\|\psi_k\|^2 \leq M$. *Then the U- cross Gram operator associated to* $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$ *is a well-defined, bounded and compact operator.*

Proof. Suppose that $G_{U,\Phi,\Psi} = \{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$. For a given sequence $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ we have

$$\begin{aligned}
\|G_{U,\Phi,\Psi}\{c_k\}_{k=1}^{\infty}\|^2 &= \sum_{j=1}^{\infty} |\sum_{k=1}^{\infty} c_k \langle U\psi_k, \phi_j\rangle|^2 \\
&\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |c_k|^2 \sum_{k=1}^{\infty} |\langle U\psi_k, \phi_j\rangle|^2 \\
&= \sum_{k=1}^{\infty} |c_k|^2 \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\langle U\psi_k, \phi_j\rangle|^2 \\
&\leq B' \sum_{k=1}^{\infty} |c_k|^2 \sum_{k=1}^{\infty} \|U\psi_k\|^2 \\
&\leq B' M \|U\|^2 \sum_{k=1}^{\infty} |c_k|^2.
\end{aligned}$$

By above assertion, $G_{U,\Phi,\Psi}\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ and therefore $G_{U,\Phi,\Psi}$ is well-defined and bounded.
Now suppose that $\{e_k\}_{k=1}^{\infty}$ is the canonical orthonormal basis for $\ell^2(\mathbb{N})$. Then

$$\begin{aligned}
\left(\sum_{k=1}^{\infty} \|G_{U,\Phi,\Psi}(e_k)\|^2\right)^{\frac{1}{2}} &= \left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\langle U\psi_k, \phi_j\rangle|^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{B'}\left(\sum_{k=1}^{\infty} \|U\psi_k\|^2\right)^{\frac{1}{2}} \leq \|U\|\sqrt{B'M}.
\end{aligned}$$

Therefore $G_{U,\Phi,\Psi}$ is a Hilbert-Schmidt operator and so is compact [11]. □

The following is an intuitive example to perceive the above theorem.

Example 2.2. *Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Consider the sequences $\{\psi_k\}_{k=1}^{\infty} = \{e_1, \frac{1}{2}e_2, \frac{1}{3}e_3, ...\}$, $\{\phi_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_3, e_4, ...\}$ and $U \in B(H)$; $Ux = \sum_{n=1}^{\infty} \frac{1}{n^2}\langle x, e_n\rangle e_n$. Suppose that $G_{U,\Phi,\Psi}$ is the U-cross Gram operator associated to $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$. A simple calculation shows that*

$$G_{U,\Phi,\Psi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2^3} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3^3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4^3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{5^3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and

$$\sum_{k=1}^{\infty} \|G_{U,\Phi,\Psi}(e_k)\|^2 = 2 + \sum_{k=2}^{\infty} \frac{1}{k^6} < \infty,$$

which shows that $G_{U,\Phi,\Psi}$ is a Hilbert-Schmidt operator and so is compact.

Proposition 2.1. *Suppose that $\{\psi_k\}_{k=1}^{\infty}$, $\{\phi_k\}_{k=1}^{\infty}$ are Bessel sequences and $U \in B(H)$. Then the following statements are satisfied:*
(i) If $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis, $\{\phi_k\}_{k=1}^{\infty}$ is a frame for H and U is an injective operator, then $G_{U,\Phi,\Psi}$ is a bounded injective operator.
(ii) If $\{\psi_k\}_{k=1}^{\infty}$ is a frame, $\{\phi_k\}_{k=1}^{\infty}$ is a Riesz basis for H and U is a surjective operator, then $G_{U,\Phi,\Psi}$ is a bounded surjective operator.

Proof. Because $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are Bessel sequences, we conclude that $G_{U,\Phi,\Psi}$ is a well-defined and bounded operator.
(i) Suppose that

$$G_{U,\Phi,\Psi}\{c_k\}_{k=1}^{\infty} = G_{U,\Phi,\Psi}\{b_k\}_{k=1}^{\infty}, \quad \{c_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}).$$

Then $T_{\Phi}^{*}UT_{\Psi}\{c_k\}_{k=1}^{\infty} = T_{\Phi}^{*}UT_{\Psi}\{b_k\}_{k=1}^{\infty}$. Since $\{\phi_k\}_{k=1}^{\infty}$ is a frame for H, T_{Φ}^{*} is an injective operator and so $UT_{\Psi_k}\{c_k\}_{k=1}^{\infty} = UT_{\Psi_k}\{b_k\}_{k=1}^{\infty}$. Because U is injective and T_{Ψ} is invertible, $\{c_k\}_{k=1}^{\infty} = \{b_k\}_{k=1}^{\infty}$ and we get the proof.
(ii) For a given sequence $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$, because T_{Φ}^{*} and U are surjective operators, there exists $k \in H$ such that $T_{\Phi}^{*}Uk = \{c_k\}_{k=1}^{\infty}$. Since $\{\psi_k\}_{k=1}^{\infty}$ is a frame, T_{Ψ} is a surjective operator and so there exists $\{b_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ such that $T_{\Phi}^{*}UT_{\psi}\{b_k\}_{k=1}^{\infty} = \{c_k\}_{k=1}^{\infty}$ and so $G_{U,\Phi,\Psi}\{b_k\}_{k=1}^{\infty} = \{c_k\}_{k=1}^{\infty}$. □

Here we propose a practical example to realize the above theorem.

Example 2.3. *Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Consider the sequences $\{\psi_k\}_{k=1}^{\infty} = \{\sqrt{2}e_1, \sqrt{2}e_2, \sqrt{2}e_3, ...\}$, $\{\phi_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, e_3, e_3, ...\}$ and $U \in B(H)$; $Ux = \sum_{n=1}^{\infty} \frac{1}{2n}\langle x, e_n\rangle e_n$.*
A simple calculation shows that $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis and $\{\phi_k\}_{k=1}^{\infty}$ is a frame for H and $G_{U,\Phi,\Psi}$ is a bounded injective operator.

$$G_{U,\Phi,\Psi} = \begin{bmatrix} \frac{\sqrt{2}}{2} & o & 0 & 0 & 0 & \cdots \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{\sqrt{2}}{4} & 0 & 0 & 0 & \cdots \\ 0 & \frac{\sqrt{2}}{4} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{\sqrt{2}}{6} & 0 & 0 & \cdots \\ 0 & 0 & \frac{\sqrt{2}}{6} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

3. Invertibility of the U-cross Gram matrix associated to its sequences

In this section, we investigate the invertibility of the operator $G_{U,\Phi,\Psi}$ associated to its sequences and by examples we show that the invertibility of $G_{U,\Phi,\Psi}$ is not possible when the associated sequences are frames but not Riesz bases or at most one of them is a Riesz basis.

Theorem 3.1. *Suppose that $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are Riesz bases and $U \in B(H)$ is an invertible operator. Then $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are complete and the U-cross Gram matrix associated to $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$ defines a bounded invertible operator on $\ell^2(\mathbb{N})$.*

Proof. Because $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are Riesz bases, there exist bijective operators V and W such that $\{\psi_k\}_{k=1}^{\infty} = \{Ve_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty} = \{We_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of H. For every $k, j \in \mathbb{N}$ we have

$$\langle U\psi_k, \phi_j\rangle = \langle UVe_k, We_j\rangle = \langle W^*UVe_k, e_j\rangle.$$

i. e., the U-cross Gram matrix associated to $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ representing the bounded invertible operator W^*UV in the basis $\{e_k\}_{k=1}^{\infty}$. □

Now, the question is what can we say about the inverse of Theorem 3.1. What is clear that by having the invertibility of the matrix $G_{U,\Phi,\Psi}$ and the completeness of $\{\phi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$, it is not necessary that these two sequences be Riesz basis. But the following theorem shows that in the case that $\{\phi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ are frames, the answer is different.

Theorem 3.2. *Suppose that $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are frames for H and $U \in B(H)$. Assume that the U- cross Gram matrix associated to $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$ is bounded and invertible. Then $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are Riesz bases for H.*

Proof. Suppose that $G_{U,\Phi,\Psi}$ is the U- cross Gram operator associated to $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$. Therefore we have

$$G_{U,\Phi,\Psi} = T_\Phi^* U T_\Psi.$$

Because $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are frames for H, T_Ψ and T_Φ are bounded and surjective operators. Now we show that T_Ψ is an injective operator. For the given sequences $\{c_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$, suppose that

$$T_\Psi\{c_k\}_{k=1}^{\infty} = T_\Psi\{b_k\}_{k=1}^{\infty}.$$

Then we have

$$T_\Phi^* U T_\Psi\{c_k\}_{k=1}^{\infty} = T_\Phi^* U T_\psi\{b_k\}_{k=1}^{\infty}.$$

So

$$G_{U,\Phi,\Psi}\{c_k\}_{k=1}^{\infty} = G_{U,\Phi,\Psi}\{b_k\}_{k=1}^{\infty}.$$

Since $G_{U,\Phi,\Psi}$ is an invertible operator, we deduce that $\{c_k\}_{k=1}^{\infty} = \{b_k\}_{k=1}^{\infty}$ and therefore T_Ψ is an injective operator and so $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis for H.

Now we show that T_Φ is also a bijective operator. Since $\{\phi_k\}_{k=1}^{\infty}$ is a frame for H, by Theorem 5.4.1 in [7], T_Φ is a surjective operator. Now it is enough to show that T_Φ is an injective operator. For this, since $N(T_\Phi) = R(T_\Phi^*)^\perp$, it is enough to show that $T_\Phi^* : H \to \ell^2(\mathbb{N})$ is a surjective operator. Since $G_{U,\Phi,\Psi}$ is invertible, for a given sequence $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ there exists a sequence $\{b_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ such that

$$G_{U,\Phi,\Psi}\{b_k\}_{k=1}^{\infty} = \{c_k\}_{k=1}^{\infty}.$$

So

$$T_\Phi^* U T_\Psi\{b_k\}_{k=1}^{\infty} = \{c_k\}_{k=1}^{\infty}.$$

So T_{Φ}^* is a surjective operator and $\{\phi_k\}_{k=1}^{\infty}$ is a Riesz basis for H. □

The following Examples show that if $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are frames but not Riesz bases, or at most one of them is a Riesz bases, the Gram matrix $G_{U,\Phi,\Psi}$ can not be invertible.

Example 3.1. *Suppose that* $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal basis for* H. *Consider* $\{\psi_k\}_{k=1}^{\infty} = \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, ...\}$, $\{\phi_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, e_3, e_4, e_5, ...\}$ *be the sequences which are frames but not Riesz bases and* $U \in B(H)$; $Ux = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ *is a bounded operator on* H. *We get the U-cross Gram matrix associated to* $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$ *as follows:*

$$G_{U,\Phi,\Psi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & \\ 1 & 0 & 0 & 0 & 0 & \cdots & \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\cdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}.$$

We obtain that $\det(G_{U,\Phi,\Psi}) = 0$ *and so* $G_{U,\Phi,\Psi}$ *is not invertible.*

Example 3.2. *Suppose that* $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal basis for* H. *Consider a Riesz basis* $\{\psi_k\}_{k=1}^{\infty} = \{\sqrt{2}e_1, e_2, e_3, e_4, e_5, ...\}$ *and* $\{\phi_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, e_3, e_4, e_5, ...\}$ *which is a frame. Also consider* $U \in B(H)$; $Ux = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ *is a bounded operator on* H. *We get the U-cross Gram matrix associated to* $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$ *as follows:*

$$G_{U,\Phi,\Psi} = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We obtain that $\det(G_{U,\Phi,\Psi}) = 0$ *and so* $G_{U,\Phi,\Psi}$ *is not invertible.*

Theorem 3.3. *Suppose that $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ are Bessel sequences for H and $U \in B(H)$ is an invertible operator. Assume that the U-cross Gram matrix $G_{U,\Phi,\Psi}$, associated to*
$\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$ is bounded . Then the following statements are satisfied:
(i) If $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis for H and the operators $G_{U,\Phi,\Psi}$ and U are invertible, then $\{\phi_k\}_{k=1}^{\infty}$ is a Riesz basis for H.
(ii) Suppose that $\{\psi_k\}_{k=1}^{\infty}$ is a frame for H and $R(T_\Phi)$ is closed. Assume that $G_{U,\Phi,\Psi}$ is an injective and U is a surjective operator, then $\{\phi_k\}_{k=1}^{\infty}$ is a frame for H.
(iii) Suppose that $\{\phi_k\}_{k=1}^{\infty}$ is a frame for H and $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis . Assume that U is an invertible and $G_{U,\Phi,\Psi}$ is a surjective operator, then $\{\phi_k\}_{k=1}^{\infty}$ is Riesz basis for H.

Proof. (i) Suppose that $G_{U,\Phi,\Psi}$ is the U-cross Gram operator associated to $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$. Therefore we have

$$G_{U,\Phi,\Psi} = T_\Phi^* U T_\Psi.$$

Since $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis and U is invertible, we have

$$T_\Phi^* = G_{U,\Phi,\Psi}(T_\Psi^{-1} U^{-1}).$$

Therefore T_Φ^* is an invertible operator and we deduce that $\{\phi_k\}_{k=1}^{\infty}$ is a Riesz basis for H.
(ii) In order to show that $\{\phi_k\}_{k=1}^{\infty}$ is a frame for H it is enough to prove that T_Φ is a surjective operator. Since $\mathrm{R}(T_\Phi)$ is closed, we need to show that T_Φ^* is injective.
Suppose that

$$T_\Phi^*(f_1) = T_\Phi^*(f_2), \quad f_1, f_2 \in H.$$

Since U is a surjective operator, there exist $g_1, g_2 \in H$ such that $U(g_1) = f_1$ and $U(g_2) = f_2$. On the other hand because T_Ψ is surjective, there exist sequences $\{c_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ such that $T_\Psi\{c_k\}_{k=1}^{\infty} = g_1$ and $T_\Psi\{b_k\}_{k=1}^{\infty} = g_2$. Therefore

$$T_\Phi^* U T_\Psi\{c_k\}_{k=1}^{\infty} = T_\Psi^* U T_\Psi\{b_k\}_{k=1}^{\infty}.$$

Now since $G_{U,\Phi,\Psi}$ is injective, we deduce that $\{c_k\}_{k=1}^{\infty} = \{b_k\}_{k=1}^{\infty}$ and we get the proof.
(iii) Since $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis and U is invertible, we have

$$T_\Phi^* = G_{U,\Phi,\Psi}(T_\Psi^{-1} U^{-1}).$$

Therefore T_Φ^* is a surjective operator and so T_Φ is injective. Because $\{\phi_k\}_{k=1}^{\infty}$ is a frame for H, T_Φ is surjective also and so $\{\phi_k\}_{k=1}^{\infty}$ is a Riesz basis for H. □

By changing the role of the sequences $\{\psi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ in above theorem we deduce the same results.

Corollary 3.1. *If $\{\psi_k\}_{k=1}^{\infty}$ is a Riesz basis and $\{\phi_k\}_{k=1}^{\infty}$ is a frame sequence and U is an invertible operator, then $G_{U,\Phi,\Psi}$ can not be invertible.*

Example 3.3. *Suppose that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H. Consider a Riesz basis $\{\psi_k\}_{k=1}^{\infty} = \{e_1, \sqrt{3}e_2, e_3, e_4, e_5, ...\}$ and a frame sequence $\{\phi_k\}_{k=1}^{\infty} = \{e_2, e_2, e_3, e_4, ...\}$. Also consider $U \in B(H)$; $Ux = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ is an invertible operator. We obtain the*

U- cross Gram matrix associated to $\{\langle U\psi_k, \phi_j\rangle\}_{j,k=1}^{\infty}$,

$$G_{U,\Phi,\Psi} = \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & \cdots & \\ 0 & \sqrt{3} & 0 & 0 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & 1 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}.$$

We deduce that $det(G_{U,\Phi,\Psi}) = 0$ and so $G_{U,\Phi,\Psi}$ is not invertible.

4. Conclusions

In this paper, we investigate the U-cross Gram operator $G_{U,\Phi,\Psi}$, associated to the sequences $\{\phi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ and sufficient and necessary conditions for boundedness, invertibility, compactness of this operator are determined depending on the associated sequences. Also by some examples we conclude that the invertibility of $G_{U,\Phi,\Psi}$ is not possible when the associated sequences are frames but not Riesz bases or at most one of them is a Riesz basis.

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