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THIRD KIND HANKEL DETERMINANT FOR MULTIVALENT BOUNDED TURNING FUNCTIONS OF ORDER ALPHA

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Abstract. The objective of this paper is to obtain an upper bound of the third order Hankel determinant for the class of multivalent bounded turning functions of order α $(0 \leq \alpha < 1)$.

Key words: Bounded turning function, upper bound, Hankel determinant, positive real function, p-valent analytic function.

AMS Subject Classification: 30C45; 30C50.

1. Introduction

Let A_p (p is a fixed integer ≥ 1) denote the class of functions f of the form

$$f(z) = z^p \sum_{n=0}^{\infty} a_{p+n} z^n \ (a_p = 1), \tag{1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in \mathbb{N} = \{1, 2, 3, ...\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture also called as coefficient conjecture, which states that for a univalent function its n^{th}- Taylor's coefficient is bounded by n (see [4]). For example, the n^{th}-coefficient gives information about the area where as the second coefficient of functions in the family S yields the growth and distortion properties of the function. A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. The Hankel determinant of f (when $p = 1$) in (1) for $q,\ n \in \mathbb{N}$ was defined by Pommerenke [18] as follows, and has been extensively studied.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{2}$$

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One can easily observe that the Fekete-Szegö functional is $H_2(1)$. In recent years, the research on Hankel determinants has focused on the estimation of $|H_2(2)|$, where

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2,$$

known as the second Hankel determinant obtained for $q = 2$ and $n = 2$ in (2). Many authors obtained an upper bound of the functional $|a_2a_4 - a_3^2|$ for various subclasses of univalent and multivalent analytic functions. It is worth citing a few of them. The exact (sharp) estimates of $|H_2(2)|$ for the subclasses of S namely, bounded turning, starlike and convex functions denoted by $\mathcal{R}$, S^* and $\mathcal{K}$ respectively in the open unit disc E, that is, functions satisfying the conditions $\mathrm{Re}\{f'(z)\} > 0$, $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ and $\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$ were proved by Janteng et al. [8, 9] and obtained the bounds as 4/9, 1 and 1/8. For the class $S^*(\psi)$ of Ma-Minda starlike functions, the exact bound of the second Hankel determinant was obtained by Lee et al. [12]. Choosing $q = 2$ and $n = p+1$ in (2), the authors obtain the second Hankel determinant for the p-valent function (see [22]), given by

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2,$$

The case $q = 3$ appears to be much more difficult than the case $q = 2$. Very few papers have been devoted to the third Hankel determinant denoted by $H_3(1)$, obtained by choosing $q = 3$ and $n = 1$ in (2). Babalola [2] is the first one, who tried to estimate an upper bound of $|H_3(1)|$ for the classes $\mathcal{R}$, S^* and $\mathcal{K}$. Following this paper, Raza and Malik [19] obtained an upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. Sudharsan et al. [20] derived an upper bound of the third kind Hankel determinant for certain subclass of analytic functions, defined as $\mathcal{C}_\alpha^\beta = \mathrm{Re}\left\{\frac{(zf'(z)+\alpha z^2 f''(z))'}{f'(z)}\right\} > \beta$, where $(0 \le \alpha \le 1)$ and $(0 \le \beta < 1)$. Bansal et al. [3] modified the upper bound of $|H_3(1)|$ for some of the classes estimated by Babalola [2] to some extent. Recently, Zaprawa [23] improved all the results obtained by Babalola [2]. Further, Orhan and Zaprawa [16] obtained an upper bound to the third kind Hankel determinant for the classes $S^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively represents starlike and convex functions of order α $(0 \le \alpha < 1)$. Recently, Kowalczyk et al. [10] calculated sharp upper bound of $|H_3(1)|$ for the class of convex functions $\mathcal{K}$ and showed as $|H_3(1)| \le \frac{4}{135}$, which is more improved one than the bound obtained by Zaprawa [23]. Arif et al. [1] estimated an upper bound of the Fourth Hankel determinant for the family of bounded turning functions. Very recently, Lecko et al. [11] determined the sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2. For our discussion in this paper, we consider $H_3(p)$ for the values $q = 3$ and $n = p$ in (2), called as Hankel determinant of third order for the p-valent function, given by

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix} (a_p = 1).$$

Expanding the determinant, we have

$$H_3(p) = [a_p(a_{p+2}a_{p+4} - a_{p+3}^2) + a_{p+1}(a_{p+2}a_{p+3} - a_{p+1}a_{p+4}) + a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2)], \tag{3}$$

equivalently

$$H_3(p) = H_2(p+2) + a_{p+1}J_{p+1} + a_{p+2}H_2(p+1), \text{ where } J_{p+1} = (a_{p+2}a_{p+3} - a_{p+1}a_{p+4}).$$

Motivated by the results obtained by different authors mentioned above and who are working in this direction (see [5], [21]), in this paper, we consider and estimate an upper bound of $|H_3(p,\alpha)|$, which is donated as the third order Hankel determinant of the function f, given in (1) belongs to the class $\mathcal{R}_p(\alpha)$, consisting of p-valent bounded turning functions of order α, defined as follows.

Definition 1.1. *A function $f(z) \in A_p$ is said to be in $\mathcal{R}_p(\alpha)$ $(0 \leq \alpha < 1)$ with $p \in \mathbb{N}$, if it satisfies the condition*

$$Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \alpha,\ z \in E. \tag{4}$$

(1) For the choice of $p = 1$, $\alpha = 0$, we obtain $\mathcal{R}_1(0) = \mathcal{R}$, this class was introduced by Alexander in 1915 and a systematic study of properties of these functions was conducted by MacGregor [15], who indeed referred to numerous earlier investigations involving functions whose derivative have a positive real part (also called as bounded turning functions).

(2) Choosing $\alpha = 0$, we have $\mathcal{R}_p(0) = \mathcal{R}_p$, consisting of multivalent (p-valent) bounded turning functions.

(3) Selecting $p = 1$, we obtain $\mathcal{R}_1(\alpha) = \mathcal{R}(\alpha)$, consisting of bounded turning functions of order α $(0 \leq \alpha < 1)$.

In proving our result, we require a few sharp estimates in the form of lemmas valid for functions with positive real part.

Let $\mathcal{P}$ denote the class of functions consisting of g, such that

$$g(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{5}$$

which are analytic in E and Re$g(z) > 0$ for $z \in E$. Here g is called the Caratheodòry function [6].

Lemma 1.1. *([7]) If $g \in \mathcal{P}$, then the sharp estimate $|c_n - \mu c_k c_{n-k}| \leq 2$, $n, k \in \mathbb{N}$, with $n > k$ and $\mu \in [0,1]$.*

Lemma 1.2. *([14]) If $g \in \mathcal{P}$, then the sharp estimate $|c_n - c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N}$, with $n > k$.*

Lemma 1.3. *([17]) If $g \in \mathcal{P}$ then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $g(z) = \frac{1+z}{1-z}$, $z \in E$.*

In order to obtain our result, we referred to the classical method devised by Libera and Zlotkiewicz [13], which has been widely used by many authors.

Theorem 1.1. *If $f \in \mathcal{R}_p(\alpha)$ $(0 \leq \alpha < 1)$ with $p \in \mathbb{N}$ then*

$$|H_3(p,\alpha)| \leq 4p^2(1-\alpha)^2\left[\frac{(6p^3 + 30p^2 + 29p + 17) - 4p^2\alpha(p+4)}{(p+1)(p+2)(p+3)^2(p+4)}\right].$$

Proof. For the function $f \in \mathcal{R}_p(\alpha)$, by virtue of Definition 1.1, there exists an analytic function $g \in \mathcal{P}$ in the open unit disc E with $g(0) = 1$ and $\mathrm{Re}g(z) > 0$ such that, we have

$$\frac{f'(z) - p\alpha z^{p-1}}{p(1-\alpha)z^{p-1}} = g(z) \Leftrightarrow \; f'(z) - p\alpha z^{p-1} = p(1-\alpha)z^{p-1}g(z). \tag{6}$$

Replacing f' and g with their series expressions in (6), upon simplification, we get

$$a_{p+n} = \frac{p(1-\alpha)c_n}{p+n}, \; n, p \in \mathbb{N}. \tag{7}$$

Substituting the values of a_{p+1}, a_{p+2} , a_{p+3} and a_{p+4} from (7) in the functional given in (3), it simplifies to

$$H_3(p,\alpha) = p^2(1-\alpha)^2\Big[\frac{c_2c_4}{(p+2)(p+4)} - \frac{p(1-\alpha)c_2^3}{(p+2)^3} - \frac{c_3^2}{(p+3)^2}$$
$$- \frac{p(1-\alpha)c_1^2c_4}{(p+1)^2(p+4)} + \frac{2p(1-\alpha)c_1c_2c_3}{(p+1)(p+2)(p+3)}\Big]. \tag{8}$$

On grouping the terms in order to apply lemmas then

$$H_3(p,\alpha) = p^2t_1^2\Big[\frac{pc_4(c_2 - t_1c_1^2)}{(p+1)^2(p+4)} - \frac{c_3}{(p+3)^2}\left\{c_3 - \frac{6pt_1}{(p+1)(p+2)}c_1c_2\right\}$$
$$+ \frac{pc_2(c_4 - c_2^2)t_1}{(p+2)^3} - \frac{2p^2t_1c_2\,(c_4 - c_1c_3)}{(p+1)(p+2)(p+3)^2} +$$
$$\frac{\left\{p^5(p+6)t_1 + p^4t_2 + 10p^3t_3 + p^2t_4 + 12pt_5 + 36\right\}c_2c_4}{(p+1)^2(p+2)^3(p+3)^2(p+4)}\Big], \tag{9}$$

where $t_1 = 1-\alpha;\; t_2 = 3-2\alpha;\; t_3 = 4\alpha - 3;\; t_4 = 73\alpha - 36;\; t_5 = 3\alpha + 2.$

Applying the triangle inequality in (9), we have

$$\Big|H_3(p,\alpha)\Big| \le p^2t_1^2\Big[\frac{p|c_4||c_2 - t_1c_1^2|}{(p+1)^2(p+4)} + \frac{|c_3|}{(p+3)^2}\left|c_3 - \frac{6pt_1}{(p+1)(p+2)}c_1c_2\right| +$$
$$\frac{p|c_2||c_4 - c_2^2|t_1}{(p+2)^3} + \frac{2p^2t_1|c_2|\,|c_4 - c_1c_3|}{(p+1)(p+2)(p+3)^2} +$$
$$\frac{\left\{(p^4 + 6p^3 + 2p^2 - 30p - 36)p^2(1-\alpha) + (10p^2 + 37p + 12)p\alpha + (p^4 + 24p + 36)\right\}|c_2||c_4|}{(p+1)^2(p+2)^3(p+3)^2(p+4)}\Big]. \tag{10}$$

Upon using the lemmas given in 1.1, 1.2 and 1.3 in (10), it reduces to

$$\left|H_3(p,\alpha)\right| \le 4p^2t_1^2\Big[\frac{p}{(p+1)^2(p+4)}+\frac{1}{(p+3)^2}+\frac{p(1-\alpha)}{(p+2)^3}+\frac{2p^2(1-\alpha)}{(p+1)(p+2)(p+3)^2}$$

$$+\frac{\{(p^4+6p^3+2p^2-30p-36)p^2(1-\alpha)+(10p^2+37p+12)p\alpha+(p^4+24p+36)\}}{(p+1)^2(p+2)^3(p+3)^2(p+4)}\Big].$$

Further simplification gives

$$\left|H_3(p,\alpha)\right| \le \; 4p^2(1-\alpha)^2\left[\frac{(6p^3+30p^2+29p+17)-4p^2\alpha(p+4)}{(p+1)(p+2)(p+3)^2(p+4)}\right]. \tag{11}$$

This completes the proof of our Theorem. □

Remark 1.1. *Choosing $p=1$ and $\alpha=0$ in the inequality (11), it coincides with the result obtained by Zaprawa [23].*

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2. ACKNOWLEDGEMENT

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