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K-G-FUSION WOVEN IN HILBERT SPACES

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Abstract. In this note, we study weaving K-g-fusion frames in separable Hilbert spaces which motivated by a generalized of fusion frames. We present necessary and sufficient conditions for these woven and also construct them by a linear bounded operator. Finally, A Paley-Wiener type perturbation result for weaving K-g-fusion frames will be investigated.

Keywords: K-g-fusion frame, weaving frame, weaving g-fusion frame.

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1. Introduction and Preliminaries

Fusion frames or frame of subspaces have been introduced by Casazza and Kutyniok in [2, 3, 4]. They have defined frames for closed subspaces of given Hilbert spaces with the help of the orthogonal projections. Găvruţa presented frames for operators (or K-frames) in [12] while studying about the atomic systems with respect to a bounded operator K which had been introduced by Fechtinger and Werther in [10] and showed that atomic systems for K are equivalent with the K-frames.

Recently, Bemrose et al. in [1] were able to introduce a new concept of frames as *weaving frames* which they have potential applications in wireless sensor networks. Two frames $\{f_j\}_{j\in\mathbb{J}}$ and $\{g_j\}_{j\in\mathbb{J}}$ for a Hilbert space H are (weakly) woven if for each subset $\sigma\in\mathbb{J}$, the family $\{f_j\}_{j\in\sigma}\cup\{g_j\}_{j\in\sigma^c}$ is a frame for H. Afterwards, this topic was presented in other frames like g-frames, fusion frames and etc [13, 11, 17]. Recently, we generalized fusion frames which we called g-fusion frames and also their woven in Hilbert spaces ([14, 15, 16]). We aim at studying woven for K-g-fusion frames.

Throughout this paper, H is a separable Hilbert space and $\mathcal{B}(H)$ is the collection of all the bounded linear operators of H into H. Also, π_V is the orthogonal projection from H onto a closed subspace $V\subset H$ and $\{H_j\}_{j\in\mathbb{J}}$ is a sequence of Hilbert spaces where $\mathbb{J}$ is a subset of $\mathbb{Z}$. The following lemmas are useful in our study on fusion frames.

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Lemma 1.1. *([12]) Let $V \subseteq H$ be a closed subspace, and U be a linear bounded operator on H. Then*

$$\pi_V U^* = \pi_V U^* \pi_{\overline{UV}}.$$

If U is a unitary (U is bijective and $U^* = U^{-1}$), then $\pi_{\overline{UV}} U = U \pi_V$.

If an operator U has closed range, then there exists a right-inverse operator $U^\dagger$ (pseudo-inverse of U) in the following senses.

Lemma 1.2. *([6]) Let $U \in \mathcal{B}(H_1, H_2)$ be a bounded operator with closed range $\mathcal{R}(U)$. Then there exists a bounded operator $U^\dagger \in \mathcal{B}(H_2, H_1)$ for which*

$$UU^\dagger x = x, \quad x \in \mathcal{R}(U).$$

Lemma 1.3. *([9]). Let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$ be on given Hilbert spaces. Then the following assertions are equivalent:*

(1) *$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$;*
(2) *$L_1 L_1^* \leq \lambda^2 L_2 L_2^*$ for some$\lambda > 0$;*
(3) *there exists a mapping $U \in \mathcal{B}(H_1, H_2)$ such that $L_1 = L_2 U$.*

Moreover, if those conditions are valid, then there exists a unique operator U such that

(a) $\|U\|^2 = \inf\{\alpha > 0 \mid L_1 L_1^* \leq \alpha L_2 L_2^*\}$;
(b) $\mathcal{N}(L_1) = \mathcal{N}(U)$;
(c) $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2^*)}$.

Now, we review the notation of K-g-fusion frames and their operators.

Definition 1.1. *Let $W = \{W_j\}_{j \in \mathbb{J}}$ be a collection of closed subspaces of H, $\{v_j\}_{j \in \mathbb{J}}$ be a family of weights, i.e. $v_j > 0$, $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in \mathbb{J}$ and $K \in \mathcal{B}(H)$. We say $\Lambda := (W_j, \Lambda_j, v_j)$ is a K-g-fusion frame for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,*

$$A\|K^* f\|^2 \leq \sum_{j \in \mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leq B\|f\|^2. \tag{1}$$

When the right hand side of (1.1) holds, Λ is called a g-fusion Bessel sequence for H with bound B. If $K = Id_H$, we get the g-fusion frame for H. We say Λ is a Parseval K-g-fusion frame whenever

$$\sum_{j \in \mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 = \|K^* f\|^2.$$

The synthesis and the analysis operators of the K-g-fusion frames are defined by (for more details, we refer to [16])

$$T_\Lambda : \mathscr{H}_2 \longrightarrow H,$$
$$T_\Lambda(\{f_j\}_{j \in \mathbb{J}}) = \sum_{j \in \mathbb{J}} v_j \pi_{W_j} \Lambda_j^* f_j,$$

and

$$T_\Lambda^* : H \longrightarrow \mathscr{H}_2,$$
$$T_\Lambda^*(f) = \{v_j \Lambda_j \pi_{W_j} f\}_{j \in \mathbb{J}},$$

where

$$\mathscr{H}_2 = \big\{\{f_j\}_{j \in \mathbb{J}} \ : \ f_j \in H_j, \ \sum_{j \in \mathbb{J}} \|f_j\|^2 < \infty\big\}. \tag{2}$$

Hence, the g-fusion frame operator is given by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j} f$$

and

$$\langle S_\Lambda f, f\rangle = \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2,$$

for all $f \in H$. Therefore,

$$\langle AKK^* f, f\rangle \leq \langle S_\Lambda f, f\rangle \leq \langle Bf, f\rangle$$

or

$$AKK^* \leq S_\Lambda \leq BId_H. \tag{3}$$

In K-g-fusion frames, if $K \in \mathcal{B}(H)$ has closed range, then S_Λ is an invertible operator on $\mathcal{R}(K)$.

2. K-G-Fusion Woven

Throughout this paper, $[m] := \{1, 2, \cdots, m\}$ for each $m > 1$, $\{W_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a collection of closed subspaces of H, $\{v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a family of weights, $K \in \mathcal{B}(H)$ and $\{\Lambda_{ij}\}_{j\in\mathbb{J}, i\in[m]} \in \mathcal{B}(H, H_{ij})$ where H_{ij} are Hilbert spaces.

Definition 2.1. *A family of g-fusion frames $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ for H is said to be K-g-fusion woven if there exist universal positive constants $0 < A \leq B$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$, the family $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\sigma_i, i\in[m]}$ is a K-g-fusion frame for H with bounds A and B.*

It is easy to check that if $\{(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}}\}$ is a g-fusion Bessel sequence for H with bound B_i for each $i \in [m]$ then, for any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$, the family $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\sigma_i, i\in[m]}$ is a g-fusion Bessel sequence with the Bessel bound $\sum_{i\in[m]} B_i$. So, every g-fusion woven has a universal upper bound. In next theorem, we provide a necessary and sufficient condition for weaving K-g-fusion frames with the same method of [11].

Theorem 2.1. *Assume that $(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ and $(V_j, \Theta_j, \nu_j)_{j\in\mathbb{J}}$ are two K-g-fusion frames for H where $\Lambda_j \in \mathcal{B}(H, H_j)$ and $\Theta_j \in \mathcal{B}(H, \mathcal{H}_j)$ for any $j \in \mathbb{J}$. The following assertions are equivalent.*

(I) *$(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ and $(V_j, \Theta_j, \nu_j)_{j\in\mathbb{J}}$ are K-g-fusion woven.*

(II) *There exists $\alpha > 0$ such that for each $\sigma \subset \mathbb{J}$ there exists a bounded linear operator*

$$\Psi_\sigma : \mathscr{H}_2^\sigma \longrightarrow H,$$
$$\Psi_\sigma\{x_j\}_{j\in\mathbb{J}} = \sum_{j\in\sigma} v_j \pi_{W_j}\Lambda_j^* x_j + \sum_{j\in\sigma^c} \nu_j \pi_{V_j}\Theta_j^* x_j,$$

such that $\alpha KK^* \leq \Psi_\sigma \Psi_\sigma^*$, where

$$\mathscr{H}_2^\sigma = \Big\{\{x_j\}_{j\in\mathbb{J}} = \{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c} \ : \ f_j \in H_j,\ g_j \in \mathcal{H}_j,\ \sum_{j\in\mathbb{J}} \|x_j\|^2 < \infty\Big\}.$$

Proof. *(I) $\Rightarrow$ (II)*: Suppose that A is an universal lower frame bound for $(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ and $(V_j, \Theta_j, \nu_j)_{j\in\mathbb{J}}$. Choose $\alpha := A$ and $\Psi_\sigma := T_\sigma$ for every $\sigma \subset \mathbb{J}$, where T_σ is the synthesis operator of $(W_j, \Lambda_j, v_j)_{j\in\sigma} \cup (V_j, \Theta_j, \nu_j)_{j\in\sigma^c}$. Then, for any $\{x_j\}_{j\in\mathbb{J}} \in \mathscr{H}_2^\sigma$ we have,

$$\begin{aligned}\Psi_\sigma\{x_j\}_{j\in\mathbb{J}} &= T_\sigma\{x_j\}_{j\in\mathbb{J}} \\ &= \sum_{j\in\sigma} v_j \pi_{W_j} \Lambda_j^* x_j + \sum_{j\in\sigma^c} \nu_j \pi_{V_j} \Theta_j^* x_j,\end{aligned}$$

and also, for each $f \in H$,

$$A\|K^* f\|^2 \le \|T_\sigma^* f\|^2 = \|\Psi_\sigma^* f\|^2.$$

Thus, $\alpha K K^* \le \Psi_\sigma \Psi_\sigma^*$.

$(II) \Rightarrow (I)$: Let $\sigma \subset \mathbb{J}$ and $f \in H$, so it is easy to check that

$$\Psi_\sigma^*\{x_j\}_{j\in\mathbb{J}} = \{v_j \Lambda_j \pi_{W_j} f\}_{j\in\sigma} \cup \{\nu_j \Theta_j \pi_{V_j} f\}_{j\in\sigma^c}.$$

Therefore,

$$\begin{aligned}\alpha\|K^* f\|^2 &= \langle \alpha K K^* f, f\rangle \\ &\le \langle \Psi_\sigma^* \Psi_\sigma f, f\rangle \\ &= \|\Psi_\sigma f\|^2 \\ &= \sum_{j\in\sigma} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 + \sum_{j\in\sigma^c} \nu_j^2 \|\Theta_j \pi_{V_j} f\|^2.\end{aligned}$$

This gives that α is an universal lower frame bound of $(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ and $(V_j, \Theta_j, \nu_j)_{j\in\mathbb{J}}$. □

Example 2.1. *Let $H = \{(f_1, f_2, f_3) : f_1, f_2, f_3 \ge 0\} \subset \mathbb{R}^3$ with the standard orthonormal basis $\{e_1, e_2, e_3\}$ and $\mathbb{J} = \{1, 2, 3\}$. We define*

$$\begin{aligned}&W_1 = \operatorname{span}\{e_1\}, \quad W_2 = \operatorname{span}\{e_1, e_2\}, \quad W_3 = \operatorname{span}\{e_1, e_3\}, \\ &V_1 = \operatorname{span}\{e_2, e_1\}, \quad V_2 = \operatorname{span}\{e_2\}, \quad V_3 = \operatorname{span}\{e_2, e_3\},\end{aligned}$$

and $\Lambda_j, \Theta_j \in \mathcal{B}(H, \mathbb{C})$ for any $j \in \mathbb{J}$ so that

$$\begin{aligned}&\Lambda_1 f = \langle e_1, f\rangle, \quad \Lambda_2 f = \langle e_1 + e_2, f\rangle, \quad \Lambda_3 f = \langle e_1 + e_3, f\rangle, \\ &\Theta_1 f = \langle e_1 + e_2, f\rangle, \quad \Theta_2 f = \langle e_2, f\rangle, \quad \Theta_3 f = \langle e_2 + e_3, f\rangle,\end{aligned}$$

where $f = (f_1, f_2, f_3)$. Also, we define

$$Ke_1 = e_1 + e_2, \quad Ke_2 = e_3, \quad Ke_3 = 0.$$

Therefore, $K^* f = (f_1 + f_2, 0, f_3)$ and it is clear that $(W_j, \Lambda_j, 1)_{j\in\mathbb{J}}$ and $(V_j, \Theta_j, 1)_{j\in\mathbb{J}}$ are K-g-fusion frames with bounds 1 and 5. Now, if $\alpha := 1$ and $\Psi_\sigma := T_\sigma$, then by Theorem 2.1 it is obvious that $(W_j, \Lambda_j, 1)_{j\in\mathbb{J}}$ and $(V_j, \Theta_j, 1)_{j\in\mathbb{J}}$ are K-g-fusion woven.

In next results, we construct a K-g-fusion woven by using a bounded linear operator.

Theorem 2.2. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ be a K-g-fusion woven for H with common frame bounds A, B and assume that $U \in \mathcal{B}(H)$ has closed range so that $\mathcal{R}(K^*) \subseteq \mathcal{R}(U)$ and $KU = UK$. Then $(UW_{ij}, \Lambda_{ij}\pi_{W_{ij}} U^*, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ is also K-g-fusion woven for $\mathcal{R}(U)$ with frame bounds $A\|U^\dagger\|^{-2}$ and $B\|U\|^2$.*

Proof. By the open mapping theorem, UW_{ij} is closed for any $j \in \mathbb{J}$ and $i \in [m]$. Using Lemma 1.1, we can write for each $f \in \mathcal{R}(U)$,

$$\begin{aligned} A\|K^*f\|^2 &= A\|(U^\dagger)^*U^*K^*f\|^2 \\ &\leq A\|U^\dagger\|^2\|K^*U^*f\|^2 \\ &\leq \|U^\dagger\|^2 \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^*f\|^2 \\ &= \|U^\dagger\|^2 \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^*\pi_{UW_{ij}}f\|^2. \end{aligned}$$

The upper bound is obvious. □

Theorem 2.3. *Let K have closed range, $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J},i\in[m]}$ be a K-g-fusion woven for H with the universal bounds A, B and $U \in \mathcal{B}(H)$ so that $\mathcal{R}(U^*) \subseteq \mathcal{R}(K)$. Then $(\overline{UW_{ij}}, \Lambda_{ij}\pi_{W_{ij}}U^*, v_{ij})_{j\in\mathbb{J},i\in[m]}$ is a K-g-fusion woven for H if and only if there exists a $\delta > 0$ such that for every $f \in H$,*

$$\|U^*f\| \geq \delta\|K^*f\|.$$

Proof. Let $f \in K$ and $(\overline{UW_{ij}}, \Lambda_{ij}\pi_{W_{ij}}U^*, v_{ij})_{j\in\mathbb{J},i\in[m]}$ be a K-g-fusion woven for K with the lower bound C and $U \in \mathcal{B}(H)$ such that $\mathcal{R}(U^*) \subseteq \mathcal{R}(K)$. Thus, by Lemma 1.1, we get

$$\begin{aligned} C\|K^*f\|^2 &\leq \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^*\pi_{\overline{UW_{ij}}}f\|^2 \\ &= \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^*f\|^2 \\ &\leq B\|U^*f\|^2. \end{aligned}$$

Therefore, $\|U^*f\| \geq \sqrt{\frac{C}{B}}\|K^*f\|$. For the opposite implication, we can write for all $f \in H$,

$$\|U^*f\| = \|(K^\dagger)^*K^*U^*f\| \leq \|K^\dagger\|.\|K^*U^*f\|.$$

Hence, we have

$$\begin{aligned} A\delta^2\|K^\dagger\|^{-2}\|K^*f\|^2 &\leq A\|K^\dagger\|^{-2}\|U^*f\|^2 \\ &\leq A\|K^*U^*f\|^2 \\ &\leq \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^*f\|^2 \\ &= \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^*\pi_{\overline{UW_{ij}}}f\|^2 \\ &\leq B\|U\|^2\|f\|^2. \end{aligned}$$

So, $(\overline{UW_{ij}}, \Lambda_{ij}\pi_{W_{ij}}U^*, v_{ij})_{j\in\mathbb{J},i\in[m]}$ is a g-fusion woven for H with frame bounds $A\delta^2\|K^\dagger\|^{-2}$ and $B\|U\|^2$. □

Theorem 2.4. *Let K have closed range, $(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ be a K-g-fusion frame for H with bounds A, B and $U \in \mathcal{B}(H)$ be a unitary operator. If $\|Id_H - U\|^2\|K^\dagger\|^2 < \dfrac{A}{B}$, then $(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ and $(U^{-1}W_j, \Lambda_jU, v_j)_{j\in\mathbb{J}}$ are K-g-fusion woven for $\mathcal{R}(K)$.*

Proof. The upper bound is clear. Let $\sigma \subset \mathbb{J}$ be a partition and $f \in R(K)$. So, by Lemma 1.1 and this fact $\|f\|^2 \leq \|K^\dagger\|^2 \|K^* f\|^2$, we can write

$$\begin{aligned}
&\sum_{j\in\sigma} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 + \sum_{j\in\sigma^c} v_j^2 \|\Lambda_j U \pi_{U^{-1}W_j} f\|^2 \\
&\qquad = \sum_{j\in\sigma} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 + \sum_{j\in\sigma^c} v_j^2 \|\Lambda_j \pi_{W_j} f - (\Lambda_j \pi_{W_j} f + \Lambda_j \pi_{W_j} U f)\|^2 \\
&\qquad \geq \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 - \sum_{j\in\sigma^c} v_j^2 \|\Lambda_j \pi_{W_j} (Id_H - U) f\|^2 \\
&\qquad \geq A\|K^* f\|^2 - B\|Id_H - U\|^2 \|f\|^2 \\
&\qquad \geq A\|K^* f\|^2 - B\|Id_H - U\|^2 \|K^\dagger\|^2 \|K^* f\|^2 \\
&\qquad = \big(A - B\|Id_H - U\|^2 \|K^\dagger\|^2\big) \|K^* f\|^2.
\end{aligned}$$

Thus, $(W_j, \Lambda_j, v_j)_{j\in\sigma} \cup (U^{-1}W_j, \Lambda_j U, v_j)_{j\in\sigma^c}$ is a K-g-fusion frame. □

Proposition 2.1. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ be a K-g-fusion woven for H with common frame bounds A and B. Suppose that $0 \leq C \leq |\omega_j^{(i)}|^2 \leq D < \infty$ for any $i \in [m]$ and $j \in \mathbb{J}$, then $(W_{ij}, \omega_j^{(i)} \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ is a K-g-fusion woven for H with frame bounds AC and BD.*

Proof. For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$ and $f \in H$, we get

$$\begin{aligned}
AC\|K^* f\|^2 = \min_{i\in[m]} |\omega_j^{(i)}|^2 A\|K^* f\|^2 &\leq \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \|\omega_j^{(i)} \Lambda_{ij} \pi_{W_{ij}} f\|^2 \\
&\leq \max_{i\in[m]} |\omega_j^{(i)}|^2 B\|f\|^2 = BD\|f\|^2.
\end{aligned}$$

□

Proposition 2.2. *Let $\mathbb{I} \subset \mathbb{J}$ be arbitrary and $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{I}, i\in[m]}$ be a K-g-fusion woven for H. Then $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ is a K-g-fusion woven.*

Proof. Assume that $\sigma_i \subset \mathbb{J}$, so $\sigma_i \cap \mathbb{I} \subset \mathbb{I}$ and A is the lower bound of $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\sigma_i\cap\mathbb{I}, i\in[m]}$, then for every $f \in H$ we have

$$A\|K^* f\|^2 \leq \sum_{i\in[m]} \sum_{j\in\sigma_i\cap\mathbb{I}} v_{ij}^2 \|\Lambda_{ij} \pi_{W_{ij}} f\|^2 \leq \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \|\Lambda_{ij} \pi_{W_{ij}} f\|^2.$$

This implies the statement. □

Next theorem is shows that even if one subspace is deleted, it dose not still remain a K-g-fusion woven.

Theorem 2.5. *Let K has closed range, $\mathbb{I} \subset \mathbb{J}$ and $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ be a K-g-fusion woven for H with the bounds A, B. If*

$$C := \sum_{i\in[m]} \sum_{j\in\mathbb{I}} v_{ij}^2 \|\Lambda_{ij}\|^2 < A\|K^\dagger\|^2,$$

then $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}\setminus\mathbb{I}, i\in[m]}$ is a K-g-fusion woven for $\mathcal{R}(K)$ with frame bounds $A - C$ and B.

Proof. The upper bound is obvious. Suppose that $\{\sigma_i\}_{i\in[m]} \subset \mathbb{J}\setminus\mathbb{I}$ and $f\in\mathcal{R}(K)$, so we get

$$\begin{aligned}\sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 &= \sum_{i\in[m]}\sum_{j\in\sigma_i\cup\mathbb{I}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 - \sum_{i\in[m]}\sum_{j\in\mathbb{I}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2\\ &\geq A\|K^*f\|^2 - \sum_{i\in[m]}\sum_{j\in\mathbb{I}} v_{ij}^2\|\Lambda_{ij}\|^2\|f\|^2\\ &\geq (A - C\|K^\dagger\|^2)\|K^*f\|^2.\end{aligned}$$

□

Corollary 2.1. *Let K have closed range operator such that $\|K\|^2\leq\|K^\dagger\|^2$ and $(W_{ij}\Lambda_{ij}, v_{ij})_{j\in\mathbb{J},i\in[m]}$ be a tight K-g-fusion woven for H with the bound A. Assume that $j_0\in\mathbb{J}$. Then the following conditions are equivalent.*
(I) $\sum_{i\in[m]} v_{ij_0}^2\|\Lambda_{ij_0}\pi_{W_{ij_0}}\|^2 < A\|K^\dagger\|^2$;
(II) $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J}\setminus\{j_0\},i\in[m]}$ is a K-g-fusion woven for $\mathcal{R}(K)$.

Proof. $(I)\Rightarrow(II)$ is clear by Theorem 2.5. For the opposite implication, suppose that C,D are the frame bounds of $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J}\setminus\{j_0\},i\in[m]}$. For any $0\neq f\in H$ we have

$$\begin{aligned}C\|K^*f\|^2 &\leq \sum_{i\in[m]}\sum_{j\in\mathbb{J}\setminus\{j_0\}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2\\ &= \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 - \sum_{i\in[m]} v_{ij_0}^2\|\Lambda_{ij_0}\pi_{W_{ij_0}}f\|^2\\ &= A\|K^*f\|^2 - \sum_{i\in[m]} v_{ij_0}^2\|\Lambda_{ij_0}\pi_{W_{ij_0}}f\|^2.\end{aligned}$$

Hence,

$$0 < C \leq A - \sum_{i\in[m]} v_{ij_0}^2\frac{\|\Lambda_{ij_0}\pi_{W_{ij_0}}f\|^2}{\|K^*f\|^2} \leq A - \|K\|^{-2}\sum_{i\in[m]} v_{ij_0}^2\frac{\|\Lambda_{ij_0}\pi_{W_{ij_0}}f\|^2}{\|f\|^2}.$$

So, we conclude that $\sum_{i\in[m]} v_{ij_0}^2\|\Lambda_{ij_0}\pi_{W_{ij_0}}\|^2 < A\|K\|^2$. □

Theorem 2.6. *Let $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J},i\in[m]}$ be a K-g-fusion woven for H with the bounds A,B. For each $i\in[m]$, $j\in\mathbb{J}$ and a index set $\mathbb{I}_{ij}$, Suppose that $\{f_{ij}^{(k)}\}_{k\in\mathbb{I}_{ij}}\in\Lambda_{ij}(W_{ij})$ is a Parseval frame for H_{ij} such that for every finite subset $\mathbb{K}_{ij}\subset\mathbb{I}_{ij}$, the set $\{f_{ij}^{(k)}\}_{k\in\mathbb{I}_{ij}\setminus\mathbb{K}_{ij}}$ is a frame with the lower bound C_{ij}. Let $\widetilde{W}_{ij} := \overline{span}\{\Lambda_{ij}^* f_{ij}^{(k)}\}_{k\in\mathbb{I}_{ij}\setminus\mathbb{K}_{ij}}$ for any $i\in[m]$ and $j\in\mathbb{J}$, then $(\widetilde{W}_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J},i\in[m]}$ is a K-g-fusion woven for H with the bounds $\big(\min_{i\in[m], j\in\mathbb{J}} C_{ij}\big)A$ and B.*

Proof. Obviously, B is the upper bound of $(\widetilde{W}_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$. Assume that $f \in H$ and $\{\sigma_i\}_{i\in[m]} \in \mathbb{J}$, so

$$\begin{aligned}
\sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2 \|\Lambda_{ij}\pi_{\widetilde{W}_{ij}} f\|^2 &= \sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2 \sum_{k\in\mathbb{I}_{ij}} |\langle \Lambda_{ij}\pi_{\widetilde{W}_{ij}} f, f_{ij}^{(k)}\rangle|^2 \\
&\geq \sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2 \sum_{k\in\mathbb{I}_{ij}\setminus\mathbb{K}_{ij}} |\langle \Lambda_{ij}\pi_{\widetilde{W}_{ij}} f, f_{ij}^{(k)}\rangle|^2 \\
&= \sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2 \sum_{k\in\mathbb{I}_{ij}\setminus\mathbb{L}_{ij}} |\langle \Lambda_{ij}\pi_{W_{ij}} f, f_{ij}^{(k)}\rangle|^2 \\
&\geq \sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2 C_{ij} \|\Lambda_{ij}\pi_{W_{ij}} f\|^2 \\
&\geq \big(\min_{\substack{i\in[m]\\ j\in\mathbb{J}}} C_{ij}\big) \sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f\|^2 \\
&\geq \big(\min_{\substack{i\in[m]\\ j\in\mathbb{J}}} C_{ij}\big) A\|K^* f\|^2.
\end{aligned}$$

□

Theorem 2.7. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}}$ is a K-g-fusion frame for H for each $i \in [m]$. Suppose that for a partition collection of disjoint finite sets $\{\tau_i\}_{i\in[m]}$ of $\mathbb{J}$ and for any $\varepsilon > 0$ there exists a partition $\{\sigma_i\}_{i\in[m]}$ of the set $\mathbb{J} \setminus \bigcup_{i\in[m]} \tau_i$ such that $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in(\sigma_i\cup\tau_i), i\in[m]}$ has a lower K-g-fusion frame bound less than ε. Then $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ is not a woven.*

Proof. We can write $\mathbb{J} = \cup_{j\in\mathbb{N}}\mathbb{J}_j$, where $\mathbb{J}_j$ are disjoint index sets. Assume that $\tau_{1j} = \emptyset$ for all $i \in [m]$ and $\varepsilon = 1$. Then, there exists a partition $\{\sigma_{i1}\}_{i\in[m]}$ of $\mathbb{J}$ such that $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in(\sigma_{i1}\cup\tau_{i1}), i\in[m]}$ has a lower bound (also, optimal lower bound) less than 1. Thus, there is a $f_1 \in H$ such that

$$\sum_{i\in[m]}\sum_{j\in(\sigma_{i1}\cup\tau_{i1})} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f_1\|^2 < \|K^* f_1\|^2.$$

Since

$$\sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f_1\|^2 < \infty,$$

so, there is a $k_1 \in \mathbb{N}$ such that

$$\sum_{i\in[m]}\sum_{j\in\mathbb{K}_1} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f_1\|^2 < \|K^* f_1\|^2,$$

where, $\mathbb{K}_1 = \cup_{i\geq k_1+1}\mathbb{J}_j$.

Continuing this way, for $\varepsilon = \frac{1}{n}$ and a partition $\{\tau_{nj}\}_{i\in[m]}$ of $\mathbb{J}_1 \cup \cdots \cup \mathbb{J}_{k_n-1}$ such that

$$\tau_{ni} = \tau_{(n-1)i} \cup \big(\sigma_{(n-1)i} \cap (\mathbb{J}_1 \cup \cdots \cup \mathbb{J}_{k_n-1})\big)$$

for all $i \in [m]$, there exists a partition $\{\sigma_{ni}\}_{i\in[m]}$ of $\mathbb{J} \setminus (\mathbb{J}_1 \cup \cdots \cup \mathbb{J}_{k_n-1})$ such that $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in(\sigma_{ni}\cup\tau_{ni}), i\in[m]}$ has a lower bound less than $\frac{1}{n}$. Therefore, there is a $f_n \in H$ and $k_n \in \mathbb{N}$ such that $k_n > k_{n-1}$ and

$$\sum_{i\in[m]}\sum_{j\in\mathbb{K}_n} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f_n\|^2 < \frac{1}{n}\|K^* f_1\|^2,$$

where, $\mathbb{K}_n = \cup_{i\geq k_n+1}\mathbb{J}_j$. Choose a partition $\{\varsigma_i\}_{i\in[m]}$ of $\mathbb{J}$, where $\varsigma_i := \cup_{j\in\mathbb{N}}\{\tau_{ji}\} = \tau_{(n+1)i}\cup(\varsigma_i\cap\mathbb{J}\setminus(\mathbb{J}_1\cup\cdots\cup\mathbb{J}_n))$. Assume that $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\varsigma_i,i\in[m]}$ is a K-g-fusion frame for H with the optimal lower bound A. Then, by the Archimedean Property, there exists a $r\in\mathbb{N}$ such that $r>\frac{2}{A}$. Now, there exists a $f_r\in H$ such that

$$\begin{aligned}\sum_{i\in[m]}\sum_{j\in\varsigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f_r\|^2 &= \sum_{i\in[m]}\sum_{j\in\tau_{(r+1)i}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f_r\|^2+\\ &\quad+\sum_{i\in[m]}\sum_{j\in\varsigma_i\cap\mathbb{J}\setminus(\mathbb{J}_1\cup\cdots\cup\mathbb{J}_r)} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f_r\|^2\\ &\leq \sum_{i\in[m]}\sum_{j\in(\tau_{ri}\cup\sigma_{ri})} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f_r\|^2+\\ &\quad+\sum_{i\in[m]}\sum_{j\in\cup_{k\geq r+1}\mathbb{J}_k} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f_r\|^2\\ &< \frac{1}{r}\|K^*f_r\|^2+\frac{1}{r}\|K^*f_r\|^2\\ &< A\|K^*f_r\|^2,\end{aligned}$$

and this is a contradiction with the lower bound of A. □

Corollary 2.2. *Let $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J},i\in[m]}$ be a K-g-fusion woven for H. Then there exists a collection of disjoint finite subsets $\{\tau_i\}_{i\in[m]}$ of $\mathbb{J}$ and $A>0$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of the set $\mathbb{J}\setminus\bigcup_{i\in[m]}\tau_i$, some the family $(W_{ij},\Lambda_{ij},v_{ij})_{j\in(\sigma_i\cup\tau_i),i\in[m]}$ is a K-g-fusion frame for H with the lower frame bound A.*

Theorem 2.8. *Let $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J}}$ be a K-g-fusion frame for H with bounds A_i and B_i for each $i\in[m]$. Suppose that there exists $N>0$ such that for all $i,k\in[m]$ with $i\neq k$, $\mathbb{I}\subset\mathbb{J}$ and $f\in H$,*

$$\sum_{j\in\mathbb{I}}\|(v_{ij}\Lambda_{ij}\pi_{W_{ij}}-v_{kj}\Lambda_{kj}\pi_{W_{kj}})f\|^2\leq N\min\Big\{\sum_{j\in\mathbb{I}}v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2,\sum_{j\in\mathbb{I}}v_{kj}^2\|\Lambda_{kj}\pi_{W_{kj}}f\|^2\Big\}.$$

Then the family $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J},i\in[m]}$ is woven with universal bounds

$$\frac{A}{(m-1)(N+1)+1}\quad and\quad B,$$

where $A:=\sum_{i\in[m]}A_i$ and $B:=\sum_{i\in[m]}B_i$.

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be a partition of $\mathbb{J}$ and $f \in H$. Therefore,

$$\begin{aligned}
\sum_{i\in[m]} A_i\|K^*f\|^2 &\le \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 \\
&= \sum_{i\in[m]}\sum_{k\in[m]}\sum_{j\in\sigma_k} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 \\
&\le \sum_{i\in[m]}\Big(\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 \\
&\quad + \sum_{\substack{k\in[m]\\k\neq i}}\sum_{j\in\sigma_k}\Big\{\|v_{ij}\Lambda_{ij}\pi_{W_{ij}}f - v_{kj}\Lambda_{kj}\pi_{W_{kj}}f\|^2 + v_{kj}^2\|\Lambda_{kj}\pi_{W_{kj}}f\|^2\Big\}\Big) \\
&\le \sum_{i\in[m]}\Big(\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 + \sum_{\substack{k\in[m]\\k\neq i}}\sum_{j\in\sigma_k}(N+1)v_{kj}^2\|\Lambda_{kj}\pi_{W_{kj}}f\|^2\Big) \\
&= \{(m-1)(N+1)+1\}\sum_{i\in[m]}\Big(\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2\Big).
\end{aligned}$$

Thus, we get

$$\frac{A}{(m-1)(N+1)+1}\|K^*f\|^2 \le \sum_{i\in[m]}\Big(\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2\Big) \le B\|f\|^2.$$

□

Bemrose et al. in [1] proved sufficient conditions for weaving frames by means of perturbation and diagonal dominance. Deepshikha and Vashisht in [7, 8] were able to present some results of perturbation on K-woven. We study a-Paley-Wiener type perturbation for weaving K-g-fusion frames.

Theorem 2.9. *Let $(W_j, \Lambda_j, w_j)_{j\in\mathbb{J}}$ and $(V_j, \Theta_j, v_j)_{j\in\mathbb{J}}$ be two K-g-fusion frames for H with frame bounds A_1, B_1 and A_2, B_2, respectively. Suppose that there exist non-negative scalers μ and $0 \le \lambda < \frac{1}{2}$ such that $(\frac{1}{2}-\lambda)A_1 > \mu$ and for each $f \in H$,*

$$\sum_{j\in\mathbb{J}}\big\|(w_j\Lambda_j\pi_{W_j} - v_j\Theta_j\pi_{V_j})f\big\|^2 \le \lambda\sum_{j\in\mathbb{J}}\big\|w_j\Lambda_j\pi_{W_j}f\big\|^2 + \mu\|K^*f\|^2.$$

Then, $(W_j, \Lambda_j, w_j)_{j\in\mathbb{J}}$ and $(V_j, \Theta_j, v_j)_{j\in\mathbb{J}}$ are K-g-fusion woven for H with universal frame bounds $(\frac{1}{2}-\lambda)A_1 - \mu$ and $B_1 + B_2$.

Proof. The upper frame bound is clear. For the lower frame bound, assume that $\sigma \subset \mathbb{J}$ and we get ,by the arithmetic-quadratic mean, for any $f \in H$

$$\begin{aligned}
&\sum_{j\in\sigma} w_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \sum_{j\in\sigma^c} v_j^2\|\Theta_j\pi_{V_j}f\|^2\\
&= \sum_{j\in\sigma} w_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \sum_{j\in\sigma^c} \|w_j\Lambda_j\pi_{W_j}f - (w_j\Lambda_j\pi_{W_j} - v_j\Theta_j\pi_{V_j})f\|^2\\
&\geq \sum_{j\in\sigma} w_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \frac{1}{2}\sum_{j\in\sigma^c} w_j^2\|\Lambda_j\pi_{W_j}f\|^2 - \sum_{j\in\sigma^c} \|(w_j\Lambda_j\pi_{W_j} - v_j\Theta_j\pi_{V_j})f\|^2\\
&= \frac{1}{2}\sum_{j\in\mathbb{J}} w_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \frac{1}{2}\sum_{j\in\sigma} w_j^2\|\Lambda_j\pi_{W_j}f\|^2 - \sum_{j\in\sigma^c} \|(w_j\Lambda_j\pi_{W_j} - v_j\Theta_j\pi_{V_j})f\|^2\\
&\geq \frac{1}{2}\sum_{j\in\mathbb{J}} w_j^2\|\Lambda_j\pi_{W_j}f\|^2 - \sum_{j\in\sigma^c} \|(w_j\Lambda_j\pi_{W_j} - v_j\Theta_j\pi_{V_j})f\|^2\\
&\geq \frac{1}{2}\sum_{j\in\mathbb{J}} w_j^2\|\Lambda_j\pi_{W_j}f\|^2 - \lambda\sum_{j\in\mathbb{J}} \|w_j\Lambda_j\pi_{W_j}f\|^2 - \mu\|K^*f\|^2\\
&\geq \Big((\frac{1}{2} - \lambda)A_1 - \mu\Big)\|K^*f\|^2.
\end{aligned}$$

This completes the proof. □

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