TWMS J. App. Eng. Math. V.11, N.4, 2021, pp. 975-987

ANALYTIC INVESTIGATION OF STEADY THIN FILM FLOW OF NON-NEWTONIAN FLUID ON VERTICAL CYLINDER FOR LIFTING AND DRAINAGE PROBLEMS

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Abstract. The work addressed in this paper is the analytic investigation of the steady thin film flow of non-Newtonian Johnson-Segalman fluid on vertical cylinder for lifting and drainage problems. The arised equation in the mathematical modelling is a highly nonlinear first order differential equation. We use the analytical methods, homotopy analysis method and a modification of differential transform method based on Adomian polynomials. Effects of important physical quantities on velocity field are demonstrated graphically with comprehensive discussions.

Keywords: Numeric approximation, Eigenvalue problem, Dispersion curve.

AMS Subject Classification: 65f15, 65L12, 65L16.

1. Introduction

Thin-film type evolution equation arises in many problems in industrial and natural settings. In Geology thin-film evolution models are employed to explain the movement of lava flows and gravity currents under water [1]. In Biophysics, the thin-film dynamics appear as membranes, as tear films in the eye [2], as description of the motion of the viscouse in a Hele-shaw cell [3] or as linings of mammalian lungs [6]. In Engineering, thin-films help in the heat mass transfer processes, they limit fluxes and they protect surfaces. Recently, the importance of non-Newtonian fluids has become prominent with developments in industries like polymer, petroleum, pulp etc. Many industrial materials fall into this category, such as solutions, melts of polymers, soap, bio-logical solutions, paints, tars, asphalts and glues. Due to complex nature of non-Newtonian fluids it is hard to establish one mathematical model that can describe characteristics of all non-Newtonian fluids. So many mathematical models are used to discuss flow of non-Newtonain flows, third grade fluids lie in one of such class and are papular among researchers due to their simpler mathematical simulation. Denson [13], Waters and Keeley [14], Tasawar Hayat

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§ Manuscript received: September 6, 2019; accepted: February 28, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.11 No.4

et al. [15] studied some problems of third grade fluids. Siddiqui et al. [11] discussed the solution for thin-film flow of third grade fluid down an inclined plane using perturbation method and Homotopy Perturbation Method (HPM). Hayat et al. [16] obtained exact solutions to same problem under certain assumptions. Authors of [10, 12] also found the results for drainage problem for thin film of a fourth grade fluid down a vertical cylinder. There are many fluid models which describe behavior of the non-Newtonian fluids, but in recent years the Johnson-–Segalman fluid have attracted many researchers as it includes important cases as Newtonian fluid, Maxwell fluid and Oldroyd B fluid. Most of the physical problems involving non-Newtonian fluids are nonlinear problems. Thus, exact solutions may not be possible and therefore one must resort to various approximation methods of solutions, such as asymptotic techniques [21, 23, 24, 25, 26], analytical and numerical methods [4, 5, 7, 8, 9]. Analytic techniques are based on either perturbation techniques [27, 29], or traditional non-perturbation methods. Perturbation techniques are based on the existence of small/large parameters, the so-called perturbation quantities [28, 29], and on the other hand selection of small parameter is very important and requires a special skill. Therefore, analytical methods which do not require a small parameter are welcome. The basic idea of the differential transform method was initially introduced by Zhou [17, 18]. The DTM is an alternative procedure for getting Taylor series solution of the equation. This method reduces the size of computations of taylor coefficients. Liao [30] took the lead to apply the homotopy [31], a basic concept in topology, to gain analytic approximations of nonlinear differential equations. More importantly, unlike all other analytic techniques, the HAM provides us with a simple way to adjust and control the convergence radius of solution series. Thus, one can always get accurate approximations by means of the HAM. There are also exist some techniques to accelerate the convergence of a series solution, such as padé technique which is widely applied. In this paper we focus on the study of a steady thin film flow of non-Newtonian fluid on vertical cylinder which has been investigated in [33] by Adomian decomposition method. We continue the same study by two analytical methods, homotopy analysis method and modified differential transform method. In modified differential differential transform method the nonlinear terms handled by the use of Adomian polynomials. On occasion we use the Pade' technique to faster the convergence.

2. Models and Methods

The filed equations governing the flow of an incompressible fluid, with assumption of neglecting the thermal effect are of the form

$$div\, \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = div\, \sigma + \rho f, \tag{2}$$

where $\mathbf{V}$ is the velocity vector, f, the body force per unit mass and ρ, the constant density and σ is the Cauchy stress tensor. Based on Johnson and Segalman model [32] the Cauchy stress tensor is related to the fluid motion

$$\sigma = -p\mathbf{I} + \mathbf{T}, \tag{3}$$

$$\mathbf{T} = 2\mu\mathbf{D} + \mathbf{S}, \tag{4}$$

$$\mathbf{S} + m[\frac{D\mathbf{S}}{Dt} + \mathbf{S}(\mathbf{W} - a\mathbf{D})^T\mathbf{S}] = 2\eta\mathbf{D}, \tag{5}$$

$$\frac{D\mathbf{S}}{Dt} = \frac{\partial \mathbf{S}}{\partial t} + [grad\mathbf{S}]\mathbf{V}, \tag{6}$$

where $\mathbf{D}$ is the symmetric part and $\mathbf{W}$ is the skew-symmetric part of the velocity gradient, that is,

$$\mathbf{D}=\frac{1}{2}[\mathbf{L}+\mathbf{L}^T],\ W=\frac{1}{2}[\mathbf{L}-\mathbf{L}^T],\ \mathbf{L}=grad\mathbf{V}, \tag{7}$$

and $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, μ and σ are viscosities, m is the relaxation time and a is slip parameter. We focus on the study of a steady thin film flow of non-Newtonian fluid on vertical cylinder which has been investigated in [33] when a non-Newtonian Johnson-Segalman fluid is falling on the outside surface of an infinitely long vertical cylinder of radius R, in the form of a thin, uniform axisymmetric film of thickness δ, in contact with stationary air. So, we seek a velocity field of the form

$$V=(0,0,w(r)). \tag{8}$$

By substituting Eq. (8) in (1) and (2) we get

$$0=-\frac{\partial p}{\partial r}+\rho f_1, \tag{9}$$

$$0=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho f_2, \tag{10}$$

$$0=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{d(rT_{rz})}{dr}+\rho f_3, \tag{11}$$

where f_1, f_2 and f_3 denote the components of f in r,θ and z directions, respectively. Using the components of S and T these equations reduce to the following equation

$$\frac{d}{dr}[r(\frac{dw}{dr})+\frac{\eta(\frac{dw}{dr}}{1+m^2(1-a^2)(\frac{dw}{dr})^2})]=-\rho gr. \tag{12}$$

In dimensionless form this equation is given by

$$\frac{dw}{dr}+\phi W_e^2(1-a^2)(\frac{dw}{dr})^3-\frac{S_t}{2}W_e^2(1-a^2)((1+\delta)^2\frac{1}{r}-r)(\frac{dw}{dr})^2=\frac{S_t}{2}((1+\delta)^2\frac{1}{r}-r), \tag{13}$$

subject to

$$w=0 \qquad at \quad r=1. \tag{14}$$

where $W_e=m\frac{U_0}{\delta}$ is the Weissenberg number, $S_t=\frac{pg\delta^2}{\mu_{eff}U_0}$ represents Stokes number and $\mu_{eff}=(\mu+\eta)$. Also, we consider the lifting problem of the same fluid on a infinite vertical cylinder where the equation

$$\frac{dw}{dr}+\phi W_e^2(1-a^2)(\frac{dw}{dr})^3-\frac{S_t}{2}W_e^2(1-a^2)(r-\frac{(1+\delta)^2}{r})(\frac{dw}{dr})^2=\frac{S_t}{2}(r-\frac{(1+\delta)^2}{r}), \tag{15}$$

along with boundary condition

$$w=1 \quad at \quad r=1, \tag{16}$$

is derived.

2.1. **Homotopy analysis method.** Here we describe the main points of HAM method. Consider the following equation

$$\mathcal{N}[u(x,t)] = 0 \tag{17}$$

where $\mathcal{N}$ is a nonlinear operator and x, t are spatial and temporal independent variables and $u(x,t)$ is unknown function. By means of generalizing the traditional Homotopy method, the zero order deformation equation is constructed as

$$(1-q)\mathcal{L}[\phi(x,t;q) - u_0(x,t)] = q\hbar\mathcal{N}[\phi(x,t;q)], \tag{18}$$

where $\mathcal{L}$ is a linear operator, $q \in [0,1]$ is the embedding parameter, $\hbar$ is a nonzero auxiliary parameter and $u_0(x,t)$ is an initial guess of $u(x,t)$. When $q = 0$ and $q = 1$ it holds

$$\phi(x,t;0) = u_0(x,t), \ \mathcal{N}[\phi(x,t;1)] = 0,$$

respectively, thus as the embedding parameter q increase from 0 to 1, the solution $\phi(x,t;q)$ of (17) varies continuously from the initial approximation $u_0(x,t)$to the exact solution $u(x,t)$. Expending $\phi(x,t;q)$ in Taylor series with respect to q, we have

$$\phi(x,t;q) = u_0(x,t) + \Sigma_{m=1}^{+\infty} u_m(x,t)q^m, \tag{19}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}. \tag{20}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are so properly chosen, then, as proved in [19], the series (19) converges at $q = 1$ and one has

$$u(x,t) = u_0(x,t) + \Sigma_{m=1}^{+\infty} u_m(x,t). \tag{21}$$

Define the vector

$$\vec{u_n} = (u_0(x,t), u_1(x,t), ..., u_n(x,t)).$$

Differentiating Eq.(18) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order defomation equation,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar\Re_m[\vec{u_{m-1}}(x,t)], \tag{22}$$

where

$$\Re_m(\vec{u_{m-1}}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;p)]}{\partial p^{m-1}}|_{q=0}, \tag{23}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2. \end{cases} \tag{24}$$

In this way it is easily to obtain $u_m(x,t)$ for $m \geq 1$ and so (21) is an accurate approximation of the original equation (17).

2.2. **Modified differential transform method.** Consider a general form of a nonlinear non-homogeneous partial differential equation

$$Du(x,t) + Nu(x,t) = f(x,t), \tag{25}$$

with the following initial conditions

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x), \tag{26}$$

where D is a linear differential operator, such as $D = \frac{\partial^2}{\partial t^2}$, N represents the general nonlinear differential operator and $f(x,t)$ the source term. If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, let

$$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}, \tag{27}$$

then the inverse transform of $U_k(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{28}$$

and combining (27) and (28), we obtain

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}t^k. \tag{29}$$

The standard ADM [20, 22] yields the solution $u(x,t)$ by the series

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \tag{30}$$

and the nonlinear term $N(u)$ is approximated by the series

$$N(u) = \sum_{n=0}^{\infty} A_n,$$

where the A_n are the Adomian polynomials determined by the definitional formula [20, 22]

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[f(\sum_{i=0}^{\infty} \lambda^i u_i)]]_{\lambda=0}, \qquad n = 0, 1, \cdots . \tag{31}$$

Now, considering (27), (28) and (31) we deduce that kth differential transform component of $N(u)$, $\tilde{A}_k$, can be obtained from the corresponding Adomian polynomial of this term, A_k, by replacing each u_k with $U_k(x)$. Therefore, taking the differential transform of (25), we have the following system of algebraic equations

$$(k+1)(k+2)U_{k+2}(x) + \tilde{A}_k = F_k(x), \tag{32}$$

where $F_k(x)$ is the kth differential transform component of $f(x,t)$.

3. Results

3.1. **Homotopy analysis method solutions.** In this section, we apply the homotopy analysis method to solve the equations (13) and (15). By applied the idea of homotopy analysis method, we see that, in view of Eq. (13), and the initial condition given in Eq. (14), it is convenient to choose the initial approximation

$$w_0 = \frac{S_t}{2}((1+\delta)^2 \ln r - \frac{r^2}{2} + \frac{1}{2}). \tag{33}$$

The nonlinear operator is suggested as

$$N[\phi(r;q)] = \frac{\partial \phi(r;q)}{\partial r} + \phi W_e^2(1-a^2)(\frac{\partial \phi(r;q)}{\partial r})^3 - \frac{S_t}{2}W_e^2(1-a^2)(\frac{(1+\delta)^2}{r} - r)(\frac{\partial \phi(r;q)}{\partial r})^2 - \frac{S_t}{2}(\frac{(1+\delta)^2}{r} - r), \tag{34}$$

and the linear operator

$$L[\phi(r;q)] = \frac{\partial \phi(r;q)}{\partial r}, \tag{35}$$

with the property $L(c) = 0$ where c is the integration constant. Using the above definitions, we construct the zeroth-order deformation equation

$$(1-q)L[\phi(r;q) - W_0(r)] = q\hbar N[\phi(r;q)]. \quad (36)$$

Obviously,when $q = 0$and $q = 1$,

$$\phi(r;0) = w_0(r), \qquad \phi(r;1) = w(r).$$

Therefore, as the embedding parameter q increases from 0 to 1, $\phi(r;q)$ varies from the intial guess $w_0(r)$ to the solution $w(r)$. Then, we obtain the mth-order deformation equation

$$L[w_m(r) - \chi_m w_{m-1}(r)] = \hbar \Re_m[(\overrightarrow{w_{m-1}}(r))], \quad (37)$$

subject to intial condition $w_m(1) = 0$, where

$$\begin{aligned}
\Re_m(\overrightarrow{w_{m-1}}(r)) = & \frac{\partial w_{m-1}(r)}{\partial r} \\
& + \phi W_e^2(1-a^2)(\Sigma_{k=0}^{m-1}(\Sigma_{i=0}^{k}\frac{\partial w_i(r)}{\partial r}\frac{\partial w_{k-i}(r)}{\partial r})\frac{\partial w_{m-1-k}(r)}{\partial r} \\
& - \frac{S_t}{2}W_e^2(1-a^2)(\frac{(1+\delta)^2}{r} - r)(\Sigma_{k=0}^{m-1}\frac{\partial w_k(r)}{\partial r}\frac{\partial w_{m-1-k}(r)}{\partial r}) \\
& - \frac{S_t}{2}(\frac{(1+\delta)^2}{r} - r)(1-\chi_m).
\end{aligned} \quad (38)$$

Now, the solution of the mth-order deformation equation (37) for $m \geq 1$ becomes

$$w_m(r) = \chi_m w_{m-1}(r) + \hbar L^{-1}[\Re_m(\overrightarrow{w_{m-1}}(r))]. \quad (39)$$

From (33) and (39), we now successively obtain

$$\begin{aligned}
w_0(r) = & \frac{S_t}{2}(\frac{(1+\delta)^2}{r} - r), \\
w_1(r) = & \frac{-1}{32}(1-a^2)hS_t^3W_e^2 + \frac{3}{16}(1-a^2)h(1+\delta)^2S_t^3W_e^2 \\
& - \frac{1}{16}(1-a^2)h(1+\delta)^6S_t^3W_e^2 + \frac{1}{16}(1-a^2)\phi hS_t^3(\frac{(1+\delta)^2}{r} - r)^2W_e^2 \\
& + \frac{(1-a^2)h(1+\delta)^6S_t^3W_e^2}{16r^2} - \frac{3}{16}(1-a^2)h(1+\delta)^2S_t^3r^2W_e^2 \\
& - \frac{1}{16}(1-a^2)\phi hS_t^3(\frac{(1+\delta)^2}{r} - r)^2r^2W_e^2 + \frac{1}{32}(1-a^2)hS_t^3r^4W_e^2 \\
& + \frac{3}{8}(1-a^2)h(1+\delta)^4S_t^3W_e^2\ln r + \frac{1}{8}(1-a^2)\phi h(1+\delta)^2S_t^3(\frac{(1+\delta)^2}{r} - r)^2W_e^2\ln r, \\
& \vdots
\end{aligned}$$

Here we will use five terms in evaluating the approximate solution

$$w(r) = \Sigma_{n=0}^{\infty} = w_0(r) + w_1(r) + w_2(r) + \cdots . \quad (40)$$

Similarly, for lifting problem to solve Eq.(15) by means of HAM, we choose the initial approximation

$$w_0(r) = 1 + \frac{S_t}{2}(\frac{r^2}{2} - (1+\delta)^2\ln r - \frac{1}{2}). \quad (41)$$

Eq.(15) suggests the nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial\phi(r;q)}{\partial r} + \phi W_e^2(1-a^2)(\frac{\partial\phi(r;q)}{\partial r})^3 - \frac{S_t}{2}W_e^2(1-a^2)(r-\frac{(1+\delta)^2}{r})(\frac{\partial\phi(r;q)}{\partial r})^2 - \frac{S_t}{2}(r-\frac{(1+\delta)^2}{r}) \quad (42)$$

and the linear operator

$$L[\phi(r;q)] = \frac{\partial\phi(r;q)}{\partial r}. \quad (43)$$

Similar to the previous case we have the following mth-order deformation equation

$$L[w_m(r) - \chi_m w_{m-1}(r)] = \hbar\Re_m[w_{m-1}^{\rightarrow}(r)], \quad (44)$$

subject to initial condition $w_m(1) = 0$ where

$$\begin{aligned}\Re_m(w_{m-1}^{\rightarrow}(r)) &= \frac{\partial w_{m-1}(r)}{\partial r} \\ &+ \phi W_e^2(1-a^2)(\Sigma_{k=0}^{m-1}(\Sigma_{i=0}^{k}\frac{\partial w_i(r)}{\partial r}\frac{\partial w_{k-i}(r)}{\partial r})w_{m-1-k}(r)) \\ &- \frac{S_t}{2}W_e^2(1-a^2)(r-\frac{(1+\delta)^2)}{r})(\Sigma_{k=0}^{m-1}\frac{\partial w_k(r)}{\partial r}\frac{\partial w_{m-1-k}(r)}{\partial r}) \\ &- \frac{S_t}{2}(r-\frac{(1+\delta)^2}{r})(1-\chi_m).\end{aligned} \quad (45)$$

Now ,the solution of the mth-order deformation equation (44) for $m \geq 1$ becomes

$$w_m(r) = \chi_m w_{m-1}(r) + \hbar L^{-1}[\Re_m(w_{m-1}^{\rightarrow}(r))]. \quad (46)$$

From (41) and (46), we now successively obtain

$$\begin{aligned}w_0(r) &= 1 + \frac{S_t}{2}(\frac{r^2}{2} - (1+\delta)^2\ln r - \frac{1}{2}), \\ w_1(r) &= \frac{1}{32}(1-a^2)hS_t^3W_e^2 - \frac{3}{16}(1-a^2)h(1+\delta)^2S_t^3W_e^2 \\ &+ \frac{1}{16}(1-a^2)h(1+\delta)^6S_e^3W_e^2 - \frac{1}{16}(1-a^2)\phi hS_t^3(r-\frac{(1+\delta)^2}{r})^2W_e^2 \\ &- \frac{(1-a^2)h(1+\delta)^6S_e^3W_e^2}{16r^2} + \frac{3}{16}(1-a^2)h(1+\delta)^2S_t^3r^2W_e^2 \\ &+ \frac{1}{16}(1-a^2)\phi hS_t^3(r-\frac{(1+\delta)^2}{r})^2r^2W_e^2 \\ &- \frac{1}{32}(1-a^2)hS_e^3r^4W_e^2 - \frac{3}{8}(1-a^2)h(1+\delta)^4S_t^3W_e^2\ln r \\ &- \frac{1}{8}(1-a^2)\phi h(1+\delta)^2S_t^3(r-\frac{(1+\delta)^2}{r})^2W_e^2\ln r, \\ &\vdots\end{aligned}$$

Then the series solution expression can be written in the form,

$$w(r) = \Sigma_{n=0}^{\infty}w_n(r) = w_0(r) + w_1(r) + w_2(r) + \cdots, \quad (47)$$

3.2. **Modified differential transform method solutions.** In this section, we apply the modified differential transform method, described in Sec. 2. to solve the equations (13) and (15). For this purpose we set $\frac{dw}{dr} = u(r)$, in Eq. (13), so we have

$$u(r)+\phi W_e^2(1-a^2)(u(r))^3-\frac{S_t}{2}W_e^2(1-a^2)((1+\delta)^2\frac{1}{r}-r)(u(r))^2 = \frac{S_t}{2}((1+\delta)^2\frac{1}{r}-r). \quad (48)$$

To get a series solution of the form

$$u(r) = U_0 + U_1(r-1) + U_2(r-1)^2 + U_3(r-1)^3 + \dots, \quad (49)$$

taking differential transform from both sides of (48) and using the strategies described in Sec. 2 for non-linear terms we deduce

$$U_k + \phi W_e^2(1-a^2)\tilde{A}_k - \frac{S_t}{2}W_e^2(1-a^2)[(1+\delta)^2\sum_{r=0}^{k}(-1)^r\tilde{B}_{k-r} - \sum_{r=0}^{1}\tilde{B}_{k-r}] = \frac{S_t}{2}[(1+\delta)^2(-1)^k - M_k], \quad (50)$$

where

$$M_k = \begin{cases} 1, & k = 0, 1, \\ 0, & m \geq 2, \end{cases}$$

and $\tilde{A}_k$, $\tilde{B}_k$ are differential transform components obtained from the Adomian polynomials for the corresponding nonlinearities, $u^3(r)$ and $u^2(r)$, respectively. So, the following differential transform components are obtained

$$\begin{aligned} \tilde{A}_0 &= U_0^3, & \tilde{B}_0 &= U_0^2, \\ \tilde{A}_1 &= 3U_1U_0^2, & \tilde{B}_1 &= 2U_0U_1, \\ \tilde{A}_2 &= 3U_0U_1^2 + 3U_0^2U_2, & \tilde{B}_2 &= U_1^2 + 2U_0U_2, \\ \tilde{A}_3 &= U_1^3 + 3U_0^2U_3, & \tilde{B}_3 &= 2U_0U_3 + 2U_1U_2, \end{aligned} \quad (51)$$

therefore starting with $U_0 = 0$ solving (50) recursively one can obtain

$$U(r) = \Sigma_{K=0}^{\infty}U_k(r-1)^k = U_0 + U_1(r-1) + U_2(r-1)^2 + U_3(r-1)^3 + \dots, \quad (52)$$

and then integration gives the series solution. For lifting problem the computations are similar and so we omit the details.

4. Results and discussion

In this paper the analytic investigation of the steady thin film flow of non-Newtonian Johnson-Segalman fluid on vertical cylinder for lifting and drainage problems, using homotopy analysis method and modified differential transform method, have been discussed. The exact solutions of this non-linear equation, if available, facilitates the verification of numerical solvers and adds in the stability analysis of solutions. The results can be obtain for the Maxwell fluid by taking slip parameter $a = 1$. We have discussed the effect of the Stokes number S_t, the Weissenberg number W_e, the ratio of viscosities ϕ and the slip parameter a on the fluid flows. We present a comparison of HAM and MDTM and observed that the methods are in good agreement. Obtained results also are compared with existing ADM solutions. In the following tables err1, err2 and err3 represents absolute error between ADM solution and HAM, ADM solution and DTM Pad'e[5, 5] and DTM Pad'e[5, 5] and HAM respectively.

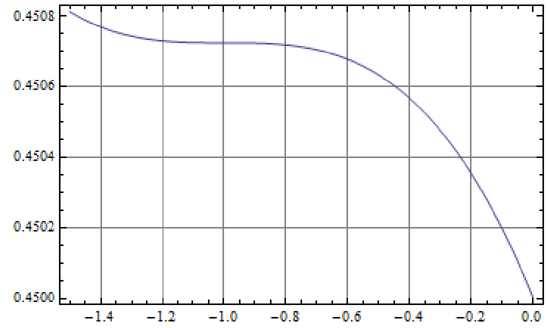

FIGURE 1. The $\hbar$-curve in drainage case.

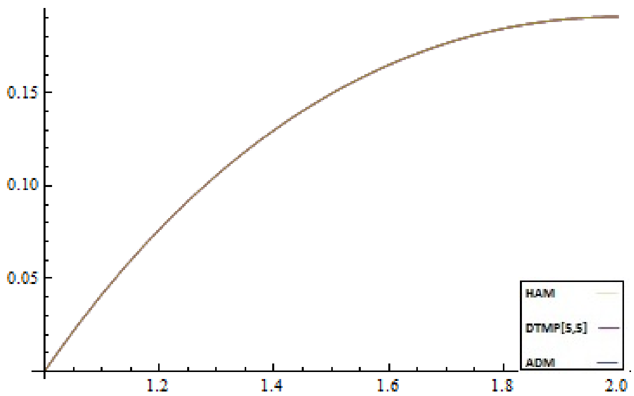

FIGURE 2. Comparison of solutions for the drainage case by taking $W_e = 0.1; \phi = 0.2; S_t = 0.3; a = 0.1;$ and $\delta = 1$.

r	ADM	HAM	DTMP[5,5]	err1	err2	err3
1.0	0.00	0.00	0.00	0.00	0.00	0.00
1.1	0.0138204	0.0138207	0.0138204	$3.42, -7$	$1.14, -11$	$3.42, -7$
1.2	0.0254777	0.025478	0.0254777	$1.07, -6$	$1.48, -11$	$1.07, -6$
1.3	0.0352392	0.0352411	0.0352392	$1.9, -6$	$1.51, -11$	$1.9, -6$
1.4	0.0433125	0.0433152	0.0433125	$2.70, -6$	$4.42, -12$	$2.70, -6$
1.5	0.0498620	0.049865	0.0498620	$3.41, -6$	$7.87, -11$	$3.41, -6$
1.6	0.0550201	0.055024	0.0550201	$3.98, -6$	$4.83, -10$	$3.98, -6$
1.7	0.0588953	0.058899	0.0588953	$4.41, -6$	$1.94, -9$	$4.41, -6$
1.8	0.0615770	0.0615817	0.0615770	$4.71, -6$	$6.21, -9$	$4.71, -6$
1.9	0.0631405	0.0631453	0.0631404	$4.87, -6$	$1.68, -8$	$4.89, -6$
2	0.0636491	0.0636540	0.0636491	$4.93, -6$	$4.04, -8$	$4.97, -6$

TABLE 1. Analytical methods estimations for the drainage case for $W_e = 0.2; \phi = 0.2; S_t = 0.1; a = 0.1;$ and $\delta = 1$.

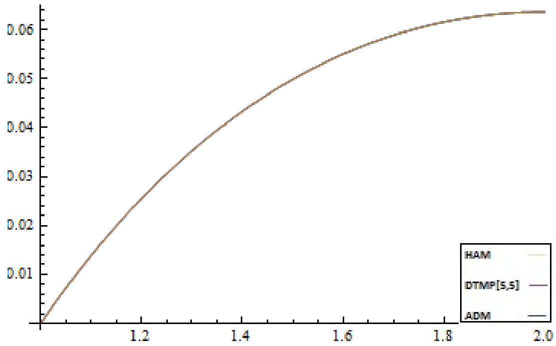

FIGURE 3. Analytical methods estimations for the drainage case for $W_e = 0.2; \phi = 0.2; S_t = 0.1; a = 0.1;$ and $\delta = 1$.

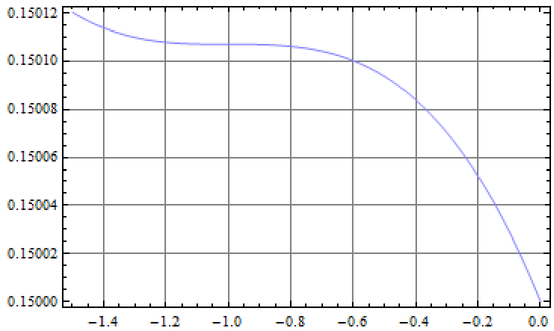

FIGURE 4. The $\hbar$-curve in drainage case.

r	ADM	HAM	$DTMP[5,5]$	$err1$	$err2$	$err3$
1.0	0.00	0.00	0.00	0.00	0.00	0.00
1.1	0.0138333	0.0138337	0.0138333	3.86, −7	2.26, −10	3.86, −7
1.2	0.0254984	0.025499	0.0254984	1.2, −6	2.95, −10	1.2, −6
1.3	0.0352645	0.035266	0.0352645	2.14, −6	3.14, −10	2.14, −6
1.4	0.0433403	0.043343	0.0433403	3.05, −6	3.05, −10	3.04, −6
1.5	0.0498912	0.0498950	0.0498912	3.84, −6	1.98, −10	3.84, −6
1.6	0.0550500	0.055054	0.0550500	4.48, −6	3.22, −10	.48, −6
1.7	0.0589254	0.058930	0.0589254	4.97, −6	2.19, −9	4.97, −6
1.8	0.0616073	0.061612	0.0616073	5.3, −6	7.61, −9	5.31, −6
1.9	0.0631707	0.063176	0.0631707	5.49, −6	2.10, −8	5.51, −6
2	0.0636794	0.0636850	0.0636793	5.55, −6	5.06, −8	5.6, −6

TABLE 2. Analytical methods estimation for the drainage case for $W_e = 0.3; \phi = 0.1; S_t = 0.1; a = 0.1;$ and $\delta = 1$.

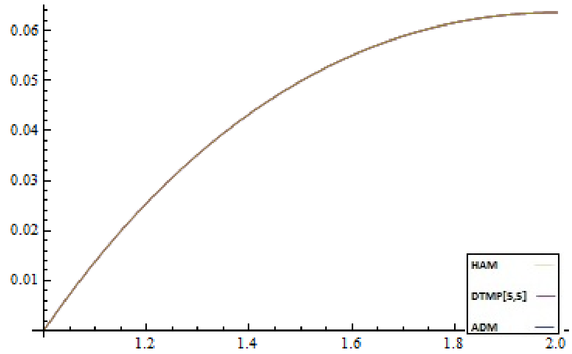

FIGURE 5. Analytical methods estimations for the drainage case for $W_e = 0.3$; $\phi = 0.1$;$S_t = 0.1$; $a = 0.1$; and $\delta = 1$.

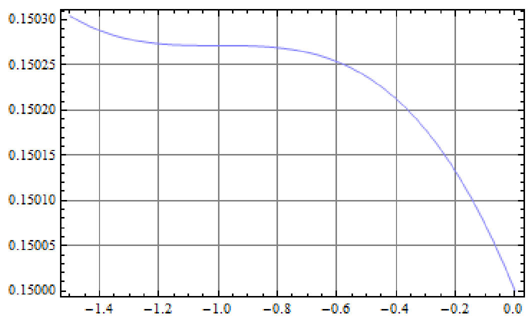

FIGURE 6. The $\hbar$-curve in lifting case.

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