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A NOTE ON EXTENDED BETA FUNCTION INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION AND ITS APPLICATIONS

NABIULLAH KHAN[1], SADDAM HUSAIN[1], §

Abstract. The main object of this paper is to introduce a new extension of beta function involving generalized Mittag-leffler function and study its important properties, like integral representation, summation formula, derivative formula, beta distribution, transform formula. Using this definition, we introduce new extended hypergeometric and confluent hypergeometric function.

Keywords: Gamma function, Beta function, Hypergeometric function, Confluent hypergeometric function, Beta Distribution

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1. Introduction

Recently, Many extensions and generalization of special functions (such as beta function, hypergeometric function and confluent hypergeometric function) have been considered by several authors (see [2, 3, 4, 5, 6, 7, 9, 10, 11]). In this paper, we study another extension of Euler beta function and investigate various formulas, such as integral representation, summation formula, derivative formula. Further, we obtain beta distribution and its some statistical formulas. We extend also the definition of hypergeometric and confluent hypergeometric function and study its various properties.
Throughout the paper, we take $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ be the sets of complex numbers, real numbers, positive real numbers and positive real number including zero respectively.

Definition 1.1. *The classical gauss hypergeometric function (see* [1]*) is defined as*

$$F(\theta_1, \theta_2; \theta_3; \omega), = \sum_{n=0}^{\infty} \frac{(\theta_1)_n (\theta_2)_n}{(\theta_3)_n} \frac{\omega^n}{n!}, \tag{1}$$

[1] Aligarh Muslim University, Faculty of Engineering and Technology, Department of Applied Mathematics, Aligarh, Uttar Pradesh, India.
e-mail:nukhanmath@gmail.com; ORCID: https://orcid.org/0000-0003-0389-7899.
e-mail: saddamhusainamu26@gmail.com; ORCID: https://orcid.org/0000-0002-1852-4748.
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where $(\theta)_n$ $(\theta \in \mathbb{C})$ *is the Pochhammer symbol defined by*

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)}. \tag{2}$$

The confluent hypergeometric function (see [1]*) is defined by*

$$\Phi(\theta_1;\theta_2;\ \omega) = \sum_{n=0}^{\infty} \frac{(\theta_1)_n}{(\theta_2)_n} \frac{\omega^n}{n!}. \tag{3}$$

Definition 1.2. *The Gamma function* $\Gamma(\omega)$ *developed by Euler* [1] *with the intent to extend the factorials to values between the integers is defined by the definite integral*

$$\Gamma(\omega) = \int_0^{\infty} e^{-t} t^{\omega-1}\, dt \quad (\Re(\omega) > 0). \tag{4}$$

Among various extensions of gamma function, we mention here the extended gamma function [2] *defined by Chaudhry and Zubair*

$$\Gamma_p(\omega) = \int_0^{\infty} t^{\omega-1} \exp\left(-t - \frac{p}{t}\right) dt \quad (R(p) > 0). \tag{5}$$

Definition 1.3. *The Euler beta function* $B(\theta_1, \theta_2)$ *(see* [1]*) is defined by*

$$B(\theta_1,\theta_2) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} dt \tag{6}$$

$$= \frac{\Gamma(\theta_1)\Gamma(\theta_2)}{\Gamma(\theta_1+\theta_2)} = \frac{(\theta_1-1)!\,(\theta_2-1)!}{(\theta_1+\theta_2-1)!}, \tag{7}$$

where

$$(\Re(\theta_1) > 0,\ \Re(\theta_2) > 0).$$

In 1997, Choudhary et al. [3] introduced an extension of beta function defined by

$$B^p(\theta_1,\theta_2) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} \exp\left(-\frac{p}{t(1-t)}\right) dt, \tag{8}$$

where

$$(R(p) \geq 0,\ \Re(\theta_1) > 0,\ \Re(\theta_2) > 0).$$

Chaudhary et al. [4] used new extended beta function $B^p(\theta_1, \theta_2)$ to introduced an extended hypergeometric and confluent hypergeometric function defined respectively as

$$F_p(\theta_1,\theta_2;\theta_3;\omega) = \sum_{n=0}^{\infty} (\theta_1)_n \frac{B_p(\theta_2+n,\theta_3-\theta_2)}{B(\theta_2,\theta_3-\theta_2)} \frac{\omega^n}{n!} \tag{9}$$

$$(p \geq 0,\ |\omega| < 1,\ \Re(\theta_3) > \Re(\theta_2) > 0),$$

and

$$\Phi_p(\theta_2;\theta_3;\omega) = \sum_{n=0}^{\infty} \frac{B_p(\theta_2+n,\theta_3-\theta_2)}{B(\theta_2,\theta_3-\theta_2)} \frac{\omega^n}{n!} \tag{10}$$

$$(p \geq 0,\ \Re(\theta_3) > \Re(\theta_2) > 0).$$

In 2018, Shadab et al. [11] introduced an extended beta function in terms of classical Mittag-Leffler function defined as

$$B^p_\alpha(\theta_1,\theta_2) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1} E_\alpha\left(-\frac{p}{t(1-t)}\right) dt \tag{11}$$

$$(\Re(\theta_1) > 0,\ \Re(\theta_2) > 0, R(p) \geq 0\,;\ \alpha \in \mathbb{R}_0^+),$$

where $E_\alpha(.)$ is the classical Mittag-Leffler function defined as [8]

$$E_\alpha(\omega) = \sum_{n=0}^{\infty} \frac{\omega^n}{\Gamma(\alpha n + 1)}, \tag{12}$$

where

$$(\omega \in \mathbb{C}, \alpha \in \mathbb{R}_0^+).$$

Shadab et al. [11] used extended beta function to introduced a new extended hypergeometric and confluent hypergeometric function defined respectively as

$$F_{p,\alpha}(\theta_1, \theta_2; \theta_3; \omega) = \sum_{n=0}^{\infty} (\theta_1)_n \frac{B_\alpha^p(\theta_2 + n, \theta_3 - \theta_2)}{B(\theta_2, \theta_3 - \theta_2)} \frac{\omega^n}{n!} \tag{13}$$

$$(\alpha \in \mathbb{R}^+,\ p \in \mathbb{R}_0^+,\ |\omega| < 1,\ \Re(\theta_3) > \Re(\theta_2) > 0),$$

and

$$\Phi_{p,\alpha}(\theta_2; \theta_3; \omega) = \sum_{n=0}^{\infty} \frac{B_\alpha^p(\theta_2 + n, \theta_3 - \theta_2)}{B(\theta_2, \theta_3 - \theta_2)} \frac{\omega^n}{n!} \tag{14}$$

$$(\alpha \in \mathbb{R}^+,\ p \in \mathbb{R}_0^+,\ |\omega| < 1,\ \Re(\theta_3) > \Re(\theta_2) > 0).$$

2. A new extension of beta function

Here we introduce a new extension of extended beta function $B_\alpha^p(\theta_1, \theta_2)$ and investigate various properties and representations

$$B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2) = \int_0^1 t^{\theta_1 - 1} (1-t)^{\theta_2 - 1} E_{\alpha,\beta}\left(-\frac{p}{t^\mu (1-t)^\nu}\right) dt \tag{15}$$

$$(\Re(\theta_1) > 0,\ \Re(\theta_2) > 0; \alpha, \beta \in \mathbb{R}_0^+; \mu, \nu \in \mathbb{R}^+),$$

where $E_{\alpha,\beta}(.)$ is the generalized Mittag-Leffler function defined as [12]

$$E_{\alpha,\beta}(\omega) = \sum_{n=0}^{\infty} \frac{\omega^n}{\Gamma(\alpha n + \beta)}, \tag{16}$$

where

$$(\alpha, \beta \in \mathbb{R}_0^+, \omega \in \mathbb{C}).$$

Remark 2.1. *If we take $\alpha = 1, \beta = 1, \mu = 1, \nu = 1$, in the integral representation of* (15) *we obtain $B^p(\theta_1, \theta_2)$* [3]

$$B_{1,1}^{p,1,1}(\theta_1, \theta_2) = B^p(\theta_1, \theta_2) = B(\theta_1, \theta_2; p). \tag{17}$$

If we take $\beta = 1, \mu = 1, \nu = 1$, in the integral representation of (15) *we obtain $B_\alpha^p(\theta_1, \theta_2)$* [11]

$$B_{\alpha,1}^{p,1,1}(\theta_1, \theta_2) = B_\alpha^p(\theta_1, \theta_2). \tag{18}$$

3. Properties of $B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2)$

In this section we obtain some interesting relation of summation formulas for $B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2)$.

Theorem 3.1. *The following integral representations holds:*

$$B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2) = 2\int_0^{\frac{\pi}{2}} \cos^{2\theta_1-1}\phi \, \sin^{2\theta_2-1}\phi \, E_{\alpha,\beta}\left[-p(sec^2\phi)^{\mu}(cosec^2\phi)^{\nu}\right] d\phi, \tag{19}$$

$$B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2) = \int_0^{\infty} \frac{u^{\theta_1-1}}{(1+u)^{\theta_1+\theta_2}} E_{\alpha,\beta}\left[-p(\frac{(1+u)^{\mu+\nu}}{u^{\mu}})\right] du, \tag{20}$$

$$B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2) = 2^{1-\theta_1-\theta_2}\int_{-1}^{1} (1+u)^{\theta_1-1}(1-u)^{\theta_2-1} E_{\alpha,\beta}\left[-\frac{2^{\mu+\nu}p}{(1+u)^{\mu}(1-u)^{\nu}}\right] du. \tag{21}$$

$$(R(p)>0,\ \Re(\theta_1)>0,\ \Re(\theta_2)>0, \alpha,\beta\in\mathbb{R}^+; \mu,\nu\in\mathbb{R}^+).$$

Proof. Let $t = cos^2\theta$, $t=\frac{u}{1+u}$, $t=\frac{1+u}{2}$, respectively in equation (15), we obtain the above representations. □

Remark 3.1. *If we take $\alpha=1,\beta=1,\mu=1,\nu=1$, in the integral representation of Theorem* (3.1)*, we obtain corresponding integrals for $B(\theta_1,\theta_2;p)$ in* [3].

If we take $\beta=1,\mu=1,\nu=1$, in the integral representation of Theorem (3.1)*, we obtain corresponding integrals for $B^{p}_{\alpha}(\theta_1,\theta_2)$ in* [11].

Theorem 3.2. *The following summation formula for $B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2)$ hold:*

$$B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2) = \sum_{k=0}^{n}\binom{n}{k} B^{p;\mu,\nu}_{\alpha,\beta}(\theta_1+k,\theta_2+n-k) \quad (n\in\mathbb{N}_0). \tag{22}$$

Proof. We find from (15) that

$$\begin{aligned} B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2) &= \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1}[t+(1-t)]E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right)dt \\ &= B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1+1,\theta_2) + B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2+1). \end{aligned} \tag{23}$$

Repeating the same argument to the above two terms in (23), we obtain

$$\begin{aligned} B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2) &= B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1+2,\theta_2) \\ &\quad + 2B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1+1,\theta_2+1) + B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2+2) \end{aligned} \tag{24}$$

Continuing this process, by using mathematical induction we get the desired result (22). □

Theorem 3.3. *The following summation formula for $B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2)$ hold:*

$$B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,1-\theta_2) = \sum_{n=0}^{\infty}\frac{(\theta_2)_n}{n!} B^{p,\mu,\nu}_{\alpha,\beta}(\theta_1+n,1) \tag{25}$$

$$(R(p)>0,\ \Re(\theta_1)>0,\ \Re(\theta_2)>0, \alpha,\beta\in\mathbb{R}^+; \mu,\nu\in\mathbb{R}^+).$$

Proof. To prove above result, we make use of the generalized binomial theorem defined as

$$(1-t)^{-\theta_2} = \sum_{n=0}^{\infty} (\theta_2)_n \frac{t^n}{n!}, \quad (|t| < 1). \tag{26}$$

We find

$$B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, 1-\theta_2) = \int_0^1 \sum_{n=0}^{\infty} (\theta_2)_n \frac{t^{\theta_1+n-1}}{n!} E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right) dt.$$

Interchanging the order of integral and summation in the above equation and using (15), we get the desired result (25). □

Theorem 3.4. *The following summation formula for* $B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2)$ *hold:*

$$B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2) = \sum_{n=0}^{\infty} B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1+n, \theta_2+1), \tag{27}$$

$$(R(p) > 0,\ \Re(\theta_1) > 0,\ \Re(\theta_2) > 0, \alpha, \beta \in \mathbb{R}^+; \mu, \nu \in \mathbb{R}^+).$$

Proof. Using the relation

$$(1-t)^{\theta_2-1} = (1-t)^{\theta_2} \sum_{n=0}^{\infty} t^n \quad (|t| < 1), \tag{28}$$

in (15), we obtain

$$B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2) = \int_0^1 (1-t)^{\theta_2} \sum_{n=0}^{\infty} t^{\theta_1+n-1} E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right) dt.$$

$$B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2) = \sum_{n=0}^{\infty} \int_0^1 (1-t)^{\theta_2} t^{\theta_1+n-1} E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right) dt,$$

Which in view of (15), we get the desired result (27). □

Remark 3.2. *In case* $\alpha = 1, \beta = 1, \mu = 1, \nu = 1$ *of* (22) *for n=1,* (25) *and* (27) *reduces to corresponding results in* [3].

In case $\beta = 1, \mu = 1, \nu = 1$ *of* (22) *for n=1,* (25) *and* (27) *reduces to corresponding results in* [11].

4. Beta distribution of $B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2)$

We now define the beta distribution of (15), and obtain its mean, variance, moment generating function and cummulative distribution.

For $B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2)$, the beta distribution is given by

$$f(t) = \begin{cases} \frac{1}{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2)} t^{\theta_1-1}(1-t)^{\theta_2-1} E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right) & (0 < t < 1), \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

$$(\theta_1, \theta_2 \in \mathbb{R};\ p, \alpha, \beta \in \mathbb{R}^+; \mu, \nu \in \mathbb{R}^+).$$

For $d \in \mathbb{R}$, the d^{th} moment of a random variable X as

$$\mathbb{E}(X^d) = \frac{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1+d,\theta_2)}{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2)}, \tag{30}$$

$$(\theta_1, \theta_2 \in \mathbb{R};\ p \geq 0, \alpha, \beta \in \mathbb{R}^+; \mu, \nu \in \mathbb{R}^+).$$

When $d = 1$, we get the mean of the distribution as a particular case of (30) given by

$$\rho = \mathbb{E}(X) = \frac{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1 + 1, \theta_2)}{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2)}. \tag{31}$$

The variance of the distribution is defined by

$$\sigma^2 = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2 = \frac{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2)\, B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1 + 2, \theta_2) - \left\{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1 + 1, \theta_2)\right\}^2}{\left\{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2)\right\}^2}. \tag{32}$$

The moment generating function of the distribution is defined as

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \mathbb{E}(X^n) = \frac{1}{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2)} \sum_{n=0}^{\infty} B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1 + n, \theta_2) \frac{t^n}{n!}. \tag{33}$$

The cummulative distribution is defined as

$$F(x) = \frac{B_{x,\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2)}{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2)}, \tag{34}$$

where

$$B_{x,\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2) = \int_0^x t^{\theta_1 - 1}(1-t)^{\theta_2 - 1}\, E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right) dt \tag{35}$$

$$(p > 0,\ -\infty < \mu, \nu < \infty)$$

is the extended incomplete beta function.

5. Generalization of Extended Hypergeometric and Confluent Hypergeometric functions

Here, we introduce a generalization of extended hypergeometric and confluent hypergeometric functions in terms of $B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2)$.

The extended hypergeometric function is defined as

$$F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2; \theta_3; \omega) = \sum_{n=0}^{\infty} (\theta_1)_n\, \frac{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_2 + n, \theta_3 - \theta_2)}{B(\theta_2, \theta_3 - \theta_2)}\, \frac{\omega^n}{n!}, \tag{36}$$

$$(p \geq 0,\ |\omega| < 1,\ \alpha, \beta, \mu, \nu > 0,\ \Re(\theta_3) > \Re(\theta_2) > 0).$$

The confluent hypergeometric function is defined as

$$\Phi_{\alpha,\beta}^{p,\mu,\nu}(\theta_2; \theta_3; \omega) = \sum_{n=0}^{\infty} \frac{B_{\alpha,\beta}^{p,\mu,\nu}(\theta_2 + n, \theta_3 - \theta_2)}{B(\theta_2, \theta_3 - \theta_2)}\, \frac{\omega^n}{n!}, \tag{37}$$

$$(p \geq 0,\ \alpha, \beta, \mu, \nu > 0,\ \Re(\theta_3) > \Re(\theta_2) > 0).$$

Remark 5.1. *In case* $\alpha = 1, \beta = 1, \mu = 1, \nu = 1$ *in* (36) *and* (37)*, we obtain corresponding result in* [4].

In case $\beta = 1, \mu = 1, \nu = 1$ *in* (36) *and* (37)*, we obtain corresponding result in* [11].

6. Integral Representation and derivative formula for extended Gauss hypergeometric functions

Theorem 6.1. *The following integral representations for the extended hypergeometric function* $F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2;\theta_3;\omega)$ *and confluent hypergeometric function* $\Phi_{\alpha,\beta}^{p,\mu,\nu}(\theta_2;\theta_3;\omega)$ *holds:*

$$\begin{aligned} F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2;\theta_3;\omega) =& \frac{1}{B(\theta_2,\theta_3-\theta_2)} \\ &\times \int_0^1 t^{\theta_2-1}(1-t)^{\theta_3-\theta_2-1}(1-\omega t)^{-\theta_1} E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right)dt, \end{aligned} \tag{38}$$

$$(p\in\mathbb{R}_0^+,\alpha,\beta,\mu,\nu\in\mathbb{R}^+;\ \textit{and}\ |\arg(1-\omega)|<\pi;\ \Re(\theta_3)>\Re(\theta_2)>0),$$

and

$$\begin{aligned} \Phi_{\alpha,\beta}^{p,\mu,\nu}(\theta_2;\theta_3;z) =& \frac{1}{B(\theta_2,\theta_3-\theta_2)} \\ &\times \int_0^1 t^{\theta_2-1}(1-t)^{\theta_3-\theta_2-1}e^{zt} E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right)dt, \end{aligned} \tag{39}$$

$$(p\in\mathbb{R}_0^+,\alpha,\beta,\mu,\nu\in\mathbb{R}^+;\ \Re(\theta_3)>\Re(\theta_2)>0).$$

Proof. By using the definition of $B_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2)$ in (15) into (36) and interchanging the order of integration and summation, which is verifed under the condition here, we have

$$\begin{aligned} F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2;\theta_3;\omega) =& \frac{1}{B(\theta_2,\theta_3-\theta_2)} \\ &\times \int_0^1 t^{\theta_2-1}(1-t)^{\theta_3-\theta_2-1} E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right)\sum_{n=0}^{\infty}(\theta_1)_n\frac{(\omega t)^n}{n!}dt. \end{aligned} \tag{40}$$

Using the binomial theorem in (28) to the summation formula in (40), we get the desired result (38).
Similarly, we can obtain (39). □

Remark 6.1. *In case* $\alpha=1,\beta=1,\mu=1,\nu=1$ *in* (38) *and* (39)*, we obtain corresponding result in* [4].
In case $\beta=1,\mu=1,\nu=1$ *in* (38) *and* (39) , *we obtain corresponding result in* [11].

Theorem 6.2. *The following derivative formula for extended Gauss hypergeometric and confluent hypergeometric function holds:*

$$\frac{d^n}{d\omega^n}\left\{F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1,\theta_2;\theta_3;\omega)\right\} = \frac{(\theta_1)_n(\theta_2)_n}{(\theta_3)_n}F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1+n,\theta_2+n;\theta_3+n;\omega)\quad (n\in\mathbb{N}_0), \tag{41}$$

and

$$\frac{d^n}{d\omega^n}\left\{\Phi_{\alpha,\beta}^{p,\mu,\nu}(\theta_2;\theta_3;\omega)\right\} = \frac{(\theta_2)_n}{(\theta_3)_n}\phi_{\alpha,\beta}^{p,\mu,\nu}(\theta_2+n;\theta_3+n;\omega)\quad (n\in\mathbb{N}_0), \tag{42}$$

where

$$(p\geq 0,\alpha,\beta,\mu,\nu\in\mathbb{R}^+;\ \Re(\theta_3)>\Re(\theta_2)>0).$$

Proof. Differentiating (36) and (37) with respect to ω and using the following formula

$$B(\theta_2,\theta_3-\theta_2)=\frac{\theta_3}{\theta_2}B(\theta_2+1,\theta_3-\theta_2)\,\textit{and}\,(\theta)_{n+1}=\theta(\theta+1)_n, \tag{43}$$

we obtain the derivative formulas (41) and (42) for n=1. Easily applying the same process, we get the desired results (41) and (42). □

Remark 6.2. *In case* $\alpha = 1, \beta = 1, \mu = 1, \nu = 1$ *in* (41) *and* (42), *we obtain corresponding result in* [4].

In case $\beta = 1, \mu = 1, \nu = 1$ *in* (41) *and* (42), *we obtain corresponding result in* [11].

7. Transformation and summation formulas

Theorem 7.1. *The following formulas for the extended hypergeometric and confluent hypergeometric function holds:*

$$F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2; \theta_3; \omega) = (1-\omega)^{-a} F_{\alpha,\beta}^{p,\mu,\nu}\left(\theta_1, \theta_3 - \theta_2; \theta_3; -\frac{\omega}{1-\omega}\right) \tag{44}$$

$$F_{\alpha,\beta}^{p,\mu,\nu}\left(\theta_1, \theta_2; \theta_3; 1 - \frac{1}{\omega}\right) = \omega^a F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_3 - \theta_2; \theta_3; 1-\omega) \tag{45}$$

$$F_{\alpha,\beta}^{p,\mu,\nu}\left(\theta_1, \theta_2; \theta_3; \frac{\omega}{1+\omega}\right) = (1+\omega)^a F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_3 - \theta_2; \theta_3; -\omega) \tag{46}$$

$$\Phi_{\alpha,\beta}^{p,\mu,\nu}(\theta_2, \theta_3; \omega) = e^{\omega} \Phi_{\alpha,\beta}^{p,\mu,\nu}(\theta_3 - \theta_2; \theta_3; -\omega). \tag{47}$$

$$(p \in \mathbb{R}_0^+, \mu, \nu \in \mathbb{R}^+; \alpha, \beta \in \mathbb{R}^+; |\omega| < 1; \Re(\theta_3) > \Re(\theta_2) > 0),$$

Proof. Replacing t by 1- t and substituting

$$[1 - \omega(1-t)]^{-\theta_1} = (1-\omega)^{-\theta_1}\left(1 + \frac{\omega}{1-\omega} t\right)^{-\theta_1}$$

in (38), we obtain

$$F_{\alpha,\beta}^{p,\mu,\nu}(\theta_1, \theta_2; \theta_3; \omega) = \frac{(1-\omega)^{-\theta_1}}{B(\theta_2, \theta_3 - \theta_2)} \times \int_0^1 t^{\theta_2 - 1}(1-t)^{\theta_3 - \theta_2 - 1}\left(1 + \frac{\omega}{1-\omega} t\right)^{-\theta_1} E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right) dt, \tag{48}$$

$$= \frac{(1-\omega)^{-\theta_1}}{B(\theta_2, \theta_3 - \theta_2)} \times \int_0^1 t^{\theta_2 - 1}(1-t)^{\theta_3 - \theta_2 - 1}\left(1 - \frac{-\omega}{1-\omega} t\right)^{-\theta_1} E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right) dt, \tag{49}$$

In view of (38), we get the desired result (44).
Replacing ω by $1 - \frac{1}{\omega}$ and $\frac{\omega}{1+\omega}$ in (44) yield (45) and (46) respectively. □

Similarly as (44), we can establish (47).

Remark 7.1. *In case* $\alpha = 1, \beta = 1, \mu = 1, \nu = 1$ *in* (44) *and* (47), *we obtain corresponding result in* [4].

In case $\beta = 1, \mu = 1, \nu = 1$ *in* (44) *and* (47), *we obtain corresponding result in* [11].

Theorem 7.2. *The following summation formula hold:*

$$F^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2;\theta_3;\,1)=\frac{B^{p,\mu,\nu}_{\alpha,\beta}(\theta_2,\theta_3-\theta_1-\theta_2)}{B(\theta_2,\theta_3-\theta_2)} \tag{50}$$

$$(p\in\mathbb{R}^+_0,\alpha,\beta,\mu,\nu,\in\mathbb{R}^+;\Re(\theta_3-\theta_1-\theta_2)>0).$$

Proof. Putting $\omega=1$ in (38) and using the definition (15), we obtain desired result (50). □

Remark 7.2. *In case $\alpha=1,\beta=1,\mu=1,\nu=1$ with p=0 in* (50) *, we obtain Gauss summation formula for ${}_2F_1$.*

$${}_2F_1(\theta_1,\theta_2;\theta_3;1)=\frac{\Gamma(\theta_3)\Gamma(\theta_3-\theta_1-\theta_2)}{\Gamma(\theta_3-\theta_1)\Gamma(\theta_3-\theta_2)},$$

$$(\Re(\theta_3-\theta_1-\theta_2)>0).$$

8. A generating function for $F^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2;\theta_3;\,\omega)$

Theorem 8.1. *The following generating function for $F^{p,\mu,\nu}_{\alpha,\beta}(\theta_1,\theta_2;\theta_3;\,\omega)$ hold:*

$$\sum_{k=0}^{\infty}(\theta_1)_k\,F^{p,\mu,\nu}_{\alpha,\beta}(\theta_1+k,\theta_2;\theta_3;\omega)\,\frac{t^k}{k!}=(1-t)^{-\theta_1}\,F^{p,\mu,\nu}_{\alpha,\beta}\left(\theta_1,\theta_2;\theta_3;\,\frac{\omega}{1-t}\right) \tag{51}$$

$$(\alpha,\beta,\mu,\nu\in\mathbb{R}^+;p\in\mathbb{R}^+_0,|t|<1).$$

Proof. Let L be the left hand side (L.H.S) of (51). From (36), we have

$$L=\sum_{k=0}^{\infty}(\theta_1)_k\left(\sum_{n=0}^{\infty}\frac{(\theta_1+k)_n\,B^{p,\mu,\nu}_{\alpha,\beta}(\theta_2+n,\theta_3-\theta_2)}{B(\theta_2,\theta_3-\theta_2)}\,\frac{\omega^n}{n!}\right)\frac{t^k}{k!}. \tag{52}$$

Using the identity $(a)_n(a+n)_k=(a)_k(a+k)_n$, we get

$$L=\sum_{n=0}^{\infty}(\theta_1)_n\,\frac{B^{p,\mu,\nu}_{\alpha,\beta}(\theta_2+n,\theta_3-\theta_2)}{B(\theta_2,\theta_3-\theta_2)}\left(\sum_{k=0}^{\infty}(\theta_1+n)_k\frac{t^k}{k!}\right)\frac{\omega^n}{n!},$$

$$=\sum_{n=0}^{\infty}(\theta_1)_n\,\frac{B^{p,\mu,\nu}_{\alpha,\beta}(\theta_2+n,\theta_3-\theta_2)}{B(\theta_2,\theta_3-\theta_2)}\,(1-t)^{-\theta_1-n}\frac{\omega^n}{n!},$$

$$=(1-t)^{-\theta_1}\sum_{n=0}^{\infty}(\theta_1)_n\,\frac{B^{p,\mu,\nu}_{\alpha,\beta}(\theta_2+n,\theta_3-\theta_2)}{B(\theta_2,\theta_3-\theta_2)}\left(\frac{\omega}{1-t}\right)^n\frac{1}{n!}, \tag{53}$$

Finally by using (36) in (53), we get the right side of (51). □

Remark 8.1. *In case $\alpha=1,\beta=1,\mu=1,\nu=1$ in* (51)*, we obtain corresponding result in* [9].

In case $\beta=1,\mu=1,\nu=1$ in (51)*, we obtain corresponding result in* [11].

9. Conclusions

In this paper, we investigated a new extension of beta function by generalizing the exponential function to Mittag-Leffler function. Using the extension of beta function, we developed a new extension of generalized hypergeometric function and confluent hypergeometric function. The results presented in this paper can be specialized to yield several new and previously known definitions and their corresponding properties. We also remarked that the generating function obtained in (51) is interesting due to several special functions and polynomials, in particular, Fox-H function, Jacobi and Laguerre polynomials can be expressed in terms of hypergeometric and other related functions, which can be easily exploited to obtain other interesting generating functions.

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Nabiullah Khan is an associate professor in the department of Applied Mathematics, Faculty of Engineering and Technology in Aligarh Muslim University and his research interest includes applicable analysis, special functions specially generating functions and integral transform.

Saddam Husain is currently pursuing his Ph.D in the Department of Applied Mathematics, Faculty of Engineering and Technology in Aligarh Muslim University under the supervision of Dr. Nabiullah Khan and his research interests in special functions. He completed M.Sc in pure Mathematics at the same university.