TWMS J. App. and Eng. Math. V.12, N.1, 2022, pp. 382-387

SOME INEQUALITIES FOR THE GRAPH ENERGY OF DISTANCE LAPLACIAN MATRIX

GÜLİSTAN KAYA GÖK[1], §

ABSTRACT. In this paper, the distance laplacian energy for distance matrix is examined. Some bounds for the laplacian eigenvalues of distance matrix are expanded including the distances, the vertices and the edges. Indeed, different inequalities for the distance laplacian energy are found out.

Keywords: Distance matrix, Laplacian distance energy.

AMS Subject Classification: 05C22, 05C50

1. INTRODUCTION

Let G be a connected graph with the set of vertices $V(G) = \{v_1, v_2, ..., v_n\}$ and the set of edges $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. If any vertices v_i and v_j are adjacent, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i denoted by d_i, is the number of the vertices adjacent to v_i. Let d_{ij} be the distance between vertices v_i and v_j.

The distance matrix is the core of theories in this paper. This matrix is a symmetric, square, nonnegative matrix and it is described as a matrix of distances between each pair of vertices. The investigation of the distance matrix started in the 1970s with Graham and others. They examined a communication network problem with sequence of loops in complete graph, tree and cycle.[4] In addition, the distance matrix was applied in various fields of chemistry. For example, Wiener index was derived by the help of distance matrix. [8] Also, the graph distance energy was formed by eigenvalues of the distance matrix.[7]

The distance matrix is represented by $D_e(G) = [d_{ij}]$ in this paper. Let the eigenvalues of $D_e(G)$ be λ_1^D,λ_2^D,...,λ_n^D. The eigenvalues of this symmetric matrix are real and $\lambda_1^D \geq \lambda_2^D \geq ... \geq \lambda_n^D$. By these inequalities, the distance energy of $E_D = E_D(G)$ of a graph G is defined as [3] $E_D(G) = \sum_{i=1}^{n} |\lambda_i^D|$.

The laplacian matrix $L(G)$ is described with $L(G) = D(G) - A(G)$ where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix of the vertex degrees. $A(G)$ and $L(G)$ are all real symmetric matrices. Thus, their eigenvalues are real numbers. Define them

[1] Department of Mathematics Education, Faculty of Education, Hakkari University, Hakkari, Turkey.
e-mail: gulistankayagok@hakkari.edu.tr; ORCID: https://orcid.org/0000-0001-9059-1606.
§ Manuscript received: February 21, 2020; accepted: June 03, 2020.

first $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$. The graph laplacian energy $LE(G)$ is described by $LE = LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$. (See in [5].)

The distance laplacian energy depends on eigenvalues of distance laplacian matrix. The distance laplacian matrix equals the difference between $D = diag(d_1, d_2, ..., d_n)$ and Distance Matrix=D_e. This matrix is defined as $LM = D - D_e$ in [1]. Let the distance laplacian eigenvalues of $D(G)$ be denoted by μ_1^D,μ_2^D,...,μ_n^D. Since $LM(G)$ is symmetric, its eigenvalues are real numbers. Thus, $\mu_1^D \geq \mu_2^D \geq ... \geq \mu_n^D$. The distance laplacian energy is defined in [9] as $LE_D = LE_D(G) = \sum_{i=1}^{n} |\mu_i^D - \frac{1}{n}\sum_{j=1}^{n} D_j|$. Since $\sum_{j=1}^{n} D_j$ is the sum of the distances in the distance matrix, then it is specified by $LE_D(G) = \sum_{i=1}^{n} |\mu_i^D - \frac{2m}{n}|$ in this paper. For complete graph K_n, $LE_D(K_n) = 2n(n-1)$ in [9].

The narrative order of this study is as follows. Firstly, some lemmas are given for use in main results. After, eigenvalues of $LD(G)$ are examined and the important connections are formed in terms of the distances and the eigenvalues. Also, some bounds for distance laplacian energy are obtained using different relations. Then, interesting inequalities are found for the complement of distance laplacian energy. In addition, two inequalities are presented about laplacian distance energy of cartesian and inner product of two distance laplacian matrices.

2. Preliminaries

In order to achieve the desired inequalities, we use three existing results:

Lemma 2.1 (6). *Let $q = (q_i)$ be a sequence, $q \in R^+$ and $a = (a_i), b = (b_i) \in R^+$. Then,*

$$\sum_{i=1}^{n} q_i \sum_{i=1}^{n} q_i a_i b_i \geq \sum_{i=1}^{n} q_i a_i \sum_{i=1}^{n} q_i b_i.$$

Lemma 2.2 (6). *If $a_i, b_i \in R^+$, $1 \leq i \leq n$, then*

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r+R)\sum_{i=1}^{n} a_i b_i$$

where $ra_i \leq b_i \leq Ra_i$.

Lemma 2.3 (9). *Let G be a connected graph of order n. Then*

$$i) \sum_{i=1}^{n}(\epsilon_i) = 0,$$

$$ii) \sum_{i=1}^{n}(\epsilon_i)^2 = 2(T + m\alpha).$$

where $\epsilon_i = \mu_i^D - \frac{2m}{n}$, $T = \sum_{1 \leq i < j \leq n}(d_{ij})^2$ and $\alpha = \frac{2m}{n}$.

3. MAIN SERULTS

Eigenvalues are important for the structure of a matrix. In this section, some relations are given for eigenvalues of distance laplacian matrix. Also, different bounds for distance laplacian energy are determined by the eigenvalues of distance laplacian matrix. Some bounds are sharp. (See Theorem 3.2, 3.3. and 3.4)

Theorem 3.1. *Let G be a connected graph with the eigenvalues μ_1^D. Then,*

$$\mu_1^D \leq \sqrt{2(\frac{n-1}{n})(T+m\alpha)} + \alpha.$$

Proof. It is known that $\epsilon_1 = -\sum_{i=2}^{n}(\epsilon_i)$. Using this result, it gets

$$|\epsilon_1| \leq \sum_{i=2}^{n}|\epsilon_i| \leq \sqrt{n-1}\sqrt{\sum_{i=2}^{n}(\epsilon_i)^2}.$$

Thus,

$$(\epsilon_1)^2 \leq (n-1)[\sum_{i=1}^{n}(\epsilon_i)^2 - (\epsilon_1)^2].$$

Also, it is stated that $\frac{n}{n-1}(\epsilon_1)^2 \leq \sum_{i=1}^{n}(\epsilon_i)^2$. Lemma 2.3 says that $\frac{n}{n-1}(\epsilon_1)^2 \leq 2(T+m\alpha)$. Since $\epsilon_i = \mu_i^D - \frac{2m}{n}$, then $\mu_i^D = \epsilon_i + \frac{2m}{n} = \epsilon_i + \alpha$. Thus, $\mu_1^D = \epsilon_1 + \alpha$. Here, μ_1^D is the spectral radius (greatest eigenvalue) of G. Hence,

$$\mu_1^D \leq \sqrt{2(\frac{n-1}{n})(T+m\alpha)} + \alpha.$$

□

Theorem 3.2. *Let G be a connected graph with n nodes and m edges. Then $LE_D(G) \geq \sqrt{2n(T+m\alpha)}$.*

Proof. Let $q_i = |\epsilon_i|$, $a_i = |\epsilon_i|$, $b_i = \frac{1}{|\epsilon_i|}$. Observing the Lemma 2.1, it gets

$$\begin{aligned}\sum_{i=1}^{n}|\epsilon_i|\sum_{i=1}^{n}|\epsilon_i||\epsilon_i|\frac{1}{|\epsilon_i|} &\geq \sum_{i=1}^{n}|\epsilon_i||\epsilon_i|\sum_{i=1}^{n}|\epsilon_i|\frac{1}{|\epsilon_i|}\\ (\sum_{i=1}^{n}|\epsilon_i|)^2 &\geq \sum_{i=1}^{n}|\epsilon_i|^2\sum_{i=1}^{n}1.\end{aligned}$$

Thus, the above inequality requires $LE_D^2(G) \geq n\sum_{i=1}^{n}|\epsilon_i|^2 = 2n(T+m\alpha)$. Hence, $LE_D(G) \geq \sqrt{2n(T+m\alpha)}$.

□

Theorem 3.3. *Let G be a connected graph with the maximum degree Δ. Then,*

$$LE_D(G) \geq \frac{2(T+m\alpha) + n|\alpha(\Delta-\alpha)|}{\Delta}.$$

Proof. For $a_i = 1$, $b_i = |\epsilon_i|$, $r = |\epsilon_n|$, $R = |\epsilon_1|$ where the summation is performed over all edges of graph G. Thus, the Lemma 2.2 becomes

$$\sum_{i=1}^{n}|\epsilon_i|^2 + |\epsilon_n||\epsilon_1|\sum_{i=1}^{n}1^2 \leq (|\epsilon_1| + |\epsilon_n|)\sum_{i=1}^{n}|\epsilon_i|.$$

According to Lemma 2.3, the inequality transforms into

$$2(T+m\alpha)+n|\epsilon_n||\epsilon_1| \leq (|\epsilon_1|+|\epsilon_n|)LE_D(G).$$

From the above, it hence

$$LE_D(G) \geq \frac{2(T+m\alpha)+n|\epsilon_n||\epsilon_1|}{|\epsilon_1|+|\epsilon_n|}.$$

Since, $\mu_1 \leq \Delta$ and $\mu_n \geq 0$ then $|\epsilon_1| = |\Delta-\alpha|$ and $|\epsilon_n| = |\alpha|$. Hence,

$$LE_D(G) \geq \frac{2(T+m\alpha)+n|\alpha(\Delta-\alpha)|}{\Delta}.$$

□

Theorem 3.4. *Let G be a connected graph and $LE_D(\bar{G})$ be the complement of $LE_D(G)$. Then*

$$LE_D(G)+LE_D(\bar{G}) \leq \sqrt{2(n^2-n)-(\frac{m}{n}(n^2-n-2m))^2-2d_{ij}[2\sum d_{ij}-d_{ij}]}.$$

Proof. By the Cauchy-Schwartz inequality, $(LE_D(G)+LE_D(\bar{G}))^2 \leq (LE_D(G))^2+(LE_D(\bar{G}))^2$. Hence,

$$\begin{aligned}(LE_D(G)+LE_D(\bar{G}))^2 &\leq \sum_{i=1}^{n}(|\epsilon_i|)^2+\sum_{i=1}^{n}(|\bar{\epsilon_i}|)^2\\ &=\sum_{i=1}^{n}((\mu_i^D)^2+(\bar{\mu_i^D})^2)-\frac{4m}{n}\sum_{i=1}^{n}\mu_i^D-\frac{4\bar{m}}{n}\sum_{i=1}^{n}\bar{\mu_i^D}+\frac{4(m^2+\bar{m}^2)}{n^2}.\end{aligned}$$

Since $\sum_{i=1}^{n}(\epsilon_i) = \sum_{i=1}^{n}\mu_i^D-2m = 0$, then $\sum_{i=1}^{n}\mu_i^D = 2m$ and $\sum_{i=1}^{n}\bar{\mu_i^D} = 2\bar{m}$. This means that,

$$(LE_D(G)+LE_D(\bar{G}))^2 \leq \sum_{i=1}^{n}((\mu_i^D)^2+(\bar{\mu_i^D})^2)-\frac{4}{n^2}(m^2+\bar{m}^2).$$

Knowing that $2(m+\bar{m}) = (n^2-n)$, it yields $\bar{m} = \dfrac{n^2-n-2m}{2}$. Thus,

$$(LE_D(G)+LE_D(\bar{G}))^2 \leq \sum_{i=1}^{n}((\mu_i^D)^2+(\bar{\mu_i^D})^2)-\frac{4}{n^2}(m^2+(\frac{n^2-n-2m}{2})^2).$$

In the sequel, it is outlined that $\sum_{i=1}^{n}(\bar{\mu_i^D})^2 = 2\sum(\bar{d_{ij}})^2 = 2\sum(\sum d_{ij}-d_{ij})^2 = 2\sum((\sum d_{ij})^2-2d_{ij}\sum d_{ij}+(d_{ij})^2)$.

Also, $d_{ij} \geq 1$ for $i \neq j$ and there are $\dfrac{(n^2-n)}{2}$ vertices. Therefore, $\sum_{i=1}^{n}(\bar{\mu_i^D})^2 = (n^2-n)-2d_{ij}[2\sum d_{ij}-d_{ij}]$. Then, the inequality is summarized in the following that

$$\begin{aligned}(LE_D(G)+LE_D(\bar{G}))^2 &\leq 2\sum(d_{ij})^2+(n^2-n)-2d_{ij}[2\sum d_{ij}-d_{ij}]\\ &\leq 2(n^2-n)-2d_{ij}[2\sum d_{ij}-d_{ij}]-\frac{4m^2(n^2-n-2m)^2}{4n^2}.\end{aligned}$$

Hence,

$$LE_D(G)+LE_D(\bar{G}) \leq \sqrt{2(n^2-n)-\frac{m^2}{n^2}(n^2-n-2m)^2-2d_{ij}[2\sum d_{ij}-d_{ij}]}.$$

□

Theorem 3.5. *Let G be a connected, (n, m) graph and LD_1, LD_2 be two laplacian distance matrices. Then,*

$$LE_{LD_1 \times LD_2}(G) \leq \sqrt{4(T + m\alpha) + 2LE_{LD_1}(G)LE_{LD_2}(G)}.$$

Proof. Let $\mu_i^{D_1}$ and $\mu_j^{D_2}$ be the laplacian eigenvalues of LD_1 and LD_2, respectively. Let $\mu_i^{D_1} - \dfrac{2m}{n} = \epsilon_i$ and $\mu_j^{D_2} - \dfrac{2m}{n} = \epsilon_j$, $i, j = 1, 2, ..., n$. By the survey of properties of $LD_1 \times LD_2$ given in [2], we have

$$\begin{aligned} LE_{LD_1 \times LD_2}(G) &= \sum_{i,j=1}^{n} |\epsilon_i + \epsilon_j| \\ &= \sqrt{(\sum_{i,j=1}^{n} |\epsilon_i + \epsilon_j|)^2} \\ &\leq \sqrt{(\sum_{i=1}^{n} |\epsilon_i|)^2 + (\sum_{j=1}^{n} |\epsilon_j|)^2 + 2(\sum_{i=1}^{n} |\epsilon_i| \sum_{j=1}^{n} |\epsilon_j|)} \\ &\leq \sqrt{4(T + m\alpha) + 2LE_{LD_1}(G)LE_{LD_2}(G)}. \end{aligned}$$

□

Corollary 3.1. *Let G be a connected, (n, m) graph and LD_1, LD_2 be two laplacian distance matrices. Then,*

$$LE_{LD_1 \times LD_2}(K) \leq \sqrt{4(T + m\alpha) + 8(n(n-1))^2}.$$

Proof. Since $LE_D(K) = 2n(n-1)$, then the corollary is clear by the Theorem 3.5.

□

Theorem 3.6. *Let G be a connected, (n, m) graph and LD_1, LD_2 be two laplacian distance matrices. Then,*

$$LE_{LD_1 \otimes LD_2}(G) \leq 2(T + m\alpha).$$

Proof. By the result of properties of $D_1 \otimes D_2$ given in [2], the inequality becomes

$$\begin{aligned} LE_{LD_1 \otimes LD_2}(G) &= \sum_{i,j=1}^{n} |\epsilon_i \epsilon_j| \\ &\leq \sqrt{(\sum_{i=1}^{n} |\epsilon_i|)^2 (\sum_{j=1}^{n} |\epsilon_j|)^2} \\ &\leq \sqrt{(2(T + m\alpha))^2}. \end{aligned}$$

Thus, the proof is completed with

$$LE_{LD_1 \otimes LD_2}(G) \leq 2(T + m\alpha).$$

□

4. Conclusion

The distance matrix is the focus of theorems in this paper. Distance laplacian matrix is obtained by distance matrix. Some relations are improved for eigenvalues of distance laplacian matrix. Considering that the energy is a point of application of eigenvalues and its importance in molecular graph theory, the energy of distance laplacian matrix is studied. Some inequalities for this energy are achieved. These inequalities aim to contribute greatly to molecular graph theory.

References

[1] Aouchiche M., Hansen, P., Two Laplacians for the distance matrix of a graph, Linear Algebra Appl. 439 (2013), 21–33.
[2] Bapat, R.B. Graphs and Matrices,Indian Statistical Institute, New Delhi 110016, India.
[3] Das, K.C., Güngör, A.D., and Cevik, A.S., On Kirchhoff Index and Resistance–Distance Energy of a Graph, MATCH Commun.Math. Comput. Chem., 67 (2012), 541-556.
[4] Graham, R. L., Pollak, H. O., On the addressing problem for loop switching, The Bell System Technical Journal, 1971, 50, 2495-2519.
[5] Merris, R., Laplacian matrices of graphs: a survey, Lin. Algebra Appl. 197-198 (1994) 143-176.
[6] Milovanović, I. Ž., Milovanović, E. I., Matejić, M., Some Inequalities for General Sum-Connectivity Index, MATCH Commun.Math. Comput. Chem., 79 (2018), 477-489.
[7] Ramane, H. S., Revankar, D. S., Gutman, I., Rao, S. B., Acharya, B. D., Walikar, H. B., Bounds for the Distance Energy of a Graph, Kragujevac Journal Mathematics, 31 (2008) 59-68.
[8] Wiener, H., Structural determination of paraffin boiling points, Journal of the American Chemical Society, 1947, 69(1), 17-20.
[9] Yang, J., You, L., Gutman, I., Bounds On The Distance Laplacian Energy Of Graphs, Kragujevac Journal of Mathematics, 37(2) (2013), 245–255.

Gülistan Kaya Gök for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.9, N.4.