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# SOME SUMMATION THEOREMS FOR APPELL FUNCTION OF FIRST KIND HAVING ARGUMENTS ± 1

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Abstract. The motive of this paper is to find the closed forms of the summation theorems for Appell's double hypergeometric function of first kind with suitable convergence conditions, having the arguments ±1.

Keywords: Generalized hypergeometric function, Appell function of first kind, summation theorems of Kummer, Whipple, Dixon, Watson and Saalschütz.

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## 1. Introduction

In the usual notations, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. For definitions of Pochhammer symbol, generalized hypergeometric function ${}_pF_q$, we refer monumental work of Abramowitz and Stegun [1], Appell and Kampé de Fériet [3],Luke [8], Rainville [14]and Srivastava-Manocha [17].

The Appell's function of first kind is defined as

$$F_1[A\ ;\ B,\ C\ ;\ D\ ;\ x,\ y] = \sum_{r,s=0}^{\infty} \frac{(A)_{r+s}(B)_r(C)_s}{(D)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!} \tag{1}$$

$$= \sum_{r=0}^{\infty} \frac{(A)_r(B)_r x^r}{(D)_r r!} {}_2F_1\left[\begin{array}{r} A+r, C; \\ D+r; \end{array} y\right] = \sum_{s=0}^{\infty} \frac{(A)_s(C)_s}{(D)_s} \frac{y^s}{s!} {}_2F_1\left[\begin{array}{r} A+s, B; \\ D+s; \end{array} x\right], \tag{2}$$

$$(\max\{|x|, |y|\} < 1 \text{ and } D \in \mathbb{C}\backslash\mathbb{Z}_0^-).$$

In our investigations we shall use the following standard results.

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Series manipulation formula $\Big($[17, p.100 Lemma 1]$\Big)$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Phi(r,s)=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\Phi(r-s,s) \tag{3}$$

provided that involved double series are absolutely convergent.

Series decomposition Identity$\Big($[17, p.200 Equation (1)]$\Big)$

$$\sum_{r=0}^{\infty}\Psi(r)=\sum_{r=0}^{\infty}\Psi(2r)+\sum_{r=0}^{\infty}\Psi(2r+1) \tag{4}$$

Kummer's first summation Theorem $\Big($[6, p.852 Equation (1.3)] $\Big)$

$$ {}_2F_1\left[\begin{array}{r} \alpha,\beta; \\ 1+\alpha-\beta; \end{array} -1\right]=\frac{\Gamma(1+\alpha-\beta)\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\frac{\alpha}{2}-\beta)\Gamma(1+\alpha)} \tag{5}$$

$$\left(1+\alpha-\beta\in\mathbb{C}\backslash\mathbb{Z}_0;\Re(\beta)<1\right)$$

Put $\alpha=-r$ in equation(5), we get

$$ {}_2F_1\left[\begin{array}{r} -r,\beta; \\ 1-r-\beta; \end{array} -1\right]=\frac{\Gamma(1-r-\beta)\Gamma(1+r)}{\Gamma(1-\frac{r}{2}-\beta)\Gamma(1+\frac{r}{2})}\cos\left(\frac{\pi r}{2}\right) \tag{6}$$

where $r=0,1,2,\dots$ and $\beta\in\mathbb{C}\backslash\mathbb{Z}$.

Dixon's theorem$\Big($ [4, p.12, Equation 3.1(1)]$\Big)$

$$ {}_3F_2\left[\begin{array}{r} \lambda,\mu,\omega; \\ 1+\lambda-\mu,1+\lambda-\omega; \end{array} 1\right]=\frac{\Gamma(1+\lambda-\mu)\Gamma(1+\lambda-\omega)\Gamma(1+\frac{\lambda}{2}-\mu-\omega)\Gamma(1+\frac{\lambda}{2})}{\Gamma(1+\frac{\lambda}{2}-\mu)\Gamma(1+\frac{\lambda}{2}-\omega)\Gamma(1+\lambda-\mu-\omega)\Gamma(1+\lambda)} \tag{7}$$

$$\left(\Re(\tfrac{\lambda}{2}-\mu-\omega)>-1;1+\lambda-\mu,1+\lambda-\omega\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

Whipple's theorem$\Big($ [4, p.16, Equation 3.4(1)]$\Big)$

$$ {}_3F_2\left[\begin{array}{r} \lambda,1-\lambda,\omega; \\ \sigma,1+2\omega-\sigma; \end{array} 1\right]=\frac{\pi\Gamma(\sigma)\Gamma(1+2\omega-\sigma)}{2^{2\omega-1}\Gamma(\frac{\lambda+\sigma}{2})\Gamma(\omega+\frac{1+\lambda-\sigma}{2})\Gamma(\frac{1-\lambda+\sigma}{2})\Gamma(1+\omega-\frac{\lambda+\sigma}{2})} \tag{8}$$

$$\left(\Re(\omega)>0;\sigma,1+2\omega-\sigma\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

Watson's theorem$\Big($[4, p.16, Equation 3.3(1)]$\Big)$

$$ {}_3F_2\left[\begin{array}{r} \lambda,\mu,\omega; \\ \frac{1+\lambda+\mu}{2},2\omega; \end{array} 1\right]=\frac{\sqrt{\pi}\Gamma(\omega+\frac{1}{2})\Gamma(\frac{1+\lambda+\mu}{2})\Gamma(\omega+\frac{1-\lambda-\mu}{2})}{\Gamma(\frac{1+\lambda}{2})\Gamma(\frac{1+\mu}{2})\Gamma(\omega+\frac{1-\lambda}{2})\Gamma(\omega+\frac{1-\mu}{2})} \tag{9}$$

$$\left(\Re(1-\lambda-\mu+2\omega)>0;\tfrac{1+\lambda+\mu}{2},2\omega\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

Saalschütz's theorem $\Big($[4, p.9, Equation 2.2(1)]$\Big)$

$$ {}_3F_2\left[\begin{array}{r} -N,\lambda,\mu; \\ \omega,\lambda+\mu-\omega-N+1; \end{array} 1\right]=\frac{(\omega-\lambda)_N(\omega-\mu)_N}{(\omega)_N(\omega-\lambda-\mu)_N} \tag{10}$$

$$\left(\lambda,\mu,\omega,\lambda+\mu-\omega-N+1\in\mathbb{C}\backslash\mathbb{Z}_0^-\right);\ \text{N}=0,1,2,3\dots$$

Legendre's duplication formula

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) \tag{11}$$

$$(2z \neq 0, -1, -2, -3, \dots)$$

$$\Gamma(z+1) = z\Gamma(z) \tag{12}$$

$$\Gamma(z)\Gamma(1-z) = \tfrac{\pi}{\sin(\pi z)}; z \neq 0, \pm 1, \pm 2, \pm 3 \dots$$

$$\sin(\tfrac{3\pi}{8}) = \cos(\tfrac{\pi}{8}) = \sqrt{\left(\tfrac{\sqrt{2}+1}{2\sqrt{2}}\right)}, \sin(\tfrac{\pi}{8}) = \sqrt{\left(\tfrac{\sqrt{2}-1}{2\sqrt{2}}\right)},$$

$$\cos(\pi r) = (-1)^r, \cos(2r+1)\tfrac{\pi}{2} = 0 \text{ , where } r = 0, \pm 1, \pm 2, \dots$$

$$(\alpha)_{-r} = \tfrac{(-1)^r}{(1-\alpha)_r}, \alpha \neq 0, \pm 1 \pm 1, \pm 2, \dots$$

$$(\alpha)_{2r} = 2^{2r}(\tfrac{\alpha}{2})_r(\tfrac{\alpha+1}{2})_r$$

## 2. Section

Using series iteration technique, we can obtain

$$F_1[A\ ;\ B, B\ ;\ D\ ;\ x,\ -x] = {}_3F_2\left[\begin{array}{c} \frac{A}{2}, \frac{A+1}{2}, B; \\ \frac{D}{2}, \frac{D+1}{2}; \end{array} x^2\right], \tag{13}$$

$$\left(|x| < 1\ ;\ A \in \mathbb{C};\ B \in \mathbb{C}\backslash\mathbb{Z}\ ;\ D \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

Proof of (13)

Consider the series

$$F_1[A\ ;\ B,\ B\ ;\ D\ ;\ x,\ -x] = \sum_{r,s=0}^{\infty} \frac{(A)_{r+s}(B)_r(B)_s}{(D)_{r+s}} \frac{(-1)^s x^{r+s}}{r!s!}$$

Replacing r by (r-s) and applying Series manipulation formula (3), we get

$$F_1[A\ ;\ B,\ B\ ;\ D\ ;\ x,\ -x] = \sum_{r=0}^{\infty} \frac{(A)_r(B)_r}{(D)_r} \frac{x^r}{(1)_r} \sum_{s=0}^{r} \frac{(B+r)_{-s}(B)_s}{(1+r)_{-s}} \frac{(-1)^s}{s!}$$

$$= \sum_{r=0}^{\infty} \frac{(A)_r(B)_r x^r}{(D)_r r!} {}_2F_1\left[\begin{array}{c} -r, B; \\ 1-B-r; \end{array} -1\right].$$

Now using Kummer first summation theorem (6), we get

$$F_1[A\ ;\ B,\ B\ ;\ D\ ;\ x,\ -x] = \sum_{r=0}^{\infty} \frac{(A)_r(B)_r x^r}{(D)_r} \frac{\Gamma(1-r-B)\cos(\frac{\pi r}{2})}{\Gamma(1-\frac{r}{2}-B)\Gamma(1+\frac{r}{2})}$$

Now using series decomposition identity (4), we get

$$F_1[A\ ;\ B,\ B\ ;\ D\ ;\ x,\ -x] = \sum_{r=0}^{\infty} \frac{(A)_{2r}(B)_{2r}x^{2r}}{(D)_{2r}} \frac{\Gamma(1-2r-B)(-1)^r}{\Gamma(1-r-B)r!} + 0$$

$$= \sum_{r=0}^{\infty} \frac{(A)_{2r}(B)_{2r}x^{2r}(1-B)_{-2r}(-1)^r}{(D)_{2r}r!(1-B)_{-r}} = \sum_{r=0}^{\infty} \frac{(\frac{A}{2})_r(\frac{A+1}{2})_r(B)_r x^{2r}}{(\frac{D}{2})_r(\frac{D+1}{2})_r r!}$$

$$= {}_3F_2\left[\begin{array}{c} \frac{A}{2}, \frac{A+1}{2}, B; \\ \frac{D}{2}, \frac{D+1}{2}; \end{array} x^2\right].$$

## 3. Application of Reduction Formula (13) in Summation Theorems

When the values of parameters leading to the results which do not make sense, are tacitly excluded. Then

In the equation (13) put $A = a, B = b, D = 1 + 2b - a$ and $x = 1$, use the Dixon's summation theorem (3), we get

$$F_1\left[a; b,\ b\ ; 1+2b-a\ ; 1, -1\right] = \frac{\Gamma(\frac{1+b-2a}{2})\Gamma(1+2b-a)}{2^{2a}\Gamma(1+2b-2a)\Gamma(\frac{1+b}{2})}, \tag{14}$$

$$\left(\Re(2a-b) < 1\ ;\ (1+2b-a) \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (13) put $A = -2m, B = b, D = 1 + b - 2m$ and $x = 1$, use the Saalschütz's summation theorem (6), we get

$$F_1\left[-2m\ ;\ b,\ b\ ; 1+b-2m\ ; 1, -1\right] = \frac{(b)_{2m}}{(-b)_{2m}}, \tag{15}$$

$$\left((1+b-2m) \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m \in \mathbb{N}_0\right).$$

In the equation (13) put $A = -2m - 1, B = b, D = b - 2m$ and $x = 1$, use Saalschütz's summation theorem (6), we get

$$F_1\left[-2m-1\ ;\ b,\ b\ ; b-2m\ ; 1, -1\right] = \frac{(1+b)_{2m}}{(1-b)_{2m}}, \tag{16}$$

$$\left(\ (b-2m) \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m \in \mathbb{N}_0\right).$$

In the equation (13) put $A = \frac{2d-3}{2}$, $B = \frac{d+1}{4}$, $D = d$ and $x = 1$, use Watson's summation theorem (5), we get

$$F_1\left[\frac{2d-3}{2}; \frac{d+1}{4},\ \frac{d+1}{4}\ ;\ d\ ; 1, -1\right] = \frac{(2-\sqrt{2})(1-d)\Gamma(\frac{d}{2})\sqrt{(\pi)}}{\Gamma(\frac{2d+1}{8})\Gamma(\frac{2d+3}{8})\sin(\pi(\frac{d+3}{4}))}, \tag{17}$$

$$\left(\Re(d) < 5\ ;\ d \neq 1,\ d \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (13) put $A = \frac{2d-1}{2}$, $B = \frac{d}{4}$, $D = d$ and $x = 1$, use Watson's summation theorem (5), we get

$$F_1\left[\frac{2d-1}{2}; \frac{d}{4},\ \frac{d}{4}\ ;\ d\ ; 1, -1\right] = \frac{\sqrt{\pi}\Gamma(\frac{d+1}{2})\sqrt{(\frac{1+\sqrt{2}}{2\sqrt{2}})}}{\Gamma(\frac{2d+3}{8})\Gamma(\frac{2d+5}{8})\sin(\pi(\frac{d+2}{4}))}, \tag{18}$$

$$\left(\Re(d) < 2\ ;\ d \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (13) put $A = \frac{d-2}{2}$, $B = \frac{3d+2}{4}$, $D = d$ and $x = 1$, use Watson's summation theorem (5), we get

$$F_1\left[\frac{d-2}{2}; \frac{3d+2}{4},\ \frac{3d+2}{4}\ ;\ d\ ; 1, -1\right] = \frac{\Gamma(\frac{d+1}{2})\Gamma(\frac{6-d}{8})}{2\sqrt{\pi}\Gamma(\frac{3d+6}{8})}, \tag{19}$$

$$\left(\Re(d) < 2\ ;\ d \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

In the equation (13) put $A = a, B = \frac{3a}{2}$, $D = 2a+1$ and $x = 1$, use Watson's summation theorem (5), we get

$$F_1\left[a\,;\frac{3a}{2},\ \frac{3a}{2}\ ;\ 2a+1\ ;1,-1\right] = \frac{\sqrt{\pi}\Gamma(\frac{2a+1}{2})}{\Gamma(\frac{a+2}{4})\Gamma(\frac{3a+2}{4})\cos(\frac{\pi a}{4})}, \tag{20}$$

$$\big(\Re(a) < 2\ ;\ (2a+1) \in \mathbb{C}\backslash\mathbb{Z}_0^-\big).$$

In the equation (13) put $A = \frac{d}{2}$, $B = \frac{3d-2}{4}, D = d$ and $x = 1$, use Watson's summation theorem (5), we get

$$F_1\left[\frac{d}{2}\,;\frac{3d-2}{4},\ \frac{3d-2}{4}\ ;\ d\ ;1,-1\right] = \frac{2^{\frac{d-2}{4}}\Gamma(\frac{d+1}{2})\Gamma(\frac{2-d}{4})}{\Gamma(\frac{3d+2}{8})\Gamma(\frac{6-d}{8})}, \tag{21}$$

$$\big(\Re(d) < 2\ ;\ d \in \mathbb{C}\backslash\mathbb{Z}_0^-\big).$$

In the equation (13) put $A = \frac{d+1}{2}$, $B = \frac{3d-7}{4}, D = d$ and $x = 1$, use Watson's summation theorem (5), we get

$$F_1\left[\frac{d+1}{2}\,;\frac{3d-7}{4},\ \frac{3d-7}{4}\ ;\ d\ ;1,-1\right] = \frac{2^{\frac{d-1}{4}}\Gamma(\frac{d}{2})\Gamma(\frac{5-d}{4})}{\Gamma(\frac{3d-3}{8})\Gamma(\frac{13-d}{8})}, \tag{22}$$

$$\big(\Re(d) < 5\ ;\ d \in \mathbb{C}\backslash\mathbb{Z}_0^-\big).$$

In the equation (13) put $A = \frac{1}{2}, B = b$, $D = \frac{4b+1}{2}$ and $x = 1$, use Whipple's summation theorem (4), we get

$$F_1\left[\frac{1}{2}\,;b,\ b\ ;\ \frac{4b+1}{2}\ ;1,-1\right] = \frac{\pi\Gamma(\frac{4b+1}{2})}{2^{3b-1}\Gamma(\frac{2b+1}{2})\Gamma(\frac{b+1}{2})\Gamma(\frac{b+1}{2})}, \tag{23}$$

$$\big(\Re(b) > 0\ ;\ \tfrac{4b+1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\big).$$

## 4. Conclusions

We conclude our present analysis by observing that several interesting summation theorems for Appell function of first kind can be derived in an analogous manner. Moreover, presented summation theorems should be beneficial to those who are interested in the field of applied mathematics and applied physics.

**Remark:** Summation Theorems can be applied for some other branches of applied mathematics such as numerical improvement of weighted quadrature rules [12] whose residue is controllable by some new mathematical inequalities [9, 10] and classical symmetric hypergeometric polynomials [11].

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