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HERMITE–HADAMARD INTEGRAL INEQUALITIES FOR LOG–CONVEX INTERVAL–VALUED FUNCTIONS ON CO–ORDINATES

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Abstract. In this paper, we give the notion of interval-valued log–convex functions on the co-ordinates on the rectangle from the plane. We establish Hermite-Hadamard and related inequalities for these classes of functions. Our results are refinements of several existing results in the field of Hermite-Hadamard inequalities. Some examples are also given to justify our new results.

Keywords: Interval–valued functions, Co-ordinated convex functions, Log co-ordinated convex functions and Hermite-Hadamard inequalities.

AMS Subject Classification: 26D15, 26B25, 26D10.

1. Introduction

The Hermite–Hadamard inequality discovered by C. Hermite and J. Hadamard, (see [9], [18, pp. 137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that, if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

Both inequalities in (1) hold in the reversed direction if f is concave. We note that Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite–Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and

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generalizations have been studied.
In [8], Dragomir established the following similar inequality of Hadamard type for the co-ordinated convex functions.

Theorem 1.1. *Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is convex on co-ordinates Δ. Then following inequalities holds:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
\leq\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx\right. \\
& \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (2)
\end{aligned}$$

In [3], Alomari and Darus gave the following inequalities for co-ordinated log-convex functions.

Theorem 1.2. *Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is log-convex on co-ordinates Δ. Then following inequalities holds:*

$$\begin{aligned}
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \exp\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln f(x,y)dydx\right] \\
&\leq \sqrt[4]{f(a,c)f(a,d)f(b,c)f(b,d)}. \qquad (3)
\end{aligned}$$

For more results related to (2) and (3) we refer ([1, 2, 12, 17]) and references therein.
On the other hand, several important inequalities (Hermite–Hadamard, Ostrowski, etc.) have been studied for the interval–valued functions in recent years. In [4, 5], Chalco–Cano et al. obtained Ostrowski type inequalities for interval–valued functions by using Hukuhara derivative for interval–valued functions. In [19], Román-Flores et al. established Minkowski and Beckenbach's inequalities for interval–valued functions. For the others, please see [6, 7, 10, 19, 20]. However, inequalities were studied for more general set–valued maps. For example, in [21], Sadowska gave the Hermite–Hadamard inequality. For the other studies, you can see [13, 16].

2. Preliminaries and known results

In this section we recall some basics definitions, results, notions and properties, which are used throughout the paper. We denote $\mathbb{R}_{\mathcal{I}}^{+}$ the family of all positive intervals of $\mathbb{R}$. The Hausdorff distance between $[\underline{X}, \overline{X}]$ and $[\underline{Y}, \overline{Y}]$ is defined as

$$d([\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}]) = \max\left\{\left|\underline{X} - \underline{Y}\right|, \overline{X} - \overline{Y}\right\}.$$

The $(\mathbb{R}_{\mathcal{I}}, d)$ is a complete metric space. For more details and basic notations on interval-valued functions see ([15, 22]).

It is remarkable that Moore [14] introduced the Riemann integral for the interval-valued functions. The set of all Riemann integrable interval-valued functions and real-valued functions on $[a,b]$ are denoted by $\mathcal{IR}_{([a,b])}$ and $\mathcal{R}_{([a,b])}$, respectively. The following theorem gives relation between (IR)–integrable and Riemann integrable (R–integrable) (see [15], pp. 131):

Theorem 2.1. *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval–valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$. $F \in \mathcal{IR}_{([a,b])}$ if and only if $\underline{F}(t)$, $\overline{F}(t) \in \mathcal{R}_{([a,b])}$ and*

$$(IR)\int_a^b F(t)dt = \left[(R)\int_a^b \underline{F}(t)dt, (R)\int_a^b \overline{F}(t)dt\right].$$

In [11], Guo et al. introduced a kind of log-convex interval–valued function as follows:

Definition 2.1. [11]*Let $h : [c,d] \to \mathbb{R}$ be a non-negative function, $(0,1) \subseteq [c,d]$ and $h \neq 0$. A function $F : [a,b] \to \mathbb{R}_I^+$ is said to be logarithmically interval-valued-h-convex function if for all $x, y \in [a,b]$ and $t \in [0,1]$, we have*

$$[F(x)]^{h(t)} [F(y)]^{h(1-t)} \subseteq F(tx + (1-t)y).$$

For brevity, we can denote these classes of functions by $F \in SX(\log -h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$.

Otherwise, Guo et al. obtained the following Hermite–Hadamard inequality for interval–valued log-convex functions by using h–convex:

Theorem 2.2. [11] *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^+$ be an interval–valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $F \in \mathcal{IR}_{([a,b])}$, $h : [0,1] \to \mathbb{R}$ be a non–negative function and $h\left(\frac{1}{2}\right) \neq 0$. If $F \in SX(\log -h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, then*

$$\left[F\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}} \supseteq \exp\left[\frac{1}{b-a}(IR)\int_a^b \ln F(x)dx\right] \supseteq [F(a)F(b)]^{\int_0^1 h(t)dt}. \tag{4}$$

Remark 2.1. *(i) If $h(t) = t$, then (4) reduces to the following result:*

$$F\left(\frac{a+b}{2}\right) \supseteq \exp\left[\frac{1}{b-a}(IR)\int_a^b \ln F(x)dx\right] \supseteq \sqrt{F(a)F(b)}. \tag{5}$$

(ii) If $h(t) = t^s$, then (4) reduces to the following result:

$$\left[F\left(\frac{a+b}{2}\right)\right]^{2^{s-1}} \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq [F(a)F(b)]^{s+1}.$$

In [24], Zhang et al. established following new results for interval-valued log-convex functions.

Theorem 2.3. [24]*Let* $F:[a,b]\to\mathbb{R}_{\mathcal{I}}^{+}$, $h:[0,1]\to\mathbb{R}^{+}$ *and* $h\left(\frac{1}{2}\right)\neq 0$. *If* $F\in SX(\log -h,[a,b],\mathbb{R}_{\mathcal{I}}^{+})$ *and* $F\in\mathcal{I}^{*}\mathcal{R}_{([a,b])}$, *then following inequalities hold:*

$$\begin{aligned}\left(F\left(\frac{a+b}{2}\right)\right)^{\frac{1}{2h^{2}\left(\frac{1}{2}\right)}} &\supseteq \Delta_{1}^{\frac{1}{4h\left(\frac{1}{2}\right)}}\\ &\supseteq \left(\int_{a}^{b}F(x)^{dx}\right)^{\frac{1}{b-a}}\\ &\supseteq \Delta_{2}^{\frac{1}{2}\int_{0}^{1}h(t)dt}\\ &\supseteq (F(a)F(b))^{\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_{0}^{1}h(t)dt}, \qquad (6)\end{aligned}$$

where

$$\begin{aligned}\Delta_{1} &= F\left(\frac{3a+b}{4}\right)F\left(\frac{a+3b}{4}\right)\\ \Delta_{2} &= F(a)F(b)F^{2}\left(\frac{a+b}{2}\right).\end{aligned}$$

For more details and notations involved in Theorem 2.3, one can read [24].

Remark 2.2. *(i)If we set* $h(t)=t$ *in Theorem 2.3, we have*

$$\begin{aligned}\left(F\left(\frac{a+b}{2}\right)\right)^{2} &\supseteq \sqrt{\Delta_{1}}\supseteq\left(\int_{a}^{b}F(x)^{dx}\right)^{\frac{1}{b-a}}\\ &\supseteq \Delta_{2}^{\frac{1}{4}}\supseteq G(F(a),F(b)). \qquad (7)\end{aligned}$$

(ii) If we choose $h(t)=t^{s}$ *in Theorem 2.3, we have*

$$\begin{aligned}\left(F\left(\frac{a+b}{2}\right)\right)^{2^{2s-1}} &\supseteq \Delta_{1}^{2^{s-2}}\supseteq\left(\int_{a}^{b}F(x)^{dx}\right)^{\frac{1}{b-a}}\\ &\supseteq \Delta_{2}^{\frac{1}{2(s+1)}}\supseteq (F(a)F(b))^{\frac{1}{s+1}\left[\frac{1}{2}+\frac{1}{2^{s}}\right]}.\end{aligned}$$

Theorem 2.4. [24]*Let* $F,G:[a,b]\to\mathbb{R}_{\mathcal{I}}^{+}$, $h:[0,1]\to\mathbb{R}^{+}$ *and* $h\left(\frac{1}{2}\right)\neq 0$. *If* $F,G\in SX(\log -h,[a,b],\mathbb{R}_{\mathcal{I}}^{+})$ *and* $F,G\in\mathcal{I}^{*}\mathcal{R}_{([a,b])}$, *then following double inequalities hold:*

$$\begin{aligned}\left[F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}} &\supseteq \left(\int_{a}^{b}F(x)^{dx}.\int_{a}^{b}G(x)^{dx}\right)^{\frac{1}{b-a}}\\ &\supseteq [F(a)F(b)G(a)G(b)]^{\int_{0}^{1}h(t)dt}. \qquad (8)\end{aligned}$$

Remark 2.3. *(i) If* $h(t)=t$ *in Theorem 2.4, then we have:*

$$\begin{aligned}F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right) &\supseteq \left(\int_{a}^{b}F(x)^{dx}.\int_{a}^{b}G(x)^{dx}\right)^{\frac{1}{b-a}}\\ &\supseteq G(F(a),F(b)).G(G(a),G(b)). \qquad (9)\end{aligned}$$

(ii) If $h(t)=t^{s}$ *in Theorem 2.4, then we have:*

$$\begin{aligned}\left[F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)\right]^{2^{s-1}} &\supseteq \left(\int_{a}^{b}F(x)^{dx}.\int_{a}^{b}G(x)^{dx}\right)^{\frac{1}{b-a}}\\ &\supseteq [F(a)F(b)G(a)G(b)]^{\frac{1}{s+1}}.\end{aligned}$$

3. Interval-valued double integral

A set of numbers $\{t_{i-1}, \xi_i, t_i\}_{i=1}^m$ is called tagged partition P_1 of $[a, b]$ if

$$P_1 : a = t_0 < t_1 < \ldots < t_n = b$$

and if $t_{i-1} \leq \xi_i \leq t_i$ for all $i = 1, 2, 3, \ldots, m$. Moreover if we have $\Delta t_i = t_i - t_{i-1}$, then P_1 is said to be $\delta-$fine if $\Delta t_i < \delta$ for all i. Let $\mathcal{P}(\delta, [a, b])$ denote the set of all $\delta-$fine partitions of $[a, b]$. If $\{t_{i-1}, \xi_i, t_i\}_{i=1}^m$ is a $\delta-$fine P_1 of $[a, b]$ and if $\{s_{j-1}, \eta_j, t_j\}_{j=1}^n$ is $\delta-$fine P_2 of $[c, d]$, then rectangles

$$\Delta_{i,j} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$$

partition the rectangle $\Delta = [a, b] \times [c, d]$ and the points (ξ_i, η_j) are inside the rectangles $[t_{i-1}, t_i] \times [s_{j-1}, s_j]$. Further, by $\mathcal{P}$ (δ, Δ) we denote the set of all $\delta-$fine partitions P of Δ with $P_1 \times P_2$, where $P_1 \in \mathcal{P}(\delta, [a, b])$ and $P_2 \in \mathcal{P}(\delta, [c, d])$. Let $\Delta A_{i,\ j}$ be the area of rectangle $\Delta_{i,j}$. In each rectangle $\Delta_{i,j}$, where $1 \leq i \leq m,\ 1 \leq j \leq n$, choose arbitrary (ξ_i, η_j) and get

$$S(F, P, \delta, \Delta) = \sum_{i=1}^{m} \sum_{j=1}^{n} F(\xi_i, \eta_j) \Delta A_{i,\ j}.$$

We call $S(F, P, \delta, \Delta)$ is integral sum of F associated with $P \in \mathcal{P}(\delta, \Delta)$.
Now we recall the concept of interval-valued double integral given by Zhao et al. in [23].

Theorem 3.1. [23]*Let $F : \Delta \to \mathbb{R}_{\mathcal{I}}$. Then F is called $ID-$integrable on Δ with $ID-$integral $U = (ID) \iint\limits_{\Delta} F(t, s)dA$, if for any $\epsilon > 0$ there exist $\delta > 0$ such that*

$$d(S(F, P, \delta, \Delta)) < \epsilon$$

for any $P \in \mathcal{P}(\delta, \Delta)$. The collection of all $ID-$integrable functions on Δ will be denoted by $\mathcal{I}D_{(\Delta)}$.

Theorem 3.2. [23]*Let $\Delta = [a, b] \times [c, d]$. If $F : \Delta \to \mathbb{R}_{\mathcal{I}}$ is $ID-$integrable on Δ, then we have*

$$(ID) \iint\limits_{\Delta} F(s, t)dA = (IR) \int_a^b (IR) \int_c^d F(s, t)dsdt.$$

Example 3.1. *Let $F : \Delta = [0, 1] \times [1, 2] \to \mathbb{R}_{\mathcal{I}}^+$ be defined by*

$$F(s, t) = [st, s + t],$$

then $F(s, t)$ is integrable on Δ and $(ID) \iint\limits_{\Delta} F(t, s)dA = \left[\frac{3}{4}, 2\right]$.

4. Main Results

In this section, we define interval-valued co-ordinated log-convex function and prove some inequalities of Hermite-Hadamard type by using our new definition. Throughout this section we will use $\Delta = [a, b] \times [c, d]$, where $a < b$ and $c < d$, $a, b, c, d \in \mathbb{R}$.

Definition 4.1. *A function $F : \Delta \to \mathbb{R}_{\mathcal{I}}^+$ is said to be interval-valued co-ordinated log-convex function, if the following inequality holds:*

$$\begin{aligned} & F(tx + (1-t)y, su + (1-s)w) \\ \supseteq\ & F(x, u)^{ts} F(x, w)^{t(1-s)} F(y, u)^{s(1-t)} F(y, w)^{(1-s)(1-t)}, \end{aligned}$$

for all $(x, y), (u, w) \in \Delta$ and $s, t \in [0, 1]$.

Lemma 4.1. *A function $F : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ is interval-valued log-convex on co-ordinates if and only if there exists two functions $F_x : [a, b] \to \mathbb{R}_{\mathcal{I}}^{+},\ F_x(w) = F(x, w)$ and $F_y : [c, d] \to \mathbb{R}_{\mathcal{I}}^{+},\ F_y(u) = F(u, y)$ are interval-valued log-convex.*

Proof. The proof of this lemma follows immediately by the definition of interval-valued co-ordinated convex function. □

Lemma 4.2. *Let $F : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued function such that $F(x, y) = \left[\underline{F}, \overline{F}\right]$, then F is called interval-valued log-convex function if and only if $\underline{F}$ is log-convex and $\overline{F}$ is log-concave function.*

Proof. This proof is similar to Theorem 3.7 in [22], so it is omitted. □

In what follows, without causing confusion, we will delete notations of (R), (IR) and (ID). We start with the following Theorem.

Theorem 4.1. *If $F : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ is a continuous and interval-valued co-ordinated log-convex function on Δ such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then following inequalities holds:*

$$\begin{aligned}
& F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\supseteq\ & \exp\left[\frac{1}{2(b-a)} \int_a^b \ln F\left(x, \frac{c+d}{2}\right) dx + \frac{1}{2(d-c)} \int_c^d \ln F\left(\frac{a+b}{2}, y\right) dy\right] \\
\supseteq\ & \exp\left[\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \ln F(x, y) dy dx\right] \\
\supseteq\ & \exp\left[\frac{1}{4(b-a)} \int_a^b \ln F(x, c) dx + \frac{1}{4(b-a)} \int_a^b \ln F(x, d) dx \right. \\
& \left. + \frac{1}{4(d-c)} \int_c^d \ln F(a, y) dy + \frac{1}{4(d-c)} \int_c^d \ln F(b, y) dy\right] \\
\supseteq\ & \left[F(a, c) F(a, d) F(b, c) F(b, d)\right]^{\frac{1}{4}}.
\end{aligned} \tag{10}$$

Proof. Since F is an interval-valued co-ordinated log-convex function on co-ordinates Δ, then $F_x : [c, d] \to \mathbb{R}_{\mathcal{I}}^{+},\ F_x(y) = F(x, y)$ is an interval-valued log-convex function on $[c, d]$ and for all $x \in [a, b]$. From inequality (5), we have

$$\ln F_x\left(\frac{c+d}{2}\right) \supseteq \frac{1}{d-c} \int_c^d \ln F_x(y)\, dy \supseteq \ln \sqrt{F_x(c) F_x(d)},$$

that can be written as

$$\ln F\left(x, \frac{c+d}{2}\right) \supseteq \frac{1}{d-c} \int_c^d \ln F(x, y)\, dy \supseteq \ln \sqrt{F(x, c) F(x, d)}. \tag{11}$$

Integrating (11) with respect to x over $[a, b]$ and dividing both sides by $(b-a)$, we have

$$\begin{aligned}
& \frac{1}{b-a} \int_a^b \ln F\left(x, \frac{c+d}{2}\right) dx \\
\supseteq\ & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \ln F(x, y)\, dy dx \\
\supseteq\ & \frac{1}{2(b-a)} \left[\int_a^b \ln F(x, c) dx + \int_a^b \ln F(x, d) dx\right].
\end{aligned} \tag{12}$$

Similarly, $F_y = [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$, $F_y(x) = F(x,y)$ is an interval-valued log-convex function on $[a,b]$ and $y \in [c,d]$, we have

$$\begin{aligned} & \frac{1}{d-c}\int_c^d \ln F\left(\frac{a+b}{2}, y\right) dy \\ \supseteq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_a^b \ln F(x,y)dydx \\ \supseteq\ & \frac{1}{2(d-c)}\left[\int_c^d \ln F(a,y)dy + \int_c^d \ln F(b,y)dy\right]. \end{aligned} \tag{13}$$

By adding (12) and (13) and using Theorem 2.1, we have second and third inequality in (10). We also have from (5),

$$\ln F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \supseteq \frac{1}{b-a}\int_a^b \ln F\left(x, \frac{c+d}{2}\right) dx \tag{14}$$

$$\ln F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \supseteq \frac{1}{d-c}\int_c^d \ln F\left(\frac{a+b}{2}, y\right) dy. \tag{15}$$

By adding (14) and (15) and using Theorem 2.1, we have first inequality in (10). At the end, again from (5) and Theorem 2.1, we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b \ln F(x,c)dx &\supseteq \ln\sqrt{F(a,c)F(b,c)}, \\ \frac{1}{b-a}\int_a^b \ln F(x,d)dx &\supseteq \ln\sqrt{F(a,d)F(b,d)}, \\ \frac{1}{d-c}\int_c^d \ln F(a,y)dy &\supseteq \ln\sqrt{F(a,c)F(a,d)}, \\ \frac{1}{d-c}\int_c^d \ln F(b,y)dy &\supseteq \ln\sqrt{F(b,c)F(b,d)} \end{aligned}$$

and proof is completed. □

Example 4.1. *Suppose that* $[a,b] = [0,1]$ *and* $[c,d] = [1,2]$. *Let* $F : [a,b] \times [c,d] \to \mathbb{R}_{\mathcal{I}}^{+}$ *be given as* $F(x,y) = [e^{x+y}, x+y]$, *for all* $x \in [a,b]$ *and* $y \in [c,d]$. *We have*

$$\ln F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = [2, 0.6931],$$

$$\frac{1}{2}\left[\frac{1}{b-a}\int_a^b \ln F\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d \ln F\left(\frac{a+b}{2}, y\right) dy\right] = [2, 0.6825],$$

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln F(x,y)dydx = [2, 0.6711],$$

$$\begin{aligned} &\frac{1}{4}\left[\frac{1}{b-a}\int_a^b \ln F(x,c)dx + \frac{1}{b-a}\int_a^b \ln F(x,d)dx \right. \\ &\left. + \frac{1}{d-c}\int_c^d \ln F(a,y)dy + \frac{1}{d-c}\int_c^d \ln F(b,y)dy\right] = [2, 0.6478], \end{aligned}$$

$$[F(a,c)F(a,d)F(b,c)F(b,d)]^{\frac{1}{4}}. = [2, 0.6212].$$

Hence from (10), we have $[2, 0.6931] \supseteq [2, 0.6825] \supseteq [2, 0.6711] \supseteq [2, 0.6478] \supseteq [2, 0.6212]$.

Remark 4.1. *If* $\overline{F} = \underline{F}$, *then Theorem 4.1 reduces to* [3, Corollary 3.1].

Theorem 4.2. *If $F : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ is continuous and interval–valued co-ordinated log-convex function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then following inequalities holds:*

$$
\begin{aligned}
& F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\supseteq\ & \exp\left[\frac{1}{2(b-a)}\int_a^b \ln F\left(x,\frac{c+d}{2}\right)dx + \frac{1}{2(d-c)}\int_c^d \ln F\left(\frac{a+b}{2},y\right)dy\right] \\
\supseteq\ & \exp\left[\frac{1}{4(b-a)}\int_a^b \ln F\left(x,\frac{3c+d}{4}\right)F\left(x,\frac{c+3d}{4}\right)dx\right. \\
& \left.+\frac{1}{4(d-c)}\int_c^d \ln F\left(\frac{3a+b}{4},y\right)F\left(\frac{a+3b}{4},y\right)dy\right] \\
\supseteq\ & \exp\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dydx\right] \\
\supseteq\ & \exp\left[\frac{1}{8(b-a)}\int_a^b \ln F(x,c)\,F(x,d)\,F^2\left(x,\frac{c+d}{2}\right)dx\right. \\
& \left.+\frac{1}{8(d-c)}\int_c^d \ln F(a,y)\,F(b,y)\,F^2\left(\frac{a+b}{2},y\right)dy\right] \\
\supseteq\ & \exp\left[\frac{1}{2(b-a)}\int_a^b \ln F(x,c)\,dx + \frac{1}{2(b-a)}\int_a^b \ln F(x,d)\,dx\right. \\
& \left.\frac{1}{2(d-c)}\int_c^d \ln F(a,y)\,dy + \frac{1}{2(d-c)}\int_c^d \ln F(b,y)\,dy\right] \\
\supseteq\ & \left[F(a,c)\,F(b,c)\,F(a,d)\,F(b,d)\,F\left(\frac{a+b}{2},c\right)\right. \\
& \left.\times F\left(\frac{a+b}{2},d\right)F\left(a,\frac{c+d}{2}\right)F\left(b,\frac{c+d}{2}\right)\right] \\
\supseteq\ & \sqrt{F(a,c)\,F(b,c)\,F(a,d)\,F(b,d)}.
\end{aligned}
\tag{16}
$$

Proof. Since F is an interval-valued co-ordinated log-convex function on co-ordinates Δ, then $F_x : [c,d] \to \mathbb{R}_{\mathcal{I}}^{+},\ F_x(y) = F(x,y)$ is an interval-valued log-convex function on $[c,d]$ and for all $x \in [a,b]$. From inequality (7), we have

$$
\begin{aligned}
\ln F_x^2\left(\frac{c+d}{2}\right) &\supseteq \ln\sqrt{F_x\left(\frac{3c+d}{4}\right)F_x\left(\frac{c+3d}{4}\right)} \\
&\supseteq \frac{1}{d-c}\int_c^d \ln F_x(y)\,dy \\
&\supseteq \ln\left[F_x(c)\,F_x(d)\,F_x^2\left(\frac{c+d}{2}\right)\right]^{\frac{1}{4}} \\
&\supseteq \ln F_x(c)\,F_x(d),
\end{aligned}
$$

that can be written as

$$\begin{aligned}
\ln F^2\left(x,\frac{c+d}{2}\right) &\supseteq \ln\sqrt{F\left(x,\frac{3c+d}{4}\right)F\left(x,\frac{c+3d}{4}\right)} \\
&\supseteq \frac{1}{d-c}\int_c^d \ln F(x,y)\,dy \\
&\supseteq \ln\left[F(x,c)\,F(x,d)\,F^2\left(x,\frac{c+d}{2}\right)\right]^{\frac{1}{4}} \\
&\supseteq \ln F(x,c)\,F(x,d), \qquad (17)
\end{aligned}$$

Integrating (17) with respect to x over $[a,b]$ and dividing both sides by $(b-a)$, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \ln F^2\left(x,\frac{c+d}{2}\right)dx \\
\supseteq\ &\frac{1}{2(b-a)}\int_a^b \ln F\left(x,\frac{3c+d}{4}\right)F\left(x,\frac{c+3d}{4}\right)dx \\
\supseteq\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln F(x,y)\,dydx \\
\supseteq\ &\frac{1}{4(b-a)}\int_a^b \ln F(x,c)\,F(x,d)\,F^2\left(x,\frac{c+d}{2}\right)dx \\
\supseteq\ &\frac{1}{b-a}\int_a^b \ln F(x,c)\,F(x,d)\,dx. \qquad (18)
\end{aligned}$$

Similarly, $F_y=[a,b]\to\mathbb{R}_{\mathcal{I}}^{+}$, $F_y(x)=F(x,y)$ is an interval-valued log-convex function on $[a,b]$ and $y\in[c,d]$, we have

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d \ln F^2\left(\frac{a+b}{2},y\right)dy \\
\supseteq\ &\frac{1}{2(d-c)}\int_c^d \ln F\left(\frac{3a+b}{4},y\right)F\left(\frac{a+3b}{4},y\right)dy \\
\supseteq\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln F(x,y)\,dydx \\
\supseteq\ &\frac{1}{4(d-c)}\int_c^d \ln F(a,y)\,F(b,y)\,F^2\left(\frac{a+b}{2},y\right)dx \\
\supseteq\ &\frac{1}{d-c}\int_c^d \ln F(a,y)\,F(b,y)\,dy. \qquad (19)
\end{aligned}$$

By adding (18) and (19) and using Theorem 2.1, we have second and third inequality in (16). We also have from (7),

$$\ln F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \supseteq \frac{1}{b-a}\int_a^b \ln F^2\left(x,\frac{c+d}{2}\right)dx \qquad (20)$$

$$\ln F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \supseteq \frac{1}{d-c}\int_c^d \ln F^2\left(\frac{a+b}{2},y\right)dy. \qquad (21)$$

By adding (20) and (21) and using Theorem 2.1, we have first inequality in (16). Again from (7) and Theorem 2.1, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b \ln F(x,c)dx &\supseteq \ln\left[F(a,c)F(b,c)F^2\left(\frac{a+b}{2},c\right)\right]^{\frac{1}{4}} \\
&\supseteq \ln\sqrt{F(a,c)\,F(b,c)}, \\
\frac{1}{b-a}\int_a^b \ln F(x,d)dx &\supseteq \ln\left[F(a,d)F(b,d)F^2\left(\frac{a+b}{2},d\right)\right]^{\frac{1}{4}} \\
&\supseteq \ln\sqrt{F(a,d)\,F(b,d)}, \\
\frac{1}{d-c}\int_c^d \ln F(a,y)dy &\supseteq \ln\left[F(a,c)F(a,d)F^2\left(a,\frac{c+d}{2}\right)\right]^{\frac{1}{4}} \\
&\supseteq \ln\sqrt{F(a,c)\,F(a,d)}, \\
\frac{1}{d-c}\int_c^d \ln F(b,y)dy &\supseteq \ln\left[F(b,c)F(b,d)F^2\left(b,\frac{c+d}{2}\right)\right]^{\frac{1}{4}} \\
&\supseteq \ln\sqrt{F(b,c)\,F(a,d)},
\end{aligned}$$

and proof is completed. □

Theorem 4.3. *If $F,G:\Delta\to\mathbb{R}_{\mathcal{I}}^{+}$ are continuous and interval-valued co-ordinated log-convex functions on Δ such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$ and,$G(t)=\left[\underline{G}(t),\overline{G}(t)\right]$ then following inequalities holds:*

$$\begin{aligned}
& F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\supseteq\ & \exp\left[\frac{1}{2(b-a)}\int_a^b \ln F\left(x,\frac{c+d}{2}\right)G\left(x,\frac{c+d}{2}\right)dx\right. \\
& \left.+\frac{1}{2(d-c)}\int_c^d \ln F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)dy\right] \\
\supseteq\ & \exp\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln F(x,y)G(x,y)\,dydx\right] \\
\supseteq\ & \exp\left[\frac{1}{4(b-a)}\int_a^b \ln F(x,c)G(x,c)\,dx+\frac{1}{4(b-a)}\int_a^b \ln F(x,d)G(x,d)\,dx\right. \\
& \left.+\frac{1}{4(d-c)}\int_c^d \ln F(a,y)G(a,y)\,dy+\frac{1}{4(d-c)}\int_c^d \ln F(b,y)G(b,y)\,dy\right] \\
\supseteq\ & \left[F(a,c)F(a,d)F(b,c)F(b,d)G(a,c)G(a,d)G(b,c)G(b,d)\right]^{\frac{1}{4}}. \qquad (22)
\end{aligned}$$

Proof. Since F and G are interval-valued co-ordinated log-convex functions on Δ, therefore

$$F_x(y):[c,d]\to\mathbb{R}_{\mathcal{I}}^{+},\ F_x(y)=F(x,y),G_x(y):[c,d]\to\mathbb{R}_{\mathcal{I}}^{+},\ G_x(y)=G(x,y),$$

and

$$F_y(x):[a,b]\to\mathbb{R}_{\mathcal{I}}^{+},\ F_y(x)=F(x,y),\ G_y:[a,b]\to\mathbb{R}_{\mathcal{I}}^{+},\ G_y(x)=G(x,y)$$

are interval-valued convex functions on $[c,d]$ and $[a,b]$ respectively for all $x\in[a,b],\ y\in[c,d]$.

From inequality (9), we have

$$\begin{aligned}\ln F_x\left(\frac{c+d}{2}\right)G_x\left(\frac{c+d}{2}\right) &\supseteq \left[\frac{1}{d-c}\int_c^d \ln F_x(y)\,G_x(y)\,dy\right]\\ &\supseteq \ln\left(\sqrt{F_x(c)F_x(d)G_x(c)G_x(d)}\right),\end{aligned}$$

that can be written as

$$\begin{aligned}\ln F\left(x,\frac{c+d}{2}\right)\ln G\left(x,\frac{c+d}{2}\right) &\supseteq \left[\frac{1}{d-c}\int_c^d \ln F(x,y)\,G(x,y)\,dy\right]\\ &\supseteq \ln\left(\sqrt{F(x,c)F(x,d)G(x,c)G(x,d)}\right). \qquad (23)\end{aligned}$$

Integrating (23) with respect to x over $[a,b]$ and dividing both sides by $(b-a)$, we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \ln F\left(x,\frac{c+d}{2}\right)G\left(x,\frac{c+d}{2}\right)dx\\ \supseteq\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln F(x,y)\,G(x,y)\,dydx\\ \supseteq\ &\frac{1}{2(b-a)}\left[\int_a^b \ln F(x,c)G(x,c)dx+\int_a^b \ln F(x,d)G(x,d)dx\right]. \qquad (24)\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\frac{1}{d-c}\int_c^d \ln F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)dy\\ \supseteq\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_a^b \ln F(x,y)G(x,y)dydx\\ \supseteq\ &\frac{1}{2(d-c)}\left[\int_c^d \ln F(a,y)G(a,y)dy+\int_c^d \ln F(b,y)G(b,y)dy\right]. \qquad (25)\end{aligned}$$

By adding (24) and (25) and using Theorem 2.1, we have second and third inequality in (22). We also have from (9),

$$\ln F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right) \supseteq \frac{1}{b-a}\int_a^b \ln F\left(x,\frac{c+d}{2}\right)G\left(x,\frac{c+d}{2}\right)dx \qquad (26)$$

$$\ln F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right) \supseteq \frac{1}{d-c}\int_c^d \ln F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)dy \qquad (27)$$

By adding (26) and (27) and using Theorem 2.1, we have first inequality in (22). At the end, again from (9) and Theorem 2.1, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b \ln F(x,c)G(x,c)dx &\supseteq \ln\sqrt{F(a,c)F(b,c)G(a,c)G(b,c)},\\ \frac{1}{b-a}\int_a^b \ln F(x,d)G(x,d)dx &\supseteq \ln\sqrt{F(a,d)F(b,d)G(a,d)G(b,d)},\\ \frac{1}{d-c}\int_c^d \ln F(a,y)G(a,y)dy &\supseteq \ln\sqrt{F(a,c)F(a,d)G(a,c)G(a,d)},\\ \frac{1}{d-c}\int_c^d \ln F(b,y)G(b,y)dy &\supseteq \ln\sqrt{F(b,c)F(b,d)G(b,c)G(b,d)}\end{aligned}$$

and proof is completed. □

Remark 4.2. *If* $\overline{F} = \underline{F}$*, then resultant results coincides with the* [1, Theorem 2.4].

5. Conclusion

In this study, the notion of interval-valued log convex functions on the co-ordinates on the rectangle from the plane is given. Some new inequalities of Hermite-Hadamard type for these new classes of functions are proved. It is also proved that the results proved in this paper are potential generalization of the existing comparable results in the literature. As future directions, one may finds the similar inequalities through different types of convexities.

References

[1] Ali, M. A., Abbas, M., Budak, H. & Kashuri, A., (2020). Some New Hermite-Hadamard integral inequalities in Multiplicative Calculus. TWMS Journal of Applied Engineering and Mathematics, (In Press).

[2] Alomari, M., & Darus, M. (2008), Co-ordinated s-convex function in the first sense with some Hadamard-type inequalities. International Journal of Contemporary Mathematical Sciences, 3, (32), pp. 1557-1567.

[3] Alomari, M., & Darus, M. (2009), On the Hadamard's inequality for log-convex functions on the coordinates. Journal of Inequalities and Applications, 2009, (1), 283147.

[4] Chalco-Cano, Y., Flores-Franulič, A., & Román-Flores, H., (2012), Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. Computational & Applied Mathematics, 31, (3), pp. 457-472.

[5] Chalco-Cano, Y., Lodwick, W. A., & Condori-Equice, W., (2015), Ostrowski type inequalities and applications in numerical integration for interval-valued functions. Soft Computing, 19, (11), pp. 3293-3300.

[6] Costa, T. M., (2017), Jensen's inequality type integral for fuzzy-interval-valued functions, Fuzzy Sets and Systems, 327, pp. 31-47.

[7] Costa, T. M., & Román-Flores, H., (2017), Some integral inequalities for fuzzy-interval-valued functions. Information Sciences, 420, pp. 110-125.

[8] Dragomir, S. S., (2001), On the Hadamard's inequlality for convex functions on the co-ordinates in a rectangle from the plane. Taiwanese Journal of Mathematics, pp. 775-788.

[9] Dragomir, S. S., & Pearce, C. E. M., (2004), Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000, Online: http://www. staff. vu. edu. au/RGMIA/monographs/hermite hadamard. html.

[10] Flores-Franulič, A., Chalco-Cano, Y., & Román-Flores, H., (2013, June), An Ostrowski type inequality for interval-valued functions. In 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), pp. 1459-1462, IEEE.

[11] Guo, Y., Ye, G., Zhao, D., & Liu, W., (2019), Some Integral Inequalities for Log-h-Convex Interval-Valued Functions. IEEE Access, 7, pp. 86739-86745.

[12] Latif, M. A., & Alomari, M., (2009), Hadamard-type inequalities for product two convex functions on the co-ordinates. International Mathematical Forum 4, (47), pp. 2327-2338.

[13] Mitroi, F. C., Nikodem, K., & Wąsowicz, S., (2013), Hermite–Hadamard inequalities for convex set-valued functions. Demonstratio Mathematica, 46, (4), pp. 655-662.

[14] Moore, R. E., (1966), Interval analysis, Vol. 4, Englewood Cliffs: Prentice-Hall.

[15] Moore, R. E., Kearfott, R. B., & Cloud, M. J., (2009), Introduction to interval analysis, Vol. 110, Siam.

[16] Nikodem, K., Sánchez, J. L., & Sánchez, L., (2014), Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps. Mathematica Aeterna, 4, (8), pp. 979-987.

[17] Ozdemir, M. E., Set, E., & Sarıkaya, M. Z., (2011), Some new hadamard type inequalities for co-ordinated. Hacettepe Journal of Mathematics and Statistics, 40, (2), pp. 219-229.

[18] Peajcariaac, J. E., & Tong, Y. L., (1992), Convex functions, partial orderings, and statistical applications. Academic Press.

[19] Román-Flores, H., Chalco-Cano, Y., & Lodwick, W. A., (2018), Some integral inequalities for interval-valued functions. Computational and Applied Mathematics, 37, (2), pp. 1306-1318.

[20] Román-Flores, H., Chalco-Cano, Y., & Silva, G. N., (2013, June), A note on Gronwall type inequality for interval-valued functions. In 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), pp. 1455-1458, IEEE.
[21] Sadowska, E. (1997), Hadamard Inequality and a Refinement of Jensen Inequality for Set-Valued Functions, Results in Mathematics, 32, (3-4), pp.332-337.
[22] Zhao, D., An, T., Ye, G., & Liu, W., (2018), New Jensen and Hermite–Hadamard type inequalities for h-convex interval-valued functions, Journal of Inequalities and Applications, 2018, (1), 302.
[23] Zhao, D., An, T., Ye, G., & Liu, W., (2019), Chebyshev type inequalities for interval-valued functions. Fuzzy Sets and Systems.
[24] Zhiyue, Z., Ali, M. A., Budak, H., Sarikaya, M. Z., (2020). On Hermite Hadamard type inequalities for interval-valued multiplicative integrals available at https://www.researchgate.net/publication/342260986.

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Hüseyin Budak for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, N.4.