TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 908-918

SOME RESULTS ON GRACEFUL CENTERS OF P_n AND RELATED α-GRACEFUL GRAPHS

H. M. MAKADIA[1], V. J. KANERIA[2], P. ANDHARIA[3], D. JADEJA[4], §

Abstract. In this paper, we have proved that the graph obtained by joining two copies of a bipartite graceful graph by an edge with any two corresponding vertices of both the copies of graphs is α-graceful. We also proved path step tree and path double step tree are α-graceful and the graph $P_m \times P_n \times P_2$ is α-graceful. Graceful center of graceful graph defined. We also found some some graceful centers of path P_n. Acharya and Gill [1] proved $P_n \times P_m$ is α-graceful. In this paper we proved its generalized result.

Keywords: Graceful center of a graceful graph, universal graceful graph, α-graceful graph, Path step tree, Path double step tree.

AMS Subject Classification: 05C78.

1. Introduction

In this paper a graph $G = (V(G), E(G))$ is a pair of set of vertices and edge of G and a (p, q) graph G, we mean $p = |V(G)|$ and $q = |E(G)|$. Terms not defined here are used with standard notation from Harary [3]. A Labeling $f : V(G) \longrightarrow \{0, 1, 2, \dots, q\}$ is said to be a graceful labeling for G, if f is an injective map and its edge induced function $f^\star : E(G) \longrightarrow \{1, 2, \dots, q\}$ defined by $f^\star(uv) = |f(u) - f(v)|$, $\forall uv \in E(G)$ is a bijective map. A graph G is called a graceful graph if it admits a graceful labeling. A graceful labeling $f : V(G) \longrightarrow \{0, 1, 2, \dots, q\}$ is called an α-labeling for G, if $\exists$ an integer $k(0 \leq k < q)$ such that for any $uv \in E(G)$, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$. A graph G is called an α-graceful graph if it admits an α-labeling. An α-graceful graph is always a bipartite graph.

Let G be a graceful graph with a graceful labeling $f : V(G) \longrightarrow \{0, 1, 2, \dots, q\}$. A vertex $v \in V(G)$ is called a graceful center of G with respect to f if $f(v) = 0$ or $f(v) = q$.

[1] Lukhdhirji Engineering College, Morbi, 363642, India.
e-mail: makadia.hardik@yahoo.com; ORCID: https://orcid.org/0000-0001-5496-3994.
[2] Department of Mathematics, Saurashtra University, Rajkot, 360005, India.
e-mail: kaneriavinodray@gmail.com; ORCID: https://orcid.org/0000-0001-7872-8841.
[3] MRT and MSV Science College, Kalvibid, Bhavnager, 364002, India.
e-mail: priteshandharia@gmail.com; ORCID: https://orcid.org/0000-0002-7809-767X.
[4] VVP Engineering College, Rajkot, 360005, India.
e-mail: divyajadeja89@gmail.com; ORCID: https://orcid.org/0000-0001-8605-6897.
§ Manuscript received: March 02, 2020; accepted: July 16, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.12, No.3 © Işık University, Department of Mathematics, 2022; all rights reserved.

A graph G is said to be a universal graceful graph if for any $v \in V(G)$, there is a graceful labeling f such that either $f(v) = 0$ or $f(v) = q$.

Any graceful graph G with a graceful labeling f has at least two graceful centers. If G has precisely two graceful centers, then they are obtained in G, as they both produce the edge label q under the edge induced labeling function $f^\star : E(G) \longrightarrow \{1, 2, \dots, q\}$.

Suppose a graph G is an α-graceful graph with α-labeling $f : V(G) \longrightarrow \{0, 1, 2, \dots, q\}$ and an integer $k(0 \leq k < q)$ such that for any $uv \in E(G)$, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$. In this case $V(G)$ partition into two parts $V_1 = \{v \in V(G)/f(v) \leq k\}$ and $V_2 = \{v \in V(G)/f(v) > k\}$. Moreover, there are $w_1, w_2 \in V_1, w_3, w_4 \in V_2$ such that $f(w_1) = 0, f(w_2) = k, f(w_3) = k+1$ and $f(w_4) = q$. Defined $h : V(G) \longrightarrow \{0, 1, 2, \dots, q\}$ by $h/V_1 = k - f/V_1, h/V_2 = q+k+1-f/V_2$. Here h is an injective and its edge induced map $h^\star : E(G) \longrightarrow \{1, 2, \dots, q\}$ defined by $h^\star(uv) = |h(u)-h(v)|, \forall uv \in E(G)$ is bijective. In this case w_1, w_2, w_3, w_4 are graceful centers for G, as $f(w_1) = 0, h(w_2) = 0, f(w_4) = q$ and $h(w_3) = q$. Also G admits four α-graceful labelings $f, q-f, h$ and $q-h$.

Cycle C_{4n}, complete bipartite graph $k_{m,n}$ are universal graceful graph. C_{4n+3} and W_n are also universal graceful graphs, but do not admits α-labeling, as they are not bipartite graphs.

Take $n \geq 3$, paths $P_i(i = 2, 3, \dots, n)$ with $V(P_i) = \{v_{i,j}/1 \leq j \leq i\}$, $E(P_i) = \{v_{i,j}, v_{i,j+1}/1 \leq j \leq i\}$ and arrange them vertically. Join $v_{i,1}$ with $v_{i+1,1}$ by an edge, $\forall i = 2, 3, \dots, n-1$, such tree is called a path step tree of size n and denote it by PST_n. Take two copies of PST_n with $PST_n^l = (\{v_{l,i,j}/1 \leq j \leq i, 2 \leq i \leq n\}, \{v_{l,i,j}, v_{l,i,j+1}/1 \leq j < i, 2 \leq j < n\} \bigcup \{v_{l,i,1}, v_{l,i+1,1}/2 \leq i < n\})$ and $l = 1, 2$. The tree obtained by joining $v_{1,n,1}$ with $v_{2,n,1}$ by an edge is called path double step tree and denoted it by $PDST_n$.

Acharya and Gill [1] have investigated α-graceful labeling for the grid graph. Kaneria and Makadia [4] showed that union of two grid graphs is graceful. But M.Z. Youssef said that in the paper [7], the graph union of two grid graphs is α-graceful. Kaneria, Makadia and Viradia [5] show that union of three grids and union of finite copies of a grid is graceful. In [6], they extended it further to prove that union of finite grids is graceful as well.

2. Main Result

Theorem 2.1. *Let G be a bipartite graceful graph. The graph obtained by joining two copies of G say $G^{(1)}$ and $G^{(2)}$ by edge between any two corresponding vertices $v^{(1)} \in V(G^{(1)})$ and $v^{(2)} \in V(G^{(2)})$, for some $v \in V(G)$ is α-graceful.*

Proof. As G is bipartite, take $V(G) = V_1 \bigcup V_2$ and for any $uv \in E(G)$, either $u \in V_1, v \in V_2$ or $u \in V_2, v \in V_1$. Let $f : V(G) \longrightarrow \{0, 1, 2, \dots, q\}$ be a graceful labeling for G, where $q = |E(G)|$.

Let H be a graph obtained by joining two copies $G^{(1)}$ and $G^{(2)}$ of G by an edge between any two corresponding vertices $v^{(1)} \in V(G^{(1)})$ and $v^{(2)} \in V(G^{(2)})$ for some $v \in V(G)$.

It is observed that $V(H) = V(G^{(1)}) \bigcup V(G^{(2)})$, $E(H) = E(G^{(1)}) \bigcup E(G^{(2)}) \bigcup \{v^{(1)}v^{(2)}\}$, $|V(H)| = 2|V(G)|$ and $|E(H)| = 2q+1$. define $g : V(H) \longrightarrow \{0, 1, 2, \dots, 2q+1\}$ by $g/V_1^{(1)} = f/V_1$, $g/V_2^{(2)} = f/V_2$, $g/V_2^{(1)} = f/V_2 + (q+1)$ and $g/V_1^{(2)} = f/V_1 + (q+1)$.

Since f is one-one, g is also one-one. Take $uv \in E(G)$ be any edge. Now for any $i = 1, 2,$

$$\begin{aligned} g^{\star}(u^{(i)}v^{(i)}) &= |g(u^{(i)}) - g(v^{(i)})|, \\ &= \begin{cases} q+1+f(u)-f(v), \text{ if } g(u^{(i)}) > g(v^{(i)}) \\ q+1-(f(u)-f(v)), \text{ otherwise} \end{cases} \\ &= \begin{cases} q+1+f^{\star}(uv), \text{ if } u \in V_1 \ \& \ i=2 \text{ or} u \in V_2 \ \& \ i=1 \\ q+1-f^{\star}(uv), \text{ otherwise} \end{cases} \end{aligned}$$

Since, range of $f^{\star}$ is $\{1, 2, \ldots, q\}$, $g^{\star}(v^{(1)}\ v^{(2)}) = q+1$, we must have range of $g^{\star}$ is $\{1, 2, \ldots, 2q+1\}$. Therefore, $g^{\star} : E(H) \longrightarrow \{1, 2, \ldots, 2q+1\}$ defined by $g^{\star}(uv) = |g(u) - g(v)|$, $\forall\ uv \in E(H)$ is a bijective. Hence, g is a graceful labeling for H.

Take $k = q$. Now for each $u \in V(G)$, $f(u) \leq q$ and $\min\{g(u^{(1)}), g(u^{(2)})\} \leq q$, $\max\{g(u^{(1)}),\ g(u^{(2)})\} \geq q+1$.
$\Rightarrow \min\{g(v^{(1)}), g(v^{(2)})\} \leq k < \max\{g(v^{(1)}),\ g(v^{(2)})\}$.
Observe that, for any $(u^{(i)}, w^{(i)}) \in E(H)$, $(u, w) \in E(G), \forall i = 1, 2$. Also one of u, w lies in V_1 and another of them lies in V_2. $\min\{g(u^{(i)}), g(w^{(i)})\} \leq k < \max\{g(u^{(i)}), g(w^{(i)})\}$, $\forall (u^{(i)}, w^{(i)}) \in E(H)$ and $\forall i = 1, 2$. i.e. for any $uw \in E(H)$, $\min\{g(u), g(w)\} \leq k < \max\{g(u), g(v)\}$. Therefore, g is an α-graceful labeling for H and so, H is α-graceful. □

Theorem 2.2. *Let n be an odd integer and P_n be a path on n vertices with $V(P_n) = \{v_i/1 \leq i \leq n\}$ and $E(P_n) = \{v_iv_{i+1}/1 \leq i < n-1\}$. Let $t = \frac{n+1}{2}$ then $v_1, v_2, \ldots, v_6, v_9, v_{10}, v_{19}, v_{20}, v_{t-1}$, v_t and v_{t+1} are graceful center for P_n.*

Proof. For each $i = 1, 2, \ldots, 5$, defined $f_i : V(P_n) \longrightarrow \{0, 1, 2, \ldots, n-1\}$ as follows

$$f_1(v_i) = \begin{cases} \frac{i-1}{2}, & \text{when } i \text{ is odd} \\ q - (\frac{i-2}{2}), & \text{when } i \text{ is even} \end{cases}$$

$\forall i = 1, 2, \ldots, n$;

$f_2(v_1) = q+1, f_2(v_2) = 1, f_2(v_3) = 3, f_2(v_4) = 0, f_2(v_5) = q-3,\ f_2(v_6) = 2, f_2(v_7) = q-2, f_2(v_8) = 4, f_2(v_9) = q-4, f_2(v_{10}) = 3,\ f_2(v_i) = f_2(v_{i-6}) + 3(-1)^i, \forall i = 11, 12, \ldots, n-6$ and $f_2(v_n), f_2(v_{n-1}), \ldots, f_2(v_{n-5})$ define according to table-1, where $t = \frac{n+1}{2}$.

$$f_3(v_i) = \begin{cases} 3 - (\frac{i}{2}), & \text{when } i = 2, 4, 6 \\ n + (\frac{i-7}{2}), & \text{when } i = 1, 3, 5 \\ f_3(v_{i-6}) - 3, & \text{when } i = 7, 9, 11 \\ f_3(v_{i-6}) + 2, & \text{when } i = 8, 10 \\ f_3(v_{i-6}) - 2, & \text{when } i = 13, 15 \\ f_3(v_{i-10}) + 5, & \text{when } i = 12, 14, 16 \\ f_3(v_{i-10}) + 5(-1)^i, & \text{when } i = 17, 18, \ldots, n-10, \end{cases}$$

and $f_3(v_n), f_3(v_{n-1}), \dots, f_3(v_{n-9})$ define according to table-2.

$$f_4(v_i) = \begin{cases} 5-(\frac{i}{2}), & \text{when } i=2,4,\dots,10 \\ n+(\frac{i-11}{2}), & \text{when } i=1,3,\dots,9 \\ f_4(v_{i-10}-5), & \text{when } i=11,13,\dots,19 \\ f_4(v_{i-10})+4, & \text{when } i=12,14,16,18 \\ f_4(v_{i-10})-4, & \text{when } i=21,23,25,27 \\ f_4(v_{i-18})+9, & \text{when } i=20,22,\dots,28 \\ f_4(v_{i-18})+9(-1)^i, & \text{when } i=29,30,\dots,n-18, \end{cases}$$

and $f_4(v_n), f_4(v_{n-1}), \dots, f_4(v_{n-17})$ define according to table-3.

$$f_5(v_i) = \begin{cases} 10-(\frac{i}{2}), & \text{when } i=2,4,\dots,20 \\ n+(\frac{i-21}{2}), & \text{when } i=1,3,\dots,19 \\ 29-(\frac{i}{2}), & \text{when } i=22,24,\dots,38 \\ f_5(v_{i-20})-10, & \text{when } i=21,23,\dots,39 \\ f_5(v_{i-38})+19, & \text{when } i=40,42,\dots,58 \\ f_5(v_{i-38})-20, & \text{when } i=41,43,\dots,57 \\ f_5(v_{i-38})+19(-1)^i, & \text{when } i=59,60,\dots,n-38; \end{cases}$$

and the set of remaining vertex labels $\{f_5(v_n), f_5(v_{n-1}), \dots, f_5(v_{n-37})\}$, choose from table-4, according to value of k, when $n \equiv k \pmod{38}$.

To define $f_6 : V(P_n) \longrightarrow \{0,1,2,\dots,n-1\}$, consider following two cases,
Case-**1** : $n \equiv 1 \pmod 4$

$$f_6(v_1) = \frac{n-1}{4},$$
$$f_6(v_2) = \frac{3n+1}{4},$$
$$f_6(v_j) = \begin{cases} 0, & \text{when } j=t \\ \frac{n-1}{2}, & \text{when } j=t+1 \\ t, & \text{when } j=t=2, \end{cases}$$
$$f_6(v_i) = \begin{cases} f_6(v-i-2)+(-1)^i, & \forall i=3,4,\dots,t-1 \\ f_6(v-i-2)-(-1)^i, & \forall i=t+3,t+4,\dots,n. \end{cases}$$

Case-**2** : $n \equiv 3 \pmod 4$

$$f_6(v_1) = \frac{3n-1}{4},$$
$$f_6(v_2) = \frac{n-3}{4}$$
$$f_6(v_j) = \begin{cases} 0, & \text{when } j=t \\ t-1, & \text{when } j=t+1 \\ t-2, & \text{when } j=t+2, \end{cases}$$
$$f_6(v_i) = f_6(v_{i-2})-(-1)^i, \forall i=3,4,\dots,t-1,t+3,t+4,\dots,n.$$

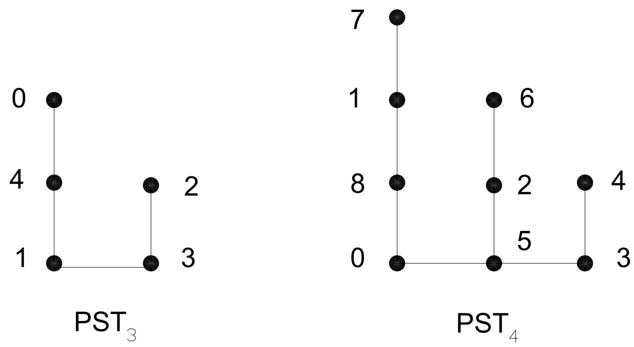
Above defined labeling pattern $f_i (i=1,2,\dots,6)$ give rise graceful labeling to P_n and so, they are graceful labelings for P_n. Since $\{f_1(v_1), f_1(v_2)\} = \{0, n-1\} = \{f_2(v_3), f_2(v_4)\} = \{f_3(v_5), f_3(v_6)\} = \{f_4(v_9), f_4(v_{10})\} = \{f_5(v_{19}), f_5(v_{20})\} = \{f_6(v_{t-1}), f_6(v_t)\}$ and symmetric structure of P_n, $v_1, v_2, \dots, v_6$, $v_9, v_{10}, v_{19}, v_{20}, v_{t-1}, v_t, v_{t+1}, v_{n-5}, v_{n-4}, \dots, v_n, v_{n-8}, v_{n-9}$, v_{n-18} and v_{n-19} are graceful centers for P_n.

□

Theorem 2.3. *For any $n \geq 3$, PST_n and $DPST_n$ are α-graceful graphs.*

Proof. Let $G = PST_n$ i.e. $V(G) = \{v_{i,j}/1 \leq j \leq i, 1 < i \leq n\}$ and $E(G) = \{v_{i,j}, v_{i,j+1}/1 \leq j < i, 1 < i \leq n\} \bigcup \{v_{i,1}, v_{i+1,1}/1 < i < n\}$. It is obvious that $p = \frac{1}{2}(n^2 + n - 2)$ and $q = \frac{1}{2}(n^2 + n - 4)$ in PST_n. To define α-graceful labeling for PST_n, use induction hypothesis. Consider $V(PST_n) = V(PST_{n-2}) \bigcup \{v_{n,j}/1 \leq j \leq n\} \bigcup \{v_{n-1,j}/1 \leq j < n\}$.

α-graceful labeling for PST_3 and PST_4 are shown in following figures



By induction hypothesis take $f : V(PST_{n-2}) \longrightarrow \{0, 1, 2, \ldots, \frac{1}{2}(n^2 - 3n - 2)\}$ as α-graceful labeling for PST_{n-2}. To define vertex labeling $g : V(PST_n) \longrightarrow \{0, 1, \ldots, \frac{1}{2}(n^2 + n - 4)\}$ take following two cases.

Case-**1 :** n is odd

$$g(v_{n,j}) = \begin{cases} (\frac{n-j}{2}), & \text{when } j = 1, 3, 5, \ldots, n \\ q - (\frac{n-1-j}{2}), & \text{when } j = 2, 4, \ldots, n-1, \end{cases}$$

$$g(v_{n-1,j}) = \begin{cases} q - (\frac{n-2+j}{2}), & \text{when } j = 1, 3, \ldots, n-2 \\ q - (\frac{n-1+j}{2}), & \text{when } j = 2, 4, \ldots, n-1, \end{cases}$$

Case-**2 :** n is even

$$g(v_{n,j}) = \begin{cases} (\frac{j-1}{2}), & \text{when } j = 1, 3, \ldots, n-1 \\ p - (\frac{j}{2}), & \text{when } j = 2, 4, \ldots, n, \end{cases}$$

$$g(v_{n-1,n-1}) = g(v_{n,n}) - 1,$$

$$g(v_{n-1,n-2}) = g(v_{n,n-1}) + 1,$$

$$g(_{n-1,j}) = g(v_{n-1,j+2}) + (-1)^j, \forall j = n-3, n-4, \ldots, 1,$$

$$g(v) = f(v) + n - \frac{1}{2} - \frac{(-1)^n}{2}, \forall v \in V(PST_{n-2}).$$

Above defined labeling pattern give rise graceful labeling to $PST_n (n \geq 3)$ as g is injective and its edge induced function $g^\star : E(PST_n) \longrightarrow \{1, 2, \ldots, \frac{1}{2}(n^2 + n - 4)\}$ defined by $g^\star(uv) = |g(u) - g(v)|$, $\forall uv \in E(PST_n)$ is bijective.

It is observed that for any $PST_n (n \geq 3)$, $g^\star(v_{2,1}, v_{2,2}) = 1$. It is also observed that, for any $uv \in E(PST_n)$, $\min\{g(u), g(v)\} \leq g(v_{2,1}) < \max\{g(u), g(v)\}$. By taking

$$\begin{aligned} k &= g(v_{2,1}),\ \text{in } PST_n \\ &= n - \frac{1}{2} - \frac{(-1)^n}{2} + g(v_{2,1}),\ \text{in } PST_{n-2} \\ &= \begin{cases} n + (n-2) + (n-4) + \ldots + 3, & \text{when } n \text{ is odd} \\ (n-1) + (n-3) + (n-5) + \ldots + 3, & \text{when } n \text{ is even} \end{cases} \\ &= \begin{cases} \frac{1}{4}(n^2 + 2n - 3), & \text{when } n \text{ is odd} \\ \frac{1}{4}(n^2 - 4), & \text{when } n \text{ is even} \end{cases} \end{aligned}$$

g is an α-labeling for $PST_n (n \geq 3)$. By applying Theorem-2.1, it is easy to get α-graceful labeling for $DPST_n$ from graceful labeling of PST_n.

□

Theorem 2.4. *$P_m \times P_n \times P_2$ is an α-graceful graph, $\forall\ m, n \in N - \{1\}$.*

Proof. Let $H = P_m \times P_n \times P_2$ and $V(H) = \{v_{i,j,k}/1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq 2\}$. Take $V(H) = \{v_{i,j,1}/1 \leq j \leq m, 1 \leq j \leq n\} \bigcup \{V_{i,j,2}/1 \leq i \leq m, 1 \leq j \leq n\} = V(P_m \times P_n^{(1)}) \bigcup V(P_m \times P_n^{(2)})$ and $E(H) = E(P_m \times P_n^{(1)}) \bigcup E(P_m \times P^2_n) \bigcup \{(v_{i,j,1}, v_{i,j,2})/1 \leq i \leq m, 1 \leq j \leq n\}$. It is obvious that $p = 2mn$ and $q = 5mn - 2(m+n)$ in H. Define $f : V(H) \longrightarrow \{0, 1, 2, \ldots, q\}$ as follows

$$f(v_{1,j,1}) = \begin{cases} q - (\frac{j-1}{2}), & \text{when } j \text{ is odd} \\ (\frac{j-2}{2}), & \text{when } j \text{ is even}\ , \end{cases}$$

$$f(v_{1,j,2}) = \begin{cases} \min\{f(v_{1,n,1}), f(v_{1,n-1,1})\} + \lfloor\frac{n+1}{2}\rfloor + (\frac{j-1}{2}), & \text{when } j \text{ is odd} \\ \max\{f(v_{1,n,1}), f(v_{1,n-1,1})\} - \lceil\frac{n+1}{2}\rceil - (\frac{j-2}{2}), & \text{when } j \text{ is even}\ , \forall 1 \leq j \leq n, \end{cases}$$

$$f(v_{2,j,2}) = \begin{cases} \max\{f(v_{1,n,2}), f(v_{1,n-1,2})\} + \lfloor\frac{n+1}{2}\rfloor + (\frac{j-1}{2}), & \text{when } j \text{ is odd} \\ \min\{f(v_{1,n,2}), f(v_{1,n-1,2})\} - \lceil\frac{3n+1}{2}\rceil - (\frac{j-2}{2}), & \text{when } j \text{ is even}\ , \forall 1 \leq j \leq n, \end{cases}$$

$$f(v_{2,j,1}) = \begin{cases} \min\{f(v_{2,n,2}), f(v_{2,n-1,2})\} + \lfloor\frac{n+1}{2}\rfloor + (\frac{j-1}{2}), & \text{when } j \text{ is odd} \\ \max\{f(v_{2,n,2}), f(v_{2,n-1,2})\} - \lceil\frac{n+1}{2}\rceil - (\frac{j-2}{2}), & \text{when } j \text{ is even}\ , \forall 1 \leq j \leq n, \end{cases}$$

$$f(v_{i,j,k}) = \begin{cases} f(v_{i-2,j,k} - 4n + 2, & \text{when } f(v_{i-2,j,k}) < \frac{q}{2} \\ f(v_{i-2,j,k} + 6n - 2, & \text{when } f(v_{i-2,j,k}) > \frac{q}{2}, \forall 3 \leq i \leq m, \forall 1 \leq j \leq n, \forall 1 \leq k \leq 2. \end{cases}$$

Above labeling pattern give rise graceful labeling to the graph $P_m \times P_n \times P_2$ and so, it is graceful. Take

$$k = \begin{cases} f(v_{m,n,1}), & \text{if } m \text{ is even and } n \text{ is odd} \\ f(v_{m,n-1,1}), & \text{if } m \text{ and } n \text{ both are even} \\ f(v_{m,n,2}), & \text{if } m \text{ and } n \text{ both are odd} \\ f(v_{m,n-1,2}), & \text{if } m \text{ is odd and } n \text{ is even}\ . \end{cases}$$

Then it is observed that for any $uv \in E(H)$, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$ and hence, H is α-graceful.

□

Theorem 2.5. *Let T be an α-graceful tree and $p = |V(T)|$. Let $f : V(T) \longrightarrow \{0, 1, 2, \ldots, p-1\}$ be an α-labeling and $k > 0$ with $min\{f(u), f(v)\} \leq k < max\{f(u), f(v)\}$, $\forall\ uv \in E(T)$. Let $V_1 = \{u \in V(T)/f(u) \leq k\}$ and $V_2 = \{u \in V(T)/f(u) > k\}$. If $|\,|v_1| - |v_2|\,| \leq 1$, then $P_n \times T$ is α-graceful.*

n≡ i (mod 6)	$f_2(v_n)$	$f_2(v_{n-1})$	$f_2(v_{n-2})$	$f_2(v_{n-3})$	$f_2(v_{n-4})$	$f_2(v_{n-5})$
i=1	t-2	t	t-1	t-4	t+1	t-3
i=3	t-2	t-1	t+1	t-3	t	t-5
i=5	t	t-3	t-1	t-2	t+2	t-4

TABLE 1. For $f_2(v_i)$

Proof. Let $q = |E(T)|$. Since $q - f$ is α-labeling for T and V_1, V_2 exchange their role in this case, without loss of generality assume that $|V_1| \geq |V_2|$.

Since, T is a tree, f and its edge induced function $f^\star : E(T) \longrightarrow \{1, 2, \ldots, q\}$ both are bijections. Let $G = P_n \times T$. It is obvious that $P = n \times p$ and $Q = (2n-1)p - n$ in G. Let $V(G) = V(T^{(1)}) \bigcup V(T^{(2)}) \bigcup \ldots \bigcup V(T^{(n)})$, where $V(T^{(i)}) = V_i^{(i)} \bigcup V_2^{(i)}$, $\forall$ $i = 1, 2, \ldots, n$.
Define $g : V(G) \longrightarrow \{0, 1, 2, \ldots, Q\}$ as follows.
$g/V_1^{(1)} = f/V_1$, $g/V_2^{(1)} = f/V_2 + (n-1)(2q-1)$, $g/V_1^{(2)} = (2q+1)(n-1) - f/V_1$, $g/V_2^{(2)} = n(2q+1) - g/V_2^{(1)}$ and $g/V_j^{(i)} = g/V_{j-2}^{(i)} - (-1)^i(2q+1)$, $\forall$ $i = 1, 2$ and $\forall$ $j = 3, 4, \ldots, n$.

Above labeling pattern give rise graceful labeling to G and so, G is graceful. Take

$$k = \begin{cases} \max\ \{g(v)/v \in V_1^{(n)}\}, & \text{when } n \text{ is odd} \\ \max\ \{g(v)/v \in V_2^{(n)}\}, & \text{when } n \text{ is even}. \end{cases}$$

It is obvious that for any $uv \in E(G)$, $\min\{g(u), g(v)\} \leq k < \max\{g(u), g(v)\}$ and so, G is α-graceful.

□

Corollary 2.1. *Grid $P_n \times P_m$ is α-graceful.*

Proof. As P_m is α-graceful and it satisfies require condition mentioned in Theorem-2.5, $P_n \times P_m$ is α-graceful. □

Corollary 2.2. *$P_n \times PST_n$ and $P_n \times DPST_n$ are α-graceful.*

Proof. As PST_n and $DPST_n$ satisfies require condition mentioned in Theorem-2.5, they are α-graceful graphs. □

Corollary 2.3. *Let T be a graceful tree. The tree S obtained by joining two copies of T say $T^{(1)}$ and $T^{(2)}$ by an edge between any two corresponding vertices $v^{(1)} \in V(T^{(1)})$ and $v^{(2)} \in V(T^{(2)})$, for some $v \in V(T)$ and $P_n \times S$ are α-graceful.*

Proof. S is α-graceful followed by Theorem-2.1 and $P_n \times S$ is α-graceful followed by Theorem-2.5, as S satisfies require conditions mentioned in Theorem-2.5. □

k=1	{ t-10, t+8, t-9, t+7, t-8, t+6, t-7, t+5, t-6, t+4, t-5, t+3, t-4, t+2, t-3, t+1, t-2, t, t-1, t-20, t+17, t-19, t+16, t-18, t+15, t-17, t+14, t-16, t+13, t-15, t+12, t-14, t+11, t-13, t+10, t-12, t+9, t-11 }

k=3	{ t-4, t-3, t-1, t+3, t-2, t+1, t-5, t+2, t-6, t+4, t-7, t+5, t-8, t+6, t-9, t+7, t-10, t+8, t-11, t+9, t, t-21, t+18, t-20, t+17, t-19, t+16, t-18, t+15, t-17, t+14, t-16, t+13, t-15, t+12, t-14, t+11, t-13 }
k=5	{ t, t+2, t-1, t-2, t-7, t-3, t+3, t-4, t+4, t-5, t+5, t-6, t+6, t-8, t+7, t-9, t+8, t-10, t+9, t-11, t+10, t-12, t+1, t-22, t+19, t-21, t+18, t-20, t+17, t-19, t+16, t-18, t+15, t-17, t+14, t-16, t+13, t-15 }
k=7	{ t-2, t-3, t, t-4, t+3, t-5, t+1, t-1, t-6, t+4, t-7, t+5, t-8, t+6, t-9, t+7, t-10, t+8, t-11, t+9, t-12, t+10, t-13, t+11, t+2, t-23, t+20, t-22, t+19, t-21, t+18, t-20, t+17, t-19, t+16, t-18, t+15, t-17 }
k=9	{ t-2, t-1, t-3, t, t-4, t+1, t-5, t+2, t-6, t+4, t-7, t+5, t-8, t+6, t-9, t+7, t-10, t+8, t-11, t+9, t-12, t+10, t-13, t+11, t-14, t+12, t+3, t-24, t+21, t-23, t+20, t-22, t+19, t-21, t+18, t-20, t+17, t-19 }
k=11	{ t-10, t-1, t, t-2, t+1, t-3, t+2, t-4, t+3, t-5, t+5, t-6, t+6, t-7, t+7, t-8, t+8, t-9, t+9, t-11, t+10, t-12, t+11, t-13, t+12, t-14, t+13, t-15, t+4, t-25, t+22, t-24, t+21, t-23, t+20, t-22, t+19, t-21 }
k=13	{ t-4, t-11, t, t-1, t+1, t-2, t+2, t-3, t+3, t-5, t+4, t-6, t+6, t-7, t+7, t-8, t+8, t-9, t+9, t-10, t+10, t-12, t+11, t-13, t+12, t-14, t+13, t-15, t+14, t-16, t+5, t-26, t+23, t-25, t+22, t-24, t+21, t-23 }
k=15	{ t-4, t, t-1, t-3, t+3, t-2, t-5, t+2, t-6, t+4, t-7, t+5, t-8, t+1, t+15, t-17, t+14, t-16, t+13, t-15, t+12, t-14, t+11, t-13, t+10, t-12, t+9, t-11, t+8, t-10, t+7, t-9, t+6, t-27, t+24, t-26, t+23, t-25 }
k=17	{ t-4, t+1, t-3, t-1, t-2, t-5, t+2, t-6, t+3, t-7, t+4, t-8, t+5, t-9, t+6, t, t+16, t-18, t+15, t-17, t+14, t-16, t+13, t-15, t+12, t-14, t+11, t-13, t+10, t-12, t+9, t-11, t+8, t-10, t+7, t-28, t+25, t-27 }
k=19	{ t-2, t, t-3, t+1, t-4, t+2, t-5, t-6, t+3, t-7, t+4, t-8, t+5, t-9, t+6, t-10, t+7, t-1, t+17, t-19, t+16, t-18, t+15, t-17, t+14, t-16, t+13, t-15, t+12, t-14, t+11, t-13, t+10, t-12, t+9, t-11, t+8, t-29 }
k=21	{ t, t+1, t-4, t+2, t-5, t+3, t-1, t-3, t-6, t+4, t-7, t+5, t-8, t+6, t-9, t+7, t-10, t+8, t-11, t-2, t+22, t-20, t+21, t-19, t+20, t-18, t+19, t-17, t+18, t-16, t+17, t-15, t+16, t-14, t+15, t-13, t+14, t-12 }
k=23	{ t+2, t-4, t-2, t-1, t-5, t, t-7, t+1, t+4, t-6, t+3, t-8, t+5, t-9, t+6, t-10, t+7, t-11, t-8, t-12, t+9, t-3, t+19, t-21, t+18, t-20, t+17, t-19, t+16, t-18, t+15, t-17, t+14, t-16, t+13, t-15, t+12, t-14 }
k=25	{ t-2, t-1, t+1, t-3, t+3, t, t-5, t+2, t-6, t+4, t-7, t+5, t-8, t+6, t-9, t+7, t-10, t+8, t-11, t+9, t-12, t+10, t-13, t-4, t+20, t-22, t+19, t-21, t+18, t-20, t+17, t-19, t+16, t-18, t+15, t-17, t+14, t-16 }
k=27	{ t-2, t, t-1, t+2, t-3, t+3, t-4, t+4, t-6, t+5, t-7, t+6, t-8, t+1, t-14, t+11, t-13, t+10, t-12, t+9, t-11, t+8, t-10, t+7, t-9, t-5, t+21, t-23, t+20, t-22, t+19, t-21, t+18, t-20, t+17, t-19, t+16, t-18 }
k=29	{ t-2, t+7, t-1, t, t-3, t+1, t-4, t+2, t-5, t-7, t+3, t-8, t+4, t-9, t+5, t-10, t+6, t-11, t+8, t-12, t+9, t-13, t+10, t-14, t+11, t-15, t+12, t-6, t+22, t-21, t+21, t-23, t+20, t-22, t+19, t-21, t+18, t-20 }
k=31	{ t+8, t-2, t-1, t-3, t, t-4, t+1, t-5, t+2, t-6, t+3, t-8, t+4, t-9, t+5, t-10, t+6, t-11, t+7, t-12, t+9, t-13, t+10, t-14, t+11, t-15, t-12, t-16, t+13, t-7, t+23, t-25, t+22, t-24, t+21, t-23, t+20, t-22 }

k=33	{ t-4, t-1, t+1, t, t-5, t+2, t-6, t+3, t-7, t-3, t+9, t-2, t+4, t-9, t+5, t-10, t+6, t-11, t+7, t-12, t+8, t-13, t+10, t-14, t+11, t-15, t+12, t-16, t+13, t-17, t+14, t-8, t+24, t-26, t+23, t-25, t+22, t-24 }
k=35	{ t, t-1, t+1, t-2, t+2, t-4, t+3, t-5, t+4, t-6, t+5, t-7, t+6, t-8, t-3, t-18, t+15, t-17, t+14, t-16, t+13, t-15, t+12, t-14, t+11, t-13, t+10, t-12, t+9, t-11, t+8, t-10, t+7, t-9, t+25, t-27, t+24, t-26 }
k=37	{ t, t-3, t-1, t-2, t-6, t+1, t-4, t+2, t+11, t-5, t+3, t-7, t+4, t-8, t+5, t-9, t+6, t-11, t+7, t-12, t+8, t-13, t+9, t-14, t+10, t-15, t+12, t-16, t+13, t-17, t+14, t-18, t+15, t-19, t+16, t-10, t+26, t-28 }

Table 4: For $f_5(v_i)$

3. Conclusions

In Th-2.1, if G is a bipartite universal graceful graph, then the graph H mentioned in Th-2.1 is also a universal α-graceful graph. Every $P_n (n \leq 16)$ is universal graceful graph. Any graceful graph G has at least two graceful centers as well as any α-graceful graph has at least four graceful centers.

According to Th-2.2, we would like to make a conjecture that every P_n(n is odd) is a universal graceful graph and so, according to Th-2.1, every path P_n(n is even) is a universal α-graceful graph. Here we also make another conjecture that the graph $P_m \times P_n \times P_r$ is α-graceful.

References

[1] Acharya B. D. and Gill M. K., (1981), On the index of gracefulness of a graph and the gracefulness of two dimensional square lattice graphs, Int. J. Math., 23, pp. 81-94.

[2] Gallian J. A., (2019), The Electronics Journal of Combinatorics, ♯DS6.

[3] Harary F., (2001), Graph theory, Narosa Publishing House, New Delhi.

[4] Kaneria V. J. and Makadia H. M., (2012), Some graceful graphs, J. of Math. Research, 4(1), pp. 54-57.

[5] Kaneria V. J., Makadia H. M. and Viradia R. V., (2015), Graceful labeling for disconnected grid related graphs, Bulletion of math. sci. and Applications, 4(1), pp. 6-11.

[6] Kaneria V. J., Makadia H. M. and Viradia R. V., (2016), Various graph operations on semi smooth graceful graphs, J. of Math. and Soft Computing, 6(1), pp. 57-79.

[7] Youssef M. Z., (2015), On α-labeling of disconnected graphs, Ars Combinatoric, 123, pp. 329-338.

n ≡ k (mod 10)	$f_3(v_n)$ $f_3(v_{n-1})$	$f_3(v_{n-2})$ $f_3(v_{n-3})$	$f_3(v_{n-4})$ $f_3(v_{n-5})$	$f_3(v_{n-6})$ $f_3(v_{n-7})$	$f_3(v_{n-8})$ $f_3(v_{n-9})$
k=1	t+1	t	t-1	t+3	t+2
	t-3	t-2	t-6	t-5	t-4
k=3	t-1	t-2	t+2	t	t+4
	t+1	t-3	t-4	t-7	t-6
k=5	t-3	t-2	t+3	t+2	t+1
	t	t-1	t-5	t-4	t-8
k=7	t-3	t	t+1	t+4	t+3
	t-1	t-4	t-2	t-6	t-5
k=9	t-1	t-4	t+1	t+2	t+5
	t	t-2	t-5	t-3	t-7

TABLE 2. For $f_3(v_i)$

n≡ k (mod 18)	$f_4(v_n)$ $f_4(v_{n-1})$ $f_4(v_{n-2})$	$f_4(v_{n-3})$ $f_4(v_{n-4})$ $f_4(v_{n-5})$	$f_4(v_{n-6})$ $f_4(v_{n-7})$ $f_4(v_{n-8})$	$f_4(v_{n-9})$ $f_4(v_{n-10})$ $f_4(v_{n-11})$	$f_4(v_{n-12})$ $f_4(v_{n-13})$ $f_4(v_{n-14})$	$f_4(v_{n-15})$ $f_4(v_{n-16})$ $f_4(v_{n-17})$
k=1	t+3	t-4	t	t-10	t+6	t-7
	t-5	t+1	t-2	t+7	t-8	t+4
	t+2	t-3	t-1	t-9	t+5	t-6
k=3	t-1	t+1	t-5	t+4	t+8	t-9
	t-3	t-4	t+3	t	t-10	t+6
	t-2	t+2	t-6	t-11	t+7	t-8
k=5	t-1	t	t+2	t-6	t+1	t-11
	t+3	t-3	t-5	t+5	t-12	t+8
	t-2	t-4	t+4	t-7	t+9	t-10
k=7	t-1	t+4	t	t-6	t+6	t-13
	t-4	t-3	t-5	t+5	t-8	t+10
	t-2	t+1	t+3	t-7	t+2	t-12
k=9	t-1	t	t+1	t-6	t+6	t-9
	t-4	t-2	t-5	t+4	t-8	t+3
	t+5	t-3	t+2	t-7	t+7	t-14
k=11	t+3	t-5	t	t-2	t+7	t-8
	t-6	t+1	t-3	t+8	t-9	t+5
	t+2	t-4	t-1	t-10	t+6	t-7
k=13	t-1	t+1	t-5	t+4	t+9	t-10
	t	t-4	t+3	t-7	t-11	t+7
	t-2	t+2	t-6	t-3	t+8	t-9
k=15	t-3	t-1	t+2	t-7	t+5	t-12
	t+1	t-6	t-5	t+4	t-4	t+9
	t-2	t	t+3	t-8	t-10	t-11
k=17	t-3	t-2	t+1	t-6	t+5	t-5
	t	t-1	t-4	t+4	t-9	t+11
	t+2	t-7	t+3	t-8	t+6	t-13

TABLE 3. For $f_4(v_i)$

H. M. Makadia is an assistant professor at Lukhdhirji Engineering College Morbi, Gujarat, India. He obtained his Ph.D. with Graph Theory in Saurashtra University Rajkot, India under the guidance of Dr. V. J. Kaneriya. He is interested in academic research activities.

V. J. Kaneria is presently working as a professor of mathematics in Saurashtra University Rajkot, Gujarat, India. He has completed his Ph.D. mathematic in 2009 from the same university. His interested areas in mathematics are graph theory and number theory.

Pritesh Andharia is an assistant professor at MRT and MSV Science College, Bhavnagar, Gujarat, India. He obtained his M.Phil. in mathematics from Saurashtra University, Rajkot, Gujarat, India. He is pursuing his Ph.D. in Mathematics in Saurashtra University under the guidance of Dr. V. J. Kaneria. He is interested in academic research activities.

Divyaba Jadeja is an assistant professor at V. V. P. Engineering College Rajkot, Gujarat, India. She has obtained M.Phil. in mathematics subject from Saurashtra University Rajkot, Gujarat, India. She is pursuing her Ph.D. in Mathematics subject from Saurashtra University under the guidance of Dr. V. J. Kaneriya. She is interested in academic research activities.