TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 919-930

NUMERICAL SOLUTION OF SECOND ORDER LINEAR HYPERBOLIC TELEGRAPH EQUATION

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Abstract. This paper is of about a numerical solution of the second order linear hyperbolic telegraph equation. To solve numerically the second order linear hyperbolic telegraph equation, the cubic B-spline collocation method is used in space discretization and the fourth order one-step method is used in time discretization. By using the fourth order one-step method, it is aimed to obtain a numerical algorithm whose accuracy is higher than the current studies. The efficiency and accuracy of the proposed method is studied by two examples. The obtained results show that the proposed method has higher accuracy as intended.

Keywords: Collocation method, cubic B-spline functions, one-step method, second order linear hyperbolic telegraph equation.

AMS Subject Classification: 65M70, 35L20.

1. Introduction

The hyperbolic-type equations [37, 20, 23, 30] like as telegraph, Klein-Gordon, sine-Gordon, etc. is encountered in the many fields of engineering and science such as vibrations of structures into beams, machines and buildings and represented the fundamental equations of atomic physics. The rapidly developing of the communication systems has been increasing the importance of the studies in this fields. In transmission media, the use of telegraph equation is commonly encountered in the studies for the analysis of information transmission signals and the propagation of electrical signals. The numerical solution of this equation, which has an important place in engineering and science, by various numerical methods is proposed. Mohanty obtained the numerical solution of telegraph equation by using the implicit three-level difference scheme [28]. Dehghan and Shokri used collocation points and approximated the numerical solution using thin plate splines radial basis function [6]. El-Azad and El-Gamel investigated Rothe-Wavelet method to get the

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§ Manuscript received: May 29, 2020; accepted: June 21, 2020.
This work has been supported by the Scientific Research Council of Eskisehir Osmangazi University under project No. 2018-2090.

numerical solution of the telegraph equation [13]. The work of Dehghan and Lakestani was about the numerical method based on the Chebyshev cardinal functions for solving the telegraph equation numerically [7]. In study [39], he developed Legendre multiwavelet Galerkin method for solving the hyperbolic telegraph equation. Saadatmandi and Dehghan proposed a numerical algorithm to solve the telegraph equation [34]. Biazar et al. solved numerically the second order hyperbolic partial differential equation by variational iteration method [4]. Borhanifar and Abazari employed the unconditionally stable parallel difference scheme to solve the second order one dimensional telegraph equation [5]. The work of Lakestani and coworker included the numerical technique based on interpolating scaling functions. In study [8], they used two methods for the solution of hyperbolic telegraph equation as the boundary integral equation method and the dual reciprocity technique. Salkuyeh and Ghehsareh applied the variational iteration method to get a numerical solution for the telegraph equation [35]. Dosti and Nazemi had an approach for the numerical solution of the telegraph equation by using quartic B-spline, septic B-spline and cubic B-spline quasi-interpolation collocation methods [10, 11, 12]. In studies [31] and [21], they obtained the differential quadrature algorithms to find the numerical solution of hyperbolic telegraph equation. Ding et al. worked on the numerical solution of the telegraph equation by the finite difference method based on non-polynomial splines [9]. Hosseini et al. used the numerical method as Rothe-wavelet-Galerkin method to get the solution of telegraph equation [18]. Mittal and Rachna used the modified cubic B-spline collocation method in space discretization and the strong-stability-preserving Runge-Kutta scheme in time discretization to solve the telegraph equation [27]. The study [25] included an unconditionally stable fourth-order method for the numerical solution of telegraph equation. Srivastava et al. performed a numerical study using the reduced differential transform method [38]. Inc et al. applied the reproducing Kernel Hilbert space method to be having the numerical solution of the telegraph equation [19]. The work [15] included the sinc-collocation method for solving the telegraph equation. In study [16], they developed an algorithm which obtained by the Chebyshev wavelets method for the numerical solution of telegraph equation. Abbasbandy et al. proposed two meshfree methods based on the radial basis functions [1]. The work [32] included the cubic B-spline collocation method for the numerical solution of telegraph equation. Rashidinia and Jokar had an approximation based on the polynomial scaling functions for the numerical solution of telegraph equation [33]. The work [14] was about the numerical solution algorithm of the telegraph equation by the high-order shifted Gegenbauer pseudospectral method. Mirzaee and Bimesl analyzed the telegraph equation by an approach based on the uniformly convergent Euler matrix method [26]. The work of Zhang et al. [41] was about an unconditionally stable method for solving the telegraph equation. Sharifi and Rashidinia used the finite difference and the finite element methods to get the numerical solution of the equation in study [36]. Yuzbasi presented a study [40] about the numerical solution of equation by using Bessel collocation method. Lu and Jiang studied the symplectic schemes for telegraph equation in the reference [24]. The study [2] which is about a numerical algorithm based on modified cubic trigonometric B-spline functions to solve the hyperbolic-type wave equations took place in the literature . Zhang and his coworker solved numerically the telegraph equation by using the Galerkin method and the orthogonal property of weighted Laguerre polynomials in the study [42]. Nazir et al. introduced the cubic trigonometric B-splines approach for the numerical solution of telegraph equation in the study [29]. Hong et al. obtained the numerical solution of telegraph equation by using adaptive Monte Carlo method [17].

In this study, the cubic b-spline colocation method, which is one of the numerical solution methods frequently encountered in the literature, is used for space dicretization of the telegraph equation. And differently from the previous ones, the one-step method which is of order four is used for time discretization of the telegraph equation. The organization of this paper is as follows. First, the time discretization of telegraph equation is described in Section 2. Then, the cubic B-spline collocation method is applied to the time discretized telegraph equation in Section 3. Two examples are given to investigate the efficiency of the proposed method, and a comparison with the existed studies is made in Section 4. Finally, the conclusion is given in Section 5.

The second order linear hyperbolic telegraph equation is given by

$$U_{tt} + 2\mu U_t + \lambda^2 U = U_{xx} + f(x,t) \tag{1}$$

where μ and λ are known positive constant parameters, the initial and boundary conditions are

$$\begin{array}{lll} U(x,0) = \alpha_0(x), & U_t(x,0) = \alpha_1(x), & a \le x \le b, \\ U(a,t) = \beta_0(t), & U(b,t) = \beta_1(t), & t \ge 0. \end{array} \tag{2}$$

Suppose that $\alpha_0(x)$, $\alpha_1(x)$ and their derivatives are continuous functions and similarly, $\beta_0(t)$, $\beta_1(t)$ and their derivatives are continuous functions.

If we set $U_t(x,t) = \widetilde{U}(x,t)$ in the equation of the form (1), it can be written as follows

$$U_t = \widetilde{U}, \tag{3}$$

$$\widetilde{U}_t = U_{xx} - 2\mu\widetilde{U} - \lambda^2 U + f(x,t). \tag{4}$$

The boundary and initial conditions can be rewritten as

$$\begin{array}{lll} U(a,t) = \beta_0(t), & U(b,t) = \beta_1(t), & t \ge 0, \\ \widetilde{U}(a,t) = \frac{\partial \beta_0}{\partial t}(t), & \widetilde{U}(b,t) = \frac{\partial \beta_1}{\partial t}(t), & t \ge 0, \\ U(x,0) = \alpha_0(x), & \widetilde{U}(x,0) = \alpha_1(x), & a \le x \le b. \end{array} \tag{5}$$

Let divide $[a,b]$ by N equally subinterval with the knots $x_l = a + jh$, $l = 0,1,2,..,N$ and $t_k = k\Delta t$, $k = 0,1,2,...$ where h and Δt are mesh sizes in the space and time direction respectively.

2. Description of time discretization

In this section, we discretizate Eqs. (3) and (4) in time by using one-step method which is of order four. The Eqs. (3) and (4) are discretized in time as follows

$$U^{k+1} = U^k + \theta_1 \widetilde{U}^{k+1} + \theta_2 \widetilde{U}^k + \theta_3 \widetilde{U}_t^{k+1} + \theta_4 \widetilde{U}_t^k \tag{6}$$

$$\widetilde{U}^{k+1} = \widetilde{U}^k + \theta_1 \widetilde{U}_t^{k+1} + \theta_2 \widetilde{U}_t^k + \theta_3 \widetilde{U}_{tt}^{k+1} + \theta_4 \widetilde{U}_{tt}^k \tag{7}$$

where θ_1,θ_2, θ_3 and θ_4 are known parameters and they can be obtained by using Taylor expansion. When $\theta_1 = \frac{\Delta t}{2}$, $\theta_2 = \frac{\Delta t}{2}$, $\theta_3 = 0$ and $\theta_4 = 0$, we obtain Crank-Nicolson approximation which is second order accurate in time. When $\theta_1 = \frac{\Delta t}{2}$, $\theta_2 = \frac{\Delta t}{2}$, $\theta_3 = \frac{-(\Delta t)^2}{12}$ and $\theta_4 = \frac{(\Delta t)^2}{12}$, we have high order accurate which is fourth order accurate in time.

By substituting Eq. (3) into Eq. (6), taking derivative with respect to t in the Eq. (3), and using Eq. (4), we obtain

$$\begin{array}{l} U^{k+1} - \theta_3 \left(U_{xx}^{k+1} + f(x,t_{k+1}) - 2\mu \widetilde{U}^{k+1} - \lambda^2 U^{k+1} \right) - \theta_1 \widetilde{U}^{k+1} = \\ U^k + \theta_2 \widetilde{U}^k + \theta_4 \left(U_{xx}^k + f(x,t_k) - 2\mu \widetilde{U}^k - \lambda^2 U^k \right). \end{array} \tag{8}$$

After some simplifications, (8) can be written in the folowing form:

$$\left(1 + \theta_3 \lambda^2\right) U^{k+1} + \left(2\mu\theta_3 - \theta_1\right) \widetilde{U}^{k+1} - \theta_3 U_{xx}^{k+1} = p(x,t_k) \tag{9}$$

where

$$p(x,t_k) = (1-\theta_4\lambda^2)U^k + (-2\mu\theta_4 \\ +\theta_2)\widetilde{U}^k + \theta_4 U^k_{xx} + \theta_4 f(x,t^k) + \theta_3 f(x,t_{k+1}).$$

With similar way, substituting Eq. (4) into Eq. (7) we have

$$\begin{aligned}&\widetilde{U}^{k+1} - \theta_1(U^{k+1}_{xx} + f(x,t_{k+1}) - 2\mu\widetilde{U}^{k+1} - \lambda^2 U^{k+1})\\ &-\theta_3\widetilde{U}^{k+1}_{tt} = \widetilde{U}^k + \theta_2(U^k_{xx} + f(x,t_k) - 2\mu\widetilde{U}^k - \lambda^2 U^k)\\ &+\theta_4\widetilde{U}^k_{tt},\end{aligned} \tag{10}$$

and taking derivative with respect to t in Eq. (4) we get

$$\begin{aligned}\widetilde{U}_{tt} &= (U_{xx})_t - 2\mu\widetilde{U}_t - \lambda^2 U_t + f_t(x,t),\\ &= (U_t)_{xx} - 2\mu\widetilde{U}_t - \lambda^2 U_t + f_t(x,t),\\ &= \widetilde{U}_{xx} - 2\mu\Big(U_{xx} - 2\mu\widetilde{U} - \lambda^2 U + f(x,t)\Big)\\ &\quad -\lambda^2\widetilde{U} + f_t(x,t).\end{aligned} \tag{11}$$

By substituting Eq. (11) into (10) and after essential arrangements we get

$$\begin{aligned}&(\lambda^2\theta_1 - 2\mu\lambda^2\theta_3)U^{k+1} + (1 + 2\mu\theta_1 + \lambda^2\theta_3\\ &-4\mu^2\theta_3)\widetilde{U}^{k+1} + (-\theta_1 + 2\mu\theta_3)U^{k+1}_{xx} - \theta_3\widetilde{U}^{k+1}_{xx}\\ = \;&(-\lambda^2\theta_2 + 2\mu\lambda^2\theta_4)U^k + (\theta_2 - 2\mu\theta_4)U^k_{xx}\\ &+(1 - 2\mu\theta_2 - \lambda^2\theta_4 + 4\mu^2\theta_4)\widetilde{U}^k + \theta_4\widetilde{U}^k_{xx}\\ &+(-2\mu\theta_4 + \theta_2)f(x,t_k) + \theta_4\frac{\partial f(x,t_k)}{\partial t}\\ &+(\theta_1 - 2\mu\theta_3)f(x,t_{k+1}) + \theta_3\frac{\partial f(x,t_{k+1})}{\partial t}\end{aligned}$$

This equation can be edited as follows:

$$\begin{aligned}&(\lambda^2\theta_1 - 2\mu\lambda^2\theta_3)U^{k+1} + (1 + 2\mu\theta_1 + \lambda^2\theta_3\\ &-4\mu^2\theta_3)\widetilde{U}^{k+1} + (-\theta_1 + 2\mu\theta_3)U^{k+1}_{xx} - \theta_3\widetilde{U}^{k+1}_{xx} = k(x,t_k)\end{aligned} \tag{12}$$

where

$$\begin{aligned}k(x,t_k) &= (-\lambda^2\theta_2 + 2\mu\lambda^2\theta_4)U^k + (1 - 2\mu\theta_2\\ &\quad -\lambda^2\theta_4 + 4\mu^2\theta_4)\widetilde{U}^k + (\theta_2 - 2\mu\theta_4)U^k_{xx}\\ &\quad +\theta_4\widetilde{U}^k_{xx} + (-2\mu\theta_4 + \theta_2)f(x,t_k) + \theta_4\frac{\partial f(x,t_k)}{\partial t}\\ &\quad +(\theta_1 - 2\mu\theta_3)f(x,t_{k+1}) + \theta_3\frac{\partial f(x,t_{k+1})}{\partial t}.\end{aligned}$$

3. Cubic B-spline collocation method

Let define cubic B-spline [22, 3] for $l=-1,0,...N+1$ in the following way

$$B_l(x)=\frac{1}{h^3}\begin{cases}(x-x_{l-2})^3, & x_{l-2}\le x<x_{l-1},\\ h^3+3h^2(x-x_{l-1})\\ +3h(x-x_{l-1})^2\\ -3(x-x_{l-1})^3, & x_{l-1}\le x<x_l,\\ h^3+3h^2(x_{l+1}-x)\\ +3h(x_{l+1}-x)^2\\ -3(x_{l+1}-x)^3, & x_l\le x<x_{l+1},\\ (x_{l+2}-x)^3, & x_{l+1}\le x<x_{l+2},\\ 0, & \text{otherwise.}\end{cases} \tag{13}$$

The approximate solutions $U_N(x,t)$ and $\widetilde{U}_N(x,t)$ are expressed in terms of the cubic B-spline functions as

$$U_N(x,t)=\sum_{l=-1}^{N+1}\delta_l(t)B_l(x),\quad \widetilde{U}_N(x,t)=\sum_{l=-1}^{N+1}\sigma_l(t)B_l(x) \tag{14}$$

where δ_l and σ_l, $l=-1,0,1,...,N+1$ are unknowns time depend parameters to be determined from collocation form of Eqs. (3) and (4).

Using Eqs. (13) and (14) ,the approximate functions $U_N,\widetilde{U}_N$ and their derivatives at the knots x_l can be written as

$$\begin{aligned}U_N(x_l)&=\delta_{l-1}+4\delta_l+\delta_{l+1},\\ \widetilde{U}_N(x_l)&=\sigma_{l-1}+4\sigma_l+\sigma_{l+1},\\ U'_N(x_l)&=\tfrac{3}{h}(-\delta_{l-1}+\delta_{l+1}),\\ \widetilde{U}'_N(x_l)&=\tfrac{3}{h}(-\sigma_{l-1}+\sigma_{l+1}),\\ U''_N(x_l)&=\tfrac{6}{h^2}(\delta_{l-1}-2\delta_l+\delta_{l+1}),\\ \widetilde{U}''_N(x_l)&=\tfrac{6}{h^2}(\sigma_{l-1}-2\sigma_l-\sigma_{l+1}).\end{aligned} \tag{15}$$

Substituting (14) into Eqs. (9) and (12), and using (15), we obtain;

$$\begin{aligned}&\delta_{l-1}^{k+1}\left[1+\theta_3\lambda^2-\tfrac{6}{h^2}\theta_3\right]+\delta_l^{k+1}\left[4(1+\theta_3\lambda^2+\tfrac{12}{h^2}\theta_3\right]+\\ &\delta_{l+1}^{k+1}\left[1+\theta_3\lambda^2-\tfrac{6}{h^2}\theta_3\right]+\sigma_{l-1}^{k+1}\left[2\mu\theta_3-\theta_1\right]+\\ &\sigma_l^{k+1}\left[4(2\mu\theta_3-\theta_1)\right]+\sigma_{l+1}^{k+1}\left[2\mu\theta_3-\theta_1\right]=p(x_l,t_k)\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\delta_{l-1}^{k+1}[(\lambda^2\theta_1-2\mu\lambda^2\theta_3)+\tfrac{6}{h^2}(-\theta_1+2\mu\theta_3)]\\ &+\delta_l^{k+1}[4(\lambda^2\theta_1-2\mu\lambda^2\theta_3)-\tfrac{12}{h^2}(-\theta_1+2\mu\theta_3)]\\ &+\delta_{l+1}^{k+1}[(\lambda^2\theta_1-2\mu\lambda^2\theta_3)+\tfrac{6}{h^2}(-\theta_1+2\mu\theta_3)]\\ &+\sigma_{l-1}^{k+1}[(1+2\mu\theta_1+\lambda^2\theta_3-4\mu^2\theta_3)-\theta_3\tfrac{6}{h^2}]\\ &+\sigma_l^{k+1}[4(1+2\mu\theta_1+\lambda^2\theta_3-4\mu^2\theta_3)+\theta_3\tfrac{12}{h^2}]\\ &+\sigma_{l+1}^{k+1}[(1+2\mu\theta_1+\lambda^2\theta_3-4\mu^2\theta_3)-\theta_3\tfrac{6}{h^2}]=k(x_l,t_k)\end{aligned} \tag{17}$$

where $l=0,1,\dots,N$,

$$\begin{aligned}p(x_l,t_k)\ &=\ (1-\theta_4\lambda^2)u_l^k+(-2\mu\theta_4+\theta_2)\widetilde{U}_l^k\\ &\quad+\theta_4\left(u_{xx}\right)_l^k+\theta_4f(x_l,t_k)+\theta_3f(x_l,t_{k+1})\end{aligned}$$

and

$$\begin{aligned} k(x_l,t_k) &= (-\lambda^2\theta_2+2\mu\lambda^2\theta_4)u_l^k+(1-2\mu\theta_2 \\ &\quad -\lambda^2\theta_4+4\mu^2\theta_4)\widetilde{U}_l^k+(\theta_2-2\mu\theta_4)\left(u_{xx}\right)_l^k \\ &\quad +\theta_4(\widetilde{U}_{xx})_l^k+(-2\mu\theta_4+\theta_2)f(x_l,t_k) \\ &\quad +\theta_4 f_t(x_l,t_k)+(\theta_1-2\mu\theta_3)f(x_l,t_{k+1}) \\ &\quad +\theta_3 f_t(x_l,t_{k+1}). \end{aligned}$$

When Eqs. (16) and (17) are associated, the system of linear equations, which is $2N+2$ algebraic equations with $2N+6$ unknowns as $\boldsymbol{\delta}=(\delta_{-1},\delta_{0,...},\delta_{N+1})^T$ and $\boldsymbol{\sigma}=(\sigma_{-1},\sigma_0,...,\sigma_{N+1})^T$, is obtained.

To get unique solutions for $\boldsymbol{\delta}=(\delta_{-1},\delta_{0,...},\delta_{N+1})$ and $\boldsymbol{\sigma}=(\sigma_{-1},\sigma_0,...,\sigma_{N+1})$, we get help from the boundry conditions (5). So, $\delta_{-1}^{k+1},\delta_{N+1}^{k+1},\sigma_{-1}^{k+1}$ and σ_{N+1}^{k+1} can be eliminated from the system. Thus we obtain a linear $(2N+2)\times(2N+2)$ system of equtions which can be solved by using Gauss elimination procedure.

The values of the parameters δ_l^{k+1} and σ_l^{k+1}, $l=0,...,N+1$ can be evaluated after finding the values of the initial parameters δ_l^0 and σ_l^0, $l=-1,0,...,N+1$ by the help of employing the boundary and initial conditions:

$$\begin{array}{l} (U_N)_x(a,0)=\frac{3}{h}(-\delta_{-1}^0+\delta_1^0)=\frac{\partial\alpha_0}{\partial x}(a), \\ U_N(x_l,0)=\delta_{l-1}^0+4\delta_l^0+\delta_{l+1}^0=\alpha_0(x_l), \\ (U_N)_x(b,0)=\frac{3}{h}(-\delta_{N-1}^0+\delta_{N+1}^0)=\frac{\partial\alpha_0}{\partial x}(b), \end{array} \tag{18}$$

and

$$\begin{array}{l} (\widetilde{U}_N)_x(a,0)=\frac{3}{h}(-\sigma_{-1}^0+\sigma_1^0)=\frac{\partial\alpha_1}{\partial x}(a), \\ \widetilde{U}_N(x_l,0)=\sigma_{l-1}^0+4\sigma_l^0+\sigma_{l+1}^0=\alpha_1(x_l), \\ (\widetilde{U}_N)_x(b,0)=\frac{3}{h}(-\sigma_{-1}^0+\sigma_1^0)=\frac{\partial\alpha_1}{\partial x}(b) \end{array} \tag{19}$$

where $l=0,1,...,N$. The systems in (18) and (19) consist of $N+3$ unknown parameters in $N+3$ equation, rescpectively. So the systems can be solved by using Gauss elimination scheme.

4. Numerical examples

In this part, we applied proposed method to two examples of linear telegraph equation. To compute the maxmimum error L_∞, we used the following formula:

$$L_\infty=\|U-U_N\|_\infty=\max_l\left|U_l-(U_N)_l\right|.$$

The order of convergence is computed by the formula:

$$\text{order}=\frac{\log\left|\dfrac{(L_\infty)_{\Delta t_i}}{(L_\infty)_{\Delta t_{i+1}}}\right|}{\log\left|\dfrac{\Delta t_i}{\Delta t_{i+1}}\right|}, \tag{20}$$

where $(L_\infty)_{\Delta t_i}$ is the error norm L_∞ for time step Δt_i.

4.1. **Example.** Consider the telegraph equation (1) with $\mu = \frac{1}{2}$ and $\lambda = 1$ in the space domain $[0, 1]$. The initial and boundary conditions are given as follows:

$$u(x,0) = 0,\ u_t(x,0) = 0,$$
$$u(0,t) = 0,\ u(1,t) = 0,$$

and $f(x,t) = (2-2t+t^2)(x-x^2)\exp(-t)+2t^2\exp(-t)$. The exact solution of this example is $u(x,t) = (x-x^2)t^2\exp(-t)$.

L_∞.error is presented in Table 1 for $h = 0.01, 0.005$, $\Delta t = 0.01$ at various times $t = 1, 2, 3, 4, 5$. The estimated solutions are compared with the results obtained by [6, 10, 32, 36, 2, 29]. According to the results in Table 1, the proposed method is considerable good compared to the collocation methods based on various B-spline functions applied to solve the telegraph equation. The rate of convergence is listed in Table 2. From this table, it is seen that the order of the method is almost 4. Fig. 1 is given to see the solution profile up to time 2. It can be seen that the simulations for numerical and analytical solutions at $t = 1$ in Fig. 2.

Table 1. The error norms for example 4.1 with $\Delta t = 0.01$ at various times.

Method	h	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
Present	0.005	1.20×10^{-11}	5.49×10^{-12}	4.15×10^{-12}	2.69×10^{-12}	1.69×10^{-12}
Present	0.01	1.20×10^{-11}	5.46×10^{-12}	4.17×10^{-12}	2.67×10^{-12}	1.69×10^{-12}
[6]	0.01	1.85×10^{-5}	1.07×10^{-5}	1.82×10^{-5}	1.65×10^{-5}	1.05×10^{-5}
[10]	0.005	1.92×10^{-4}	1.14×10^{-4}	1.71×10^{-4}	2.08×10^{-4}	9.84×10^{-5}
[32]	0.005	3.40×10^{-6}	7.78×10^{-6}	1.72×10^{-6}	5.75×10^{-7}	9.70×10^{-7}
[36]	0.01	1.67×10^{-7}	4.72×10^{-7}			
[2]	0.01	4.29×10^{-5}	1.20×10^{-5}	1.23×10^{-5}	1.45×10^{-5}	1.22×10^{-6}
[29]	0.01	8.76×10^{-5}	3.29×10^{-5}	5.90×10^{-6}	3.04×10^{-5}	6.92×10^{-6}

Table 2. The rate of convergence for example 1 with $h = 0.005$.

Δt_i	order
1	
0.5	3.748
0.25	4.001
0.125	4.031
0.0625	3.996
0.03125	3.999
0.015625	4.000
0.0078125	4.000

4.2. **Example.** Consider the telegraph equation (1) with $\mu = 10$, $\lambda = 5$ in the space domain $[0, 2]$ and $f(x,t) = \mu\left(1+\tan^2(\frac{x+t}{2})\right) + \lambda^2\tan(\frac{x+t}{2})$. The initial conditons are given by

$$u(x,0) = \tan(\frac{x}{2}),$$
$$u_t(x,0) = \frac{1}{2}\left(1+\tan^2(\frac{x}{2})\right)$$

and the boundary conditions are

$$u(0,t) = \tan(\frac{t}{2}),\ u(2,t) = \tan(\frac{2+t}{2}).$$

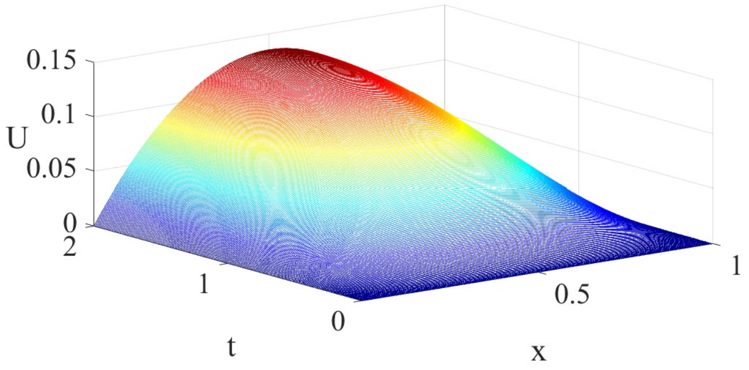

FIGURE 1. The solutions at various times with $h = 0.005$ and $\Delta t = 0.01$.

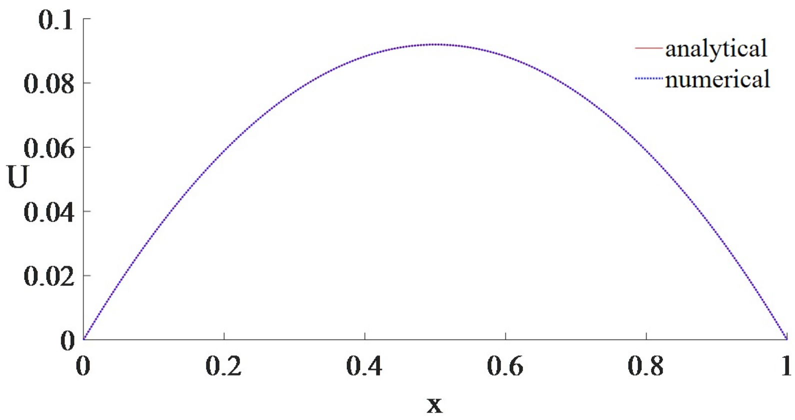

FIGURE 2. The solution profile at $t = 1$ with $h = 0.005$ and $\Delta t = 0.01$.

The analytical solution of given example is $u(x,t) = tan(\frac{x+t}{2})$. The maximum error L_∞ is tabulated in Table 3 for $t = 0.2, 0.4, 0.6, 0.8, 1$ with $h = 0.001$ and $\Delta t = 0.001, 0.005$. The solutions obtained by proposed method are compared with the results obtained by [12, 27, 36]. We observe that obtained results by the proposed meyhod are more accurate than the results obtained by the others. The simulation of solution is shown in Fig. 3. The comparisons of numerical and exact solutions is figured in Fig. 4.

TABLE 3. The error norms for example 4.2 with $h = 0.001$ at various times.

Methods	Δt	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1$
Present	0.001	1.20×10^{-8}	3.73×10^{-8}	1.23×10^{-7}	6.35×10^{-7}	1.26×10^{-5}
Present	0.005	3.00×10^{-7}	9.32×10^{-7}	3.07×10^{-6}	1.59×10^{-5}	3.15×10^{-4}
[12]	0.005	1.89×10^{-4}	3.99×10^{-4}	7.9715×10^{-4}	1.88×10^{-3}	8.01×10^{-3}
[27]	0.001	3.61×10^{-4}	1.04×10^{-4}	2.60×10^{-3}	7.63×10^{-3}	4.66×10^{-2}
[36]	0.001	6.83×10^{-5}	4.28×10^{-5}			

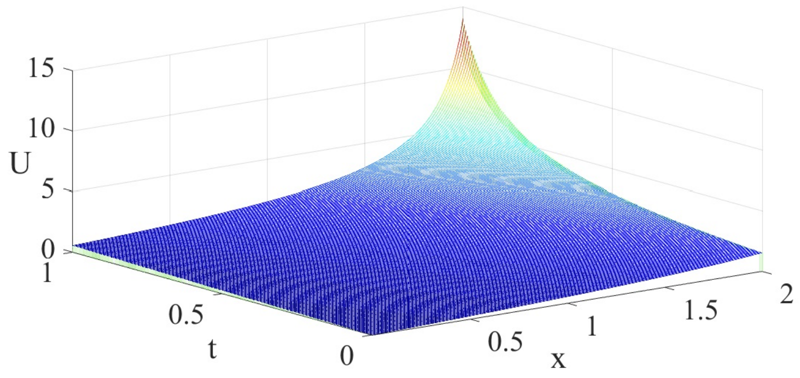

FIGURE 3. The solutions at various times with $h = 0.002$ and $\Delta t = 0.001$.

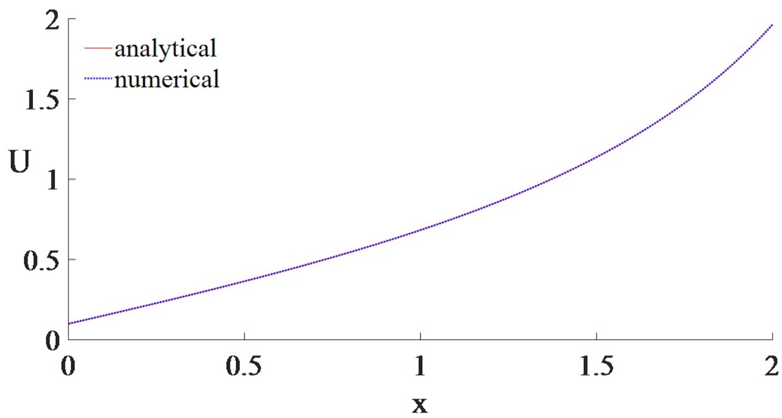

FIGURE 4. The solution profile at $t = 0.2$ with $h = 0.001$ and $\Delta t = 0.001$.

5. Conclusion

In this study, the numerical solution of second order linear hyperbolic telegraph equation is obtained numerical method which had a high accurate. The cubic B-spline collocation method is used for the space discretization of the telegraph equation. The fourth order one-step method is used for the time discretization of the telegraph equation. The reason why the proposed method differs from the existed studies is that the accuracy of the method used for the time discretization is of order $O(\Delta t^4)$. In this way, it is shown that the numerical solutions with higher accuracy than the existing studies can be obtained. The easy application and the high accuracy are the advantages of the proposed method. As a result, it can be said that it is an useful method for obtaining the numerical solutions of linear partial differential equations which has significant physical application areas like telegraph equation.

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