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A STUDY ON SPHERICAL FUZZY IDEALS OF SEMIGROUP

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Abstract. In this paper, we introduce the notion of spherical fuzzy ideals of semigroup and establish the properties of it with suitable examples. Also, we introduce the concept of spherical fuzzy sub-semigroup, spherical fuzzy left(resp.right) ideal, spherical fuzzy bi-ideal, spherical fuzzy interior ideal, and homomorphism of a spherical fuzzy ideal in semigroups with suitable illustration. We show that every spherical fuzzy left(right) ideal is a spherical fuzzy bi-ideal.

Keywords: spherical fuzzy set, fuzzy ideals, semigroup.

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1. Introduction

After the introduction of the fuzzy set by Zadeh[21], several researchers conducted experiments on the generalizations of the notion of a fuzzy set. The concept of the intuitionistic fuzzy set was introduced by Atanassov [1, 2] as a generalization of the fuzzy set. Faisal et al. [4] characterized intra-regular ordered AG groupoids in terms of generalized fuzzy ideals. Thillaigovindan and Chinnadurai [13] introduced the notion of an i-v fuzzy interior ideal, an i-v fuzzy quasi-ideal and an i-v fuzzy bi-ideal in a semi-group. Thillaigovindan et al. [14, 15, 16] introduced interval- valued fuzzy ideals of near-rings, generalized T-fuzzy bi-ideals of γ-semigroup and characterizations of near-rings by interval-valued (α, β)-fuzzy ideals. Jun et al. [5, 6] considered the fuzzification of interior ideals in semigroups and the notion of an intuitionistic fuzzy interior ideal of a semigroup S, and investigated its properties. Kuroki [9] discussed some properties of the fuzzy ideals and fuzzy bi-ideals in the semigroup. Majid et al. [10] presented the idea of neutrosophic cubic semigroups and neutrosophic cubic points. Madad et al. [11] defined the concept of generalized cubic subsemigroups (ideals) of a semigroup and investigated some of its related properties. Muhammed et al. [12] studied the idea of complex neutrosophic

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subsemigroups. Yaqoob [17] introduced the concept of interval-valued intuitionistic fuzzy sets and applied to regular LA-semigroups and established the characterizations of regular LA-semigroups. Yaqoob et al. [18] introduced the concept of interval-valued fuzzy LA-subsemigroups (left, right, two-sided, interior, bi-) ideals in LA-semigroups. Jun et al. [7] considered the fuzzification of (1,2)-ideals in semigroups and investigated its properties. Yager [19, 20] introduced the Pythagorean fuzzy set as a generalization of the fuzzy set. After its existence, several researchers also studied the properties of the fuzzy ideals of the semigroup. Gun et al. [8] introduced the new concept of spherical fuzzy set and discussed the new operations. In this paper, we discuss the properties of spherical fuzzy ideals in the semigroup.

2. Preliminaries

In this section, we discuss the details of fuzzy ideals in semigroup.

Definition 2.1. [3] *Let S be a semigroup.*
(i) M and N be subsets of S, the product of M and N is defined as
$MN = \{mn \in S \mid m \in M \text{ and } n \in N\}$.
(ii) A non- empty subset M of S is called a sub-semigroup of S if $MM \subseteq M$.
(iii) A non-empty subset M of S is called a left (resp. right) ideal of S if $SM \subseteq M$ (resp.$MS \subseteq M$).
(iv) A is called a two sided ideal of S if it is both a left ideal and right ideal of S.
(v) A sub- semigroup M of S is called a bi-ideal of S if $MSM \subseteq M$.
(vi) A sub-semigroup M of S is called a (1,2) ideal of S if $MSM^2 \subseteq M$.
(vii) A semigroup S is said to be (2,2)- regular if $m \in m^2Sm^2$ for any $m \in S$.
(viii) A semigroup S is called regular if for each element $m \in S$ there exists $x \in S$ such that $m = mxm$.
(ix) A semigroup S is said to be completely regular if, for any $m \in S$, there exists $x \in S$ such that $m = mxm$ and $mx = xm$. For a semigroup S, is completely regular if and only if(iff) S is a union of groups iff S is (2,2)-regular.
By a fuzzy set μ in a non-empty set S, we mean a function $\mu : S \to [0,1]$, and the complement of μ, denoted by $\overline{\mu}$, is the fuzzy set in S given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in S$.

Definition 2.2. [19] *Let X be a universe of discourse, A Pythagorean fuzzy set (PFS) $P = \{z, \vartheta_p(x), \omega_p(x)/z \in X\}$ where $\vartheta : X \to [0,1]$ and $\omega : X \to [0,1]$ represent the degree of membership and non-membership of the object $z \in X$ to the set P subset to the condition $0 \leq (\vartheta_p(z))^2 + (\omega_p(z))^2 \leq 1$ for all $z \in X$. For the sake of simplicity a PFS is denoted as $P = (\vartheta_p(z), \omega_p(z))$.*

Definition 2.3. [8] *Let X be a universe of discourse, A spherical fuzzy set (SFS) $P = \{z, \mu_p(x), \zeta_p(x), \psi_p(x)/z \in X\}$ where $\mu : X \to [0,1]$, $\zeta : X \to [0,1]$ and $\psi : X \to [0,1]$ represent the degree of membership, non-membership and hesitancy of the object $z \in X$ to the set P subset to the condition $0 \leq (\mu_p(z))^2 + (\zeta_p(z))^2 + (\psi_p(z))^2 \leq 1$ for all $z \in X$. For the sake of simplicity a SFS is denoted as $P = (\mu_p(z), \zeta_p(z), \psi_p(z))$.*

3. Spherical fuzzy ideals of semigroup

In this section, let S denote a semigroup unless otherwise specified.

Definition 3.1. *A spherical fuzzy set (SFS) $P = (\mu_p, \zeta_p, \psi_p)$ in S is called a spherical fuzzy sub-semigroup of S, if*
(ii) $\quad \mu_p(x_1x_2) \geq min\{\mu_p(x_1), \mu_p(x_2)\}$

(ii) $\zeta_p(x_1x_2) \geq min\{\zeta_p(x_1), \zeta_p(x_2)\}$
(iii) $\psi_p(x_1x_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$ *for all* $x_1, x_2 \in S$.

Definition 3.2. *A SFS* $P = (\mu_p, \zeta_p, \psi_p)$ *in* S *is called a spherical fuzzy left ideal of* S*, if*
(i) $\mu_p(x_1x_2) \geq \mu_p(x_2)$
(ii) $\zeta_p(x_1x_2) \geq \zeta_p(x_2)$
(iii) $\psi_p(x_1x_2) \leq \psi_p(x_2)$ *for all* $x_1, x_2 \in S$.
A spherical fuzzy right ideal of S *is defined in an analogous way. A SFS* $P = (\mu_p, \zeta_p, \psi_p)$ *in* S *is called a spherical fuzzy ideal of* S*, if it is both a spherical fuzzy left and spherical right ideal of* S*. It is clear that any spherical fuzzy left(resp. right) ideal of* S *is a spherical fuzzy sub-semigroup of* S*.*

Example 3.1. *Let* $S = \{u, v, w, x, y\}$ *be a semigroup the following cayley table.*

TABLE 1. Cayley table

$\bullet$	p	q	r	s
p	p	p	p	p
q	p	p	p	p
r	p	p	q	p
s	p	p	q	q

Define a spherical fuzzy set $P = (\mu_p, \zeta_p, \psi_p)$ *in* S *as follows.*

TABLE 2

S	$\mu(x)$	$\zeta(x)$	$\psi(x)$
p	*0.7*	*0.9*	*0.3*
q	*0.5*	*0.6*	*0.5*
r	*0.3*	*0.4*	*0.7*
s	*0.5*	*0.6*	*0.6*

Thus $P = (\mu_p, \zeta_p, \psi_p)$ *is spherical fuzzy sub-semigroup of* S*.*

Definition 3.3. *A spherical fuzzy sub-semigroup* $P = (\mu_p, \zeta_p, \psi_p)$ *of* S *is called a spherical fuzzy bi-ideal(SFBI) of* S*.*
(i) $\mu_p(x_1ux_2) \geq min\{\mu_p(x_1), \mu_p(x_2)\}$
(ii) $\zeta_p(x_1ux_2) \geq min\{\zeta_p(x_1), \zeta_p(x_2)\}$
(ii) $\psi_p(x_1ux_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$ *for all* $u, x_1, x_2 \in S$.

Example 3.2. *Using table-1 Define a spherical fuzzy set* $P = (\mu_p, \zeta_p, \psi_p)$ *in* S *as follows.*

TABLE 3

S	$\mu(x)$	$\zeta(x)$	$\psi(x)$
p	*0.9*	*0.8*	*0.3*
q	*0.4*	*0.4*	*0.5*
r	*0.2*	*0.3*	*0.7*
s	*0.6*	*0.5*	*0.4*

Thus $P = (\mu_p, \zeta_p, \psi_p)$ *is spherical fuzzy bi-ideal of* S*.*

Theorem 3.1. *If* $\{P_i\}_{i\in I}$ *is a family of SFBI of* S*, then* $\cap P_i$ *is a SFBI of* S*. Where* $\cap P_i = (\wedge\mu_{p_i}, \wedge\zeta_{p_i}, \vee\psi_{p_i})$ *and* $\wedge\mu_{p_i} = inf\,\{\mu_{p_i}(x_1)|i \in I, x_1 \in S\}$,
$\wedge\zeta_{p_i} = inf\,\{\zeta_{p_i}(x_1)|i \in I, x_1 \in S\}$, $\vee\psi_{p_i} = sup\,\{\psi_{p_i}(x_1)|i \in I, x_1 \in S\}$.

Proof. Let $x_1, x_2 \in S$. Then we have

$$\begin{aligned}
\wedge\mu_{p_i}(x_1x_2) &\geq \wedge\{min\,\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\
&= min\,\{min\,\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\
&= min\,\{min\,\{\mu_{p_i}(x_1)\}, min\,\{\mu_{p_i}(x_2)\}\}\\
&= min\,\{\wedge\mu_{p_i}(x_1), \wedge\mu_{p_i}(x_2)\}\\
\wedge\zeta_{p_i}(x_1x_2) &\geq \wedge\{min\,\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\
&= min\,\{min\,\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\
&= min\,\{min\,\{\zeta_{p_i}(x_1)\}, min\,\{\zeta_{p_i}(x_2)\}\}\\
&= min\,\{\wedge\zeta_{p_i}(x_1), \wedge\zeta_{p_i}(x_2)\}\\
\vee\psi_{p_i}(x_1x_2) &\leq \vee\{max\,\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\
&= max\,\{max\,\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\
&= max\,\{max\,\{\psi_{p_i}(x_1)\}, max\,\{\psi_{p_i}(x_2)\}\}\\
&= max\,\{\vee\psi_{p_i}(x_1), \vee\psi_{p_i}(x_2)\}.
\end{aligned}$$

Hence $\cap P_i$ is a spherical fuzzy sub-semigroup of S.
Next for $u, x_1, x_2 \in S$, we obtain

$$\begin{aligned}
\wedge\mu_{p_i}(x_1ux_2) &\geq \wedge\{min\,\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\
&= min\,\{min\,\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\
&= min\,\{min\,\{\mu_{p_i}(x_1)\}, min\,\{\mu_{p_i}(x_2)\}\}\\
&= min\,\{\wedge\mu_{p_i}(x_1), \wedge\mu_{p_i}(x_2)\}\\
\wedge\zeta_{p_i}(x_1ux_2) &\geq \wedge\{min\,\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\
&= min\,\{min\,\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\
&= min\,\{min\,\{\zeta_{p_i}(x_1)\}, min\,\{\zeta_{p_i}(x_2)\}\}\\
&= min\,\{\wedge\zeta_{p_i}(x_1), \wedge\zeta_{p_i}(x_2)\}\\
\vee\psi_{p_i}(x_1ux_2) &\leq \vee\{max\,\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\
&= max\,\{max\,\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\
&= max\,\{max\,\{\psi_{p_i}(x_1)\}, max\,\{\psi_{p_i}(x_2)\}\}\\
&= max\,\{\vee\psi_{p_i}(x_1), \vee\psi_{p_i}(x_2)\}.
\end{aligned}$$

Hence $\cap P_i$ is a SFBI of S.
This completes the proof. □

Theorem 3.2. *Every spherical fuzzy left(right) ideal of* S *is a spherical fuzzy bi-ideal of* S*.*

Proof. Let $P = (\mu_p, \zeta_p, \psi_p)$ be a spherical fuzzy left ideal of S and $u, x_1, x_2 \in S$.
Then

$$\begin{aligned}
\mu_p(x_1ux_2) &= \mu_p(x_1ux_2)\\
&\geq \mu_p(x_2)\\
\mu_p(x_1ux_2) &\geq min\{\mu_p(x_1, \mu_p(x_2))\}\\
\zeta_p(x_1ux_2) &= \zeta_p(x_1ux_2)\\
&\geq \zeta_p(x_2)\\
\zeta_p(x_1ux_2) &\geq min\{\zeta_p(x_1, \zeta_p(x_2))\}\\
\psi_p(x_1ux_2) &= \psi_p(x_1ux_2)\\
&\leq \psi_p(x_2)\\
\psi_p(x_1ux_2) &\leq max\{\psi_p(x_1, \psi_p(x_2))\}
\end{aligned}$$

Thus $P = (\mu_p, \zeta_p, \psi_p)$ is SFBI of S.
The right case is provided in an analogous way.

□

Theorem 3.3. *Every spherical fuzzy bi-ideal of a group S is constant.*

Proof. Let $P = (\mu_p, \zeta_p, \psi_p)$ be a SFBI of a group S and let x_1 be any element of S. Then

$$\begin{aligned} \mu_p(x_1) &= \mu_p(ex_1e) \\ &\geq min\{\mu_p(e), \mu_p(e)\} \\ &= \mu_p(e) \\ &= \mu_p(ee) \\ &= \mu_p(x_1x_1^{-1})(x_1^{-1}x_1) \\ &= \mu_p(x_1(x_1^{-1}x_1^{-1})x_1) \\ &\geq min\{\mu_p(x_1, \mu_p(x_1))\} \\ &\mu_p(x_1) \end{aligned}$$

$$\begin{aligned} \zeta_p(x_1) &= \zeta_p(ex_1e) \\ &\geq min\{\zeta_p(e), \zeta_p(e)\} \\ &= \zeta_p(e) \\ &= \zeta_p(ee) \\ &= \zeta_p(x_1x_1^{-1})(x_1^{-1}x_1) \\ &= \zeta_p(x_1(x_1^{-1}x_1^{-1})x_1) \\ &\geq min\{\zeta_p(x_1, \zeta_p(x_1))\} \\ &\zeta_p(x_1) \end{aligned}$$

and

$$\begin{aligned} \psi_p(x_1) &= \psi_p(ex_1e) \\ &\leq max\{\psi_p(e), \psi_p(e)\} \\ &= \psi_p(e) \\ &= \psi_p(ee) \\ &= \psi_p(x_1x_1^{-1})(x_1^{-1}x_1) \\ &= \psi_p(x_1(x_1^{-1}x_1^{-1})x_1) \\ &\leq max\{\psi_p(x_1, \psi_p(x_1))\} \\ &= \psi_p(x_1). \end{aligned}$$

Where e is the identity of S. It follows that $\mu_p(x_1) = \mu_p(e)$, $\zeta_p(x_1) = \zeta_p(e)$ and $\psi_p(x_1) = \psi_p(e)$ which means that $P = (\mu_p, \zeta_p, \psi_p)$ is constant. □

Theorem 3.4. *If a SFS $P = (\mu_p, \zeta_p, \psi_p)$ in S is a SFBI of S, then so is $\Box P = \left(\mu_p, \zeta_p, \overline{\zeta}_p\right)$.*

Proof. It is sufficient to show that $\overline{\zeta}_p$ satisfies the conditions in Definition 3.1 and Definition 3.4. For any $u, x_1, x_2 \in S$, we have

$$\begin{aligned} \overline{\zeta}_p(x_1x_2) &= 1 - \zeta_p(x_1x_2) \\ &\leq 1 - min\left\{\zeta_p(x_1), \zeta_p(x_2)\right\} \\ &= max\left\{1 - \zeta_p(x_1), 1 - \zeta_p(x_2)\right\} \\ &= max\left\{\overline{\zeta}_p(x_1), \overline{\zeta}_p(x_2)\right\} \end{aligned}$$

and

$$\begin{aligned} \overline{\zeta}_p(x_1ux_2) &= 1 - \zeta_p(x_1ux_2) \\ &\leq 1 - min\left\{\zeta_p(x_1), \zeta_p(x_2)\right\} \\ &= max\left\{1 - \zeta_p(x_1), 1 - \zeta_p(x_2)\right\} \\ &= max\left\{\overline{\zeta}_p(x_1), \overline{\zeta}_p(x_2)\right\}. \end{aligned}$$

Therefore $\Box P$ is a SFBI of S. □

Definition 3.4. *A spherical fuzzy sub-semigroup $P = (\mu_p, \zeta_p, \psi_p)$ of S is called a spherical fuzzy (1,2) ideal of S. If*
(i) $\mu_p(x_1u(x_2x_3)) \geq min\{\mu_p(x_1), \mu_p(x_2), \mu_p(x_3)\}$
(ii) $\zeta_p(x_1u(x_2x_3)) \geq min\{\zeta_p(x_1), \zeta_p(x_2), \zeta_p(x_3)\}$
(iii) $\psi_p(x_1u(x_2x_3)) \leq max\{\psi_p(x_1), \psi_p(x_2), \psi_p(x_3)\}$ $u, x_1, x_2, x_3 \in S$.

Theorem 3.5. *Every SFBI is a spherical fuzzy (1,2) ideal of S.*

Proof. Let SFS $P = (\mu_p, \zeta_p, \psi_p)$ be a SFBI of S and let $u, x_1, x_2, x_3 \in S$.
Then

$$\begin{aligned}\mu_p(x_1u(x_2x_3)) &= \mu_p((x_1ux_2)x_3)\\ &\geq min\{\mu_p(x_1ux_2), \mu_p(x_3)\}\\ &\geq min\{min\{\mu_p(x_1), \mu_p(x_2)\}, \mu_p(x_3)\}\\ &= min\{\mu_p(x_1), \mu_p(x_2), \mu_p(x_3)\}\end{aligned}$$

$$\begin{aligned}\zeta_p(x_1u(x_2x_3)) &= \zeta_p((x_1ux_2)x_3)\\ &\geq min\{\zeta_p(x_1ux_2), \zeta_p(x_3)\}\\ &\geq min\{min\{\zeta_p(x_1), \zeta_p(x_2)\}, \zeta_p(x_3)\}\\ &= min\{\zeta_p(x_1), \zeta_p(x_2), \zeta_p(x_3)\}\end{aligned}$$

and

$$\begin{aligned}\psi_p(x_1u(x_2x_3)) &= \psi_p((x_1ux_2)x_3)\\ &\leq max\{\psi_p(x_1ux_2), \psi_p(x_3)\}\\ &\leq max\{max\{\psi_p(x_1), \psi_p(x_2)\}, \psi_p(x_3)\}\\ &= max\{\psi_p(x_1), \psi_p(x_2), \psi_p(x_3)\}.\end{aligned}$$

Hence $P = (\mu_p, \zeta_p, \psi_p)$ is a spherical fuzzy (1,2) ideal of S. □

To consider the converse of theorem next theorem, we need to strengthen the condition of a semigroup S.

Theorem 3.6. *If S is a regular semigroup, then every spherical fuzzy (1,2) ideal of S is a SFBI of S.*

Proof. Assume that a semigroup S is regular and let $P = (\mu_p, \zeta_p, \psi_p)$ be a spherical fuzzy (1,2) ideal of S. Let $u, x_1, x_2, x_3 \in S$. Since S is regular, we have $x_1u \in (x_1Sx_1)S \subseteq x_1Sx_1$, which implies that $x_1u = x_1Sx_1$ for some $s \in S$.
Thus

$$\begin{aligned}\mu_p(x_1ux_2) &= \mu_p((x_1sx_1)x_2)\\ &= \mu_p(x_1s(x_1x_2))\\ &\geq min\{\mu_p(x_1), \mu_p(x_1), \mu_p(x_2)\}\\ &= min\{\mu_p(x_1), \mu_p(x_2)\}\end{aligned}$$

$$\begin{aligned}\zeta_p(x_1ux_2) &= \zeta_p((x_1sx_1)x_2)\\ &= \zeta_p(x_1s(x_1x_2))\\ &\geq min\{\zeta_p(x_1), \zeta_p(x_1), \zeta_p(x_2)\}\\ &= min\{\zeta_p(x_1), \zeta_p(x_2)\}\end{aligned}$$

and

$$\begin{aligned}\psi_p(x_1ux_2) &= \psi_p((x_1sx_1)x_2)\\ &= \psi_p(x_1s(x_1x_2))\\ &\leq max\{\psi_p(x_1), \psi_p(x_1), \psi_p(x_2)\}\\ &= max\{\psi_p(x_1), \psi_p(x_2)\}.\end{aligned}$$

Therefore $P = (\zeta_p, \psi_p)$ is PFBI of S. □

Theorem 3.7. *A SFS $P = (\mu_p, \zeta_p, \psi_p)$ is a SFBI of S if and only if the fuzzy sets μ_p, ζ_p and $\overline{\psi_p}$ are FBI of S.*

Proof. Let $P = (\mu_p, \zeta_p, \psi_p)$ be a SFBI of S. Then clearly μ_p is a FBI of S. Let $u, x_1, x_2 \in S$.
Then

$$\begin{aligned}\overline{\psi_p}(x_1x_2) &= 1 - \psi_p(x_1x_2)\\ &\geq 1 - max\{\psi_p(x_1), \psi_p(x_2)\}\\ &= min\{(1-\psi_p(x_1)), (1-\psi_p(x_2))\}\\ &= min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}\end{aligned}$$

$$\begin{aligned}\overline{\psi_p}(x_1ux_2) &= 1 - \psi_p(x_1ux_2)\\ &\geq 1 - max\{\psi_p(x_1), \psi_p(x_2)\}\\ &= min\{(1-\psi_p(x_1)), (1-\psi_p(x_2))\}\\ &= min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}.\end{aligned}$$

Hence $\overline{\psi_p}$ is a fuzzy bi-ideal of S.
Conversely, suppose that ζ_p and $\overline{\psi_p}$ are FBI of S. Let $u, x_1, x_2 \in S$.
Then

$$\begin{aligned}1 - \psi_p(x_1x_2) &= \overline{\psi_p}(x_1x_2)\\ &\geq min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}\\ &= min\{(1-\psi_p(x_1)), (1-\psi_p(x_2))\}\\ &= max\{\psi_p(x_1), \psi_p(x_2)\}\end{aligned}$$

$$\begin{aligned}1 - \psi_p(x_1ux_2) &= \overline{\psi_p}(x_1ux_2)\\ &\geq min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}\\ &= 1 - max\{\psi_p(x_1), \psi_p(x_2)\}.\end{aligned}$$

Which implies that $\psi_p(x_1x_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$ and $\psi_p(x_1ux_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$
This completes the proof. □

Definition 3.5. *A SFS $P = (\mu_p, \zeta_p, \psi_p)$ in S is called a spherical fuzzy interior ideal(SFII) of S if it satisfies*
(i) $\mu_p(x_1ux_2) \geq \mu_p(u)$
(ii) $\zeta_p(x_1ux_2) \geq \zeta_p(u)$
(iii) $\psi_p(x_1ux_2) \leq \psi_p(u)$ $u, x_1, x_2 \in S$.

Example 3.3. *Define a spherical fuzzy set $P = (\mu_p, \zeta_p, \psi_p)$ in S as follows.*

TABLE 4

S	$\mu(x)$	$\zeta(x)$	$\psi(x)$
p	0.5	0.4	0.3
q	0.3	0.2	0.4
r	0.2	0.1	0.3
s	0.3	0.2	0.6

Thus $P = (\mu_p, \zeta_p, \psi_p)$ is spherical fuzzy interior ideal of S.

Theorem 3.8. *If $\{P_i\}_{i \in I}$ is a family of SFII of S, then $\cap P_i$ is a SFII of S. Where $\cap P_i = (\wedge\mu_{p_i}, \wedge\zeta_{p_i}, \vee\psi_{p_i})$ and $\wedge\mu_{p_i}(x_1) = inf\{\mu_{p_i}(x_1) | i \in I, x_1 \in S\}$,*
$\wedge\zeta_{p_i}(x_1) = inf\{\zeta_{p_i}(x_1) | i \in I, x_1 \in S\}$, $\vee\psi_{p_i}(x_1) = sup\{\psi_{p_i}(x_1) | i \in I, x_1 \in S\}$.

Proof. Let $u, x_1, x_2 \in S$.
Then

$$\begin{aligned}\wedge\mu_{p_i}(x_1x_2) &\geq min\{min\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\ &= (\wedge\mu_{p_i}(x_1)) \wedge (\wedge\mu_{p_i}(x_2))\end{aligned}$$

$$\wedge\zeta_{p_i}(x_1x_2) \geq min\{min\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}$$

$= (\wedge\zeta_{p_i}(x_1)) \wedge (\wedge\zeta_{p_i}(x_2))$
and
$\vee\psi_{p_i}(x_1x_2) \leq max\,\{max\,\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}$
$= (\vee\psi_{p_i}(x_1)) \vee (\vee\psi_{p_i}(x_2))$
$\wedge\mu_{p_i}(x_1ux_2) \geq \wedge\mu_{p_i}(u)$
$\wedge\zeta_{p_i}(x_1ux_2) \geq \wedge\zeta_{p_i}(u)$
and
$\vee\psi_{p_i}(x_1ux_2) \leq \vee\psi_{p_i}(u)$.
Hence $\cap P_i$ is a SFII of S. □

Definition 3.6. *Let $P = (\mu_p, \zeta_p, \psi_p)$ be a SFS of S and let $\alpha \in [0, 1]$ then the sets. $\mu_{p,\alpha} = \{x_1 \in S : \mu_p(x_1)\alpha\}$, $\zeta_{p,\alpha} = \{x_1 \in S : \zeta_p(x_1)\alpha\}$ and $\psi_{p,\alpha} = \{x_1 \in S : \psi_p(x_1)\alpha\}$ are called a μ_p-level α-cut, ζ_p-level α-cut and ψ_p-level α-cut of K respectively.*

Example 3.4. *An α-cut or α- level set of SFS is*
A_α $=\{X/(\mu_A(x), \zeta_A(x), \psi_A(x)) \geq \alpha, \forall\ x \in X\}$
such that $(\mu_A(x))^2 + (\zeta_A(x))^2 + (\psi_A(x))^2 \leq 1$, $\alpha \in [0, 1]$.
Consider $X = \{p, q, r, s\}$ and a set A.
$A_x = \{(0.7, 0.6, 0.3)/p, (0.5, 0.6, 0.5)/q, (0.3, 0.4, 0.7)/r, (0.5, 0.6, 0.6)/s\}$. *Then α cut of X* $= (0.6, 0.4, 0.3)$. *Hence, $A_{(0.6,0.4,0.3)} = (0.7, 0.6, 0.3)/p$.*

Theorem 3.9. *If a SFS $P = (\mu_p, \zeta_p, \psi_p)$ in S is a SFII of S, then the μ-level α-cut $\mu_{p,\alpha}$, ζ-level α-cut $\zeta_{p,\alpha}$and ψ-level α-cut $\psi_{p,\alpha}$ of P are interior ideal of S, for every $\alpha \in Im(\mu_p) \cap Im(\zeta_p) \cap Im(\psi_p) \subseteq [0, 1]$.*

Proof. Let $\alpha \in Im(\mu_p) \cap Im(\zeta_p) \cap Im(\psi_p) \subseteq [0, 1]$.
let $x_1, x_2 \in \mu_{p,\alpha}$ then $\mu_p(x_1) \geq \alpha$ and $\mu_p(x_2) \geq \alpha$. It follows from that
$\mu_p(x_1x_2) \geq \mu_p(x_1) \wedge \mu_p(x_2) \geq \alpha$. So that $x_1, x_2 \in \mu_{p,\alpha}$.
If $x_1, x_2 \in \zeta_{p,\alpha}$ then $\zeta_p(x_1) \geq \alpha$ and $\zeta_p(x_2) \geq \alpha$. It follows from that.
$\zeta_p(x_1x_2) \geq \zeta_p(x_1) \wedge \zeta_p(x_2) \geq \alpha$. So that $x_1, x_2 \in \zeta_{p,\alpha}$.
If $x_1, x_2 \in \psi_{p,\alpha}$, then $\psi_p(x_1) \leq \alpha$ and $\psi_p(x_2) \leq \alpha$ and so $\psi_p(x_1x_2) \leq \psi_p(x_1) \vee \psi_p(x_2) \leq \alpha$, that is $x_1, x_2 \in \psi_{p,\alpha}$.
Hence $\mu_{p,\alpha}$, $\zeta_{p,\alpha}$ and $\psi_{p,\alpha}$ are sub-semigroup of S. Now let $x_1x_2 \in S$ and $u \in \mu_{p,\alpha}$. Then $\mu_p(x_1ux_2) \geq \mu_p(u) \geq \alpha$ and so $x_1ux_2 \in \mu_{p,\alpha}$.
If $u \in \zeta_{p,\alpha}$. Then $\zeta_p(x_1ux_2) \geq \zeta_p(u) \geq \alpha$ and so $x_1ux_2 \in \zeta_{p,\alpha}$.
If $u \in \psi_{p,\alpha}$. Then $\psi_p(x_1ux_2) \leq \psi_p(u) \leq \alpha$ thus $x_1ux_2 \in \psi_{p,\alpha}$.
Therefore $\mu_{p,\alpha}$,$\zeta_{p,\alpha}$ and $\psi_{p,\alpha}$ are interior ideal of S. □

Theorem 3.10. *A SFS $P = (\mu_p, \zeta_p, \psi_p)$ is and SFII of S if and only if the fuzzy set $\mu_p, \zeta_p, \overline{\psi}_p$ are fuzzy interior ideal (FII) of S.*

Proof. Let $P = (\mu_p, \zeta_p, \psi_p)$ be an SFII of S. Then clearly μ_p is FII of S. Let $u, x_1, x_2 \in S$. Then
$\overline{\psi_p}(x_1x_2) = 1 - \psi_p(x_1x_2)$
$\geq 1 - (\psi_p(x_1)) \vee \psi_p(x_2)$
$= (1 - \psi_p(x_1)) \wedge (1 - \psi_p(x_2))$
$= \overline{\psi_k}(x_1) \wedge \overline{\psi_p}(x_2)$
$\overline{\psi_p}(x_1ux_2) = 1 - \psi_p(x_1ux_2)$
$\geq 1 - (\psi_p(u))$
$= \overline{\psi_p}(u)$
$\overline{\psi_k}$ is a FII of S.
Conversely.

Suppose that ζ_p and $\overline{\psi_p}$ are FII of S. Let $u, x_1, x_2 \in S$.

$$\begin{aligned} 1-\psi_p(x_1x_2) &= \overline{\psi_p}(x_1x_2) \\ &\geq \overline{\psi_p}(x_1) \wedge \overline{\psi_p}(x_2) \\ &= (1-\psi_p(x_1)) \wedge (1-\psi_p(x_2)) \\ &= 1-\psi_p(x_1) \vee \psi_p(x_2) \\ &= 1-\psi_p(x_1ux_2) = \overline{\psi_p}(x_1ux_2) \\ &\geq \overline{\psi_p}(u) = 1-\psi_p(u) \end{aligned}$$

which implies $\psi_p(x_1x_2) \leq \psi_p(x_1) \vee \psi_p(x_2)$
and
$\psi_p(x_1ux_2) \leq \psi_p(u)$
This completes the proof. □

4. Homomorphism of spherical fuzzy ideals in semigroup

Let f be a mapping from a set A to a set B. If $P_1 = (\mu_{p_1}, \zeta_{p_1}, \psi_{p_1})$ and $P_2 = (\mu_{p_2}, \zeta_{p_2}, \psi_{p_2})$ are SFSs in A and B respectively then the preimage of B under f, denoted by $f^{-1}(p_2)$ is a SFS in P_1 defined by $f^{-1}(p_2) = \big(f^{-1}(\mu_{p_2}), f^{-1}(\zeta_{p_2}), f^{-1}(\psi_{p_2})\big)$.

Theorem 4.1. *Let $f : S \to T$ be a homomorphism of semigroup. If $P_2 = (\mu_p, \zeta_{p_2}, \psi_{p_2})$ is a SFBI of T. Then the preimage $f^{-1}(P_2) = \big(f^{-1}(\mu_{p_2}), f^{-1}(\zeta_{p_2}), f^{-1}(\psi_{p_2})\big)$ of P_2 under f is a SFBI of S.*

Proof.

$$\begin{aligned} f^{-1}(\mu_{p_2})(x_1x_2) &= \mu_{p_2}(f(x_1x_2)) \\ &= \mu_{p_2}(f(x_1), f(x_2)) \\ &\geq min\,\{\mu_{p_2}(f(x_1)), \mu_{p_2}(f(x_2))\} \\ &= min\,\big\{f^{-1}(\mu_{p_2}(x_1)), f^{-1}(\mu_{p_2}(x_2))\big\} \end{aligned}$$

$$\begin{aligned} f^{-1}(\zeta_{p_2})(x_1x_2) &= \zeta_{p_2}(f(x_1x_2)) \\ &= \zeta_{p_2}(f(x_1), f(x_2)) \\ &\geq min\,\{\zeta_{p_2}(f(x_1)), \zeta_{p_2}(f(x_2))\} \\ &= min\,\big\{f^{-1}(\zeta_{p_2}(x_1)), f^{-1}(\zeta_{p_2}(x_2))\big\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\psi_{p_2})(x_1x_2) &= \psi_{p_2}(f(x_1x_2)) \\ &= \zeta_{p_2}(f(x_1), f(x_2)) \\ &\leq max\,\{\psi_{p_2}(f(x_1)), \psi_{p_2}(f(x_2))\} \\ &= max\,\big\{f^{-1}(\psi_{p_2}(x_1)), f^{-1}(\psi_{p_2}(x_2))\big\}. \end{aligned}$$

Hence $f^{-1}(P_2) = \big(f^{-1}(\mu_{p_2}), f^{-1}(\zeta_{p_2}), f^{-1}(\psi_{p_2})\big)$ is a spherical fuzzy sub-semigroup of S for any $u, x_1, x_2 \in S$.

$$\begin{aligned} f^{-1}(\mu_{p_2})(x_1ux_2) &= \mu_{p_2}(f(x_1ux_2)) \\ &= \mu_{p_2}(f(x_1), f(u)), f(x_2)) \\ &\geq min\,\{\mu_{p_2}(f(x_1)), \mu_{p_2}(f(x_2))\} \\ &= min\,\big\{f^{-1}(\mu_{p_2}(x_1)), f^{-1}(\mu_{p_2}(x_2))\big\} \end{aligned}$$

$$\begin{aligned} f^{-1}(\zeta_{p_2})(x_1ux_2) &= \zeta_{p_2}(f(x_1ux_2)) \\ &= \zeta_{p_2}(f(x_1), f(u)), f(x_2)) \\ &\geq min\,\{\zeta_{p_2}(f(x_1)), \zeta_{p_2}(f(x_2))\} \\ &= min\,\big\{f^{-1}(\zeta_{p_2}(x_1)), f^{-1}(\zeta_{p_2}(x_2))\big\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\psi_{p_2})(x_1ux_2) &= \psi_{p_2}(f(x_1ux_2)) \\ &= \zeta_{p_2}(f(x_1), f(u)), f(x_2)) \\ &\leq max\,\{\psi_{p_2}(f(x_1)), \psi_{p_2}(f(x_2))\} \end{aligned}$$

$= max\left\{f^{-1}(\psi_{p_2}(x_1)), f^{-1}(\psi_{p_2}(x_2))\right\}.$

Therefore $f^{-1}(p_2) = \left(f^{-1}(\mu_{p_2}), f^{-1}(\zeta_{p_2}), f^{-1}(\psi_{p_2})\right)$ is a SFBI of S. □

Theorem 4.2. *Let $f : A \to B$ be a homomorphism of semigroup. If $P_2 = (\mu_p, \zeta_{p_2}, \psi_{p_2})$ is a SFII of B, then preimage $f^{-1}(P_2) = (f^{-1}(\mu_{p_2}), f^{-1}(\zeta_{p_2}), f^{-1}(\psi_{p_2}))$ of p_2 under f is a SFII of S.*

Proof. Assume that $P_2 = (\mu_{p_2}, \zeta_{p_2}, \psi_{p_2})$ is a SFII of S and let $x_1, x_2 \in S$.
Then

$$\begin{aligned}
f^{-1}(\mu_{p_2})(x_1x_2) &= \mu_{p_2}(f(x_1x_2)) \\
&= \mu_{p_2}(f(x_1)f(x_2)) \\
&\geq \mu_{p_2}(f(x_1)) \wedge \mu_{p_2}(f(x_2)) \\
&= f^{-1}(\mu_{p_1}(x_1)) \wedge f^{-1}(\mu_{p_2}(x_2)) \\
f^{-1}(\zeta_{p_2})(x_1x_2) &= \zeta_{p_2}(f(x_1x_2)) \\
&= \zeta_{p_2}(f(x_1)f(x_2)) \\
&\geq \zeta_{p_2}(f(x_1)) \wedge \zeta_{p_2}(f(x_2)) \\
&= f^{-1}(\zeta_{p_1}(x_1)) \wedge f^{-1}(\zeta_{p_2}(x_2)) \\
f^{-1}(\psi_{p_2})(x_1x_2) &= \psi_{p_2}(f(x_1x_2)) \\
&= \psi_{p_2}(f(x_1)f(x_2)) \\
&\leq \psi_{p_2}(f(x_1)) \vee \psi_{p_2}(f(x_2)) \\
&= f^{-1}(\psi_{p_1}(x_1)) \vee f^{-1}(\psi_{p_2}(x_2)).
\end{aligned}$$

Hence $f^{-1}(P_2) = (f^{-1}(\mu_{p_2}), f^{-1}(\zeta_{p_2}), f^{-1}(\psi_{p_2}))$ is a SF sub-semigroup of S for any $u.x_1, x_2 \in S$,
we have

$$\begin{aligned}
f^{-1}(\mu_{p_2})(x_1ux_2) &= \mu_{p_2}(f(x_1ux_2)) \\
&= \mu_{p_2}(f(x_1), f(u), f(x_2)) \\
&\geq \mu_{p_2}(f(u)) \\
&= f^{-1}(\mu_{p_2}(u)) \\
f^{-1}(\zeta_{p_2})(x_1ux_2) &= \zeta_{p_2}(f(x_1ux_2)) \\
&= \zeta_{p_2}(f(x_1), f(u), f(x_2)) \\
&\geq \zeta_{p_2}(f(u)) \\
&= f^{-1}(\zeta_{p_2}(u)) \\
f^{-1}(\psi_{p_2})(x_1ux_2) &= \psi_{p_2}(f(x_1ux_2)) \\
&= \psi_{p_2}(f(x_1), f(u), f(x_2)) \\
&\leq \psi_{p_2}(f(u)) \\
&= f^{-1}(\psi_{p_2}(u)).
\end{aligned}$$

Therefore $f^{-1}(P_2) = (f^{-1}(\mu_{p_2}), f^{-1}(\zeta_{p_2}), f^{-1}(\psi_{p_2}))$ is SFII of S. □

5. Conclusions

In this paper spherical fuzzy sub-semigroup, spherical fuzzy left(resp.right) ideal, spherical fuzzy ideal, spherical fuzzy bi-ideal, spherical fuzzy interior ideal and Homomorphism of spherical fuzzy ideal in semigroups are studied and investigated with some properties and suitable examples.

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