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FITTED NON-POLYNOMIAL SPLINE METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL DIFFERENCE EQUATIONS WITH INTEGRAL BOUNDARY CONDITION

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Abstract. The aim of this paper is to present fitted non-polynomial spline method for singularly perturbed differential-difference equations with integral boundary condition. The stability and uniform convergence of the proposed method are proved. To validate the applicability of the scheme, two model problems are considered for numerical experimentation and solved for different values of the perturbation parameter, ε and mesh size, h. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence and it is observed that the present method is more accurate and uniformly convergent for $h \geq \varepsilon$ where the classical numerical methods fails to give good result and it also improves the results of the methods existing in the literature.

Keywords: Singularly perturbed problems, Delay differential equation, Non-polynomial spline, Integral boundary condition.

AMS Subject Classification: 65L11, 65L12, 65L20.

1. Introduction

A differential equation in which the highest order derivative is multiplied by a small positive parameter ε is called singular perturbed problem and the parameter ε is known as the perturbation parameter [24]. Such type of problems are numerically treated by different researchers, to mention in point one can refer [4],[5][23], and [6].

A differential equation is said to be singularly perturbed delay differential equation, if it includes at least one delay term, involving unknown functions occurring with different arguments, and also, the highest derivative term is multiplied by a small parameter. Such type of delay, differential equations play a very important role in the mathematical models of science and engineering, such as, the human pupil light reflex with mixed delay type [19], variational problems in control theory with small state problem [14], models of HIV infection [7], and signal transition [12].

Any system involving a feedback control almost involves time delay. The delay occurs

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because a finite time is required to sense the information and then react to it. Finding the solution of singularly perturbed delay differential equations, whose application mentioned above, is a challenging problem. In response to these, in recent years, there has been a growing interest in numerical methods on singularly perturbed delay differential equations. The authors of [1],[8],[9] have developed various numerical schemes on uniform meshes for singularly perturbed differential equations with integral boundary conditions. The authors of [2],[13],[18] have proved that the problem of differential equations with integral boundary conditions is well posed. Cubic spline in compression approximations for singularly perturbed delay differential equation with large delay has been presented by [3]. Similarly, the authors of [15] developed fitted non-polynomial cubic spline method for singularly perturbed delay convection-diffusion equations.

The standard numerical methods used for solving singularly perturbed differential equation are sometime ill posed and fail to give analytical solution when the perturbation parameter ε is small. Therefore, it is necessary to develop suitable numerical methods which are uniformly convergent to solve this type of differential equations. In [20],[21],[26],[29] finite difference and finite element methods are proposed to solve this kind of equations with large and small shifts. Recently, the problem under consideration was done by [25],[10],[11],[17] using fitted mesh and fitted operator methods.

As far as the researchers knowledge is concerned, numerical solution of singularly perturbed boundary value problem containing integral boundary condition via fitted non-polynomial spline method is first being considered.
Thus, the purpose of this study is to develop stable, convergent and more accurate numerical method for solving singularly perturbed differential-difference equations with Integral boundary condition. Throughout our analysis C is generic positive constant that are independent of the number of mesh points N. We assume that $\bar{\Omega} = [0,2]$, $\Omega = (0,2)$, $\Omega_1 = (0,1)$, $\Omega_2 = (1,2)$, $\Omega^* = \Omega_1 \cup \Omega_2$.

2. Statement of the problem

Consider the following singularly perturbed problem

$$Ly(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), x \in (0,2), \tag{1}$$

$$y(x) = \phi(x), x \in [-1,0], \tag{2}$$

$$Ky(2) = y(2) - \varepsilon \int_0^2 g(x)y(x)dx = l, \tag{3}$$

where $\phi(x)$ is sufficiently smooth on $[-1,0]$. For all $x \in \Omega$, it is assumed that the sufficient smooth functions $a(x), b(x)$ and $c(x)$ satisfy at $a(x) > a > 0, b(x) > b \geq 0, c(x) \leq c < 0$, and $a + b + c > 0$. Furthermore, $g(x)$ is non-negative and monotonic with $\int_0^2 g(x)dx < 1$. The above assumptions ensure that $y \in X = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$.

Eqs. (1)–(3) exhibits strong boundary layer at $x = 2$ due to the small perturbation parameter, ε and interior layer at $x = 1$ due to the delay parameter [11].
As we observed from Eqs. (1)-(3), the value of $y(x-1)$ is known for the domain Ω_1 and unknown for the domain Ω_2 due to the large delay at $x = 1$. So, it is impossible to treat the problem throughout the domain $\bar{\Omega}$. Therefore, we have to treat the problem at Ω_1 and Ω_2 separately.

So, Eqs. (1)-(3) is equivalent to

$$Ly(x) = R(x), \tag{4}$$

where

$$Ly(x) = \begin{cases} L_1y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x), x \in \Omega_1, \\ L_2y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1), \\ x \in \Omega_2. \end{cases} \tag{5}$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), x \in \Omega_1, \\ f(x), x \in \Omega_2, \end{cases} \tag{6}$$

with boundary conditions

$$\begin{cases} y(x) = \phi(x), x \in [-1, 0], \\ y(1^-) = y(1^+), y'(1^-) = y'(1^+), \\ Ky(2) = y(2) - \varepsilon \int_0^2 g(x)y(x)dx = l. \end{cases} \tag{7}$$

3. Properties of Continuous solution

Lemma 3.1. *(Maximum Principle) Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0, K\psi(2) \geq 0, L_1\psi(x) \geq 0, \forall x \in \Omega_1, L_2\psi(x) \geq 0, \forall x \in \Omega_2$ and $[\psi'](1) \leq 0$ then $\psi(x) \geq 0, \forall x \in \bar{\Omega}$.*

Proof. Refer [11] □

Lemma 3.2. *(Stability Result) The solution $y(x)$ of the problem* (1)-(3)*, satisfies the bound*

$$|y(x)| \leq C \max\{|y(0)|, |Ky(2)|, \sup_{x \in \Omega*} |Ly(x)|\}, \quad x \in \overline{\Omega}$$

Proof. Refer [11] □

Lemma 3.3. *Let $y(x)$ be the solution of* (1)-(3)*. Then we have the following bounds:*

$$|y^{(k)}(x)|_{\Omega^*} \leq C\varepsilon^{-k}, \; for \quad k = 1, 2, 3.$$

Proof. Refer [11] □

4. Numerical Scheme Formulation

The linear ordinary differential equation in Eq.(1) cannot, in general, be solved analytically because of the dependence of $a(x), b(x)$ and $c(x)$ on the spatial coordinate x.
We divide the interval $[0, 2]$ into $2N$ equal parts with constant mesh length h. Let $0 = x_0, x_1, ..., x_N = 1, x_{N+1}, x_{N+2}, ..., x_{2N} = 2$ be the mesh points. Then, we have $x_i = ih, \quad i = 0, 1, 2, ..., 2N$.

From Eqs.(4)–(6), we have

$$\begin{cases} -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) - c(x)\phi(x-1), & x \in \mathit{\Omega}_1, \\ y_0 = \phi_0, \quad y(1) = \theta. \end{cases} \tag{8}$$

Now, the domain $[0, 1]$ is discretized into N equal number of subintervals, each of length h. Let $0 = x_0 < x_1 < ... < x_N = 1$ be the points such that $x_i = ih, i = 0, 1, 2, ..., N$.

We can rewrite Eq. (8) as

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = R(x), \quad x \in \mathit{\Omega}_1, \tag{9}$$

where $p(x) = -a(x), q(x) = -b(x), R(x) = c(x)\phi(x-1) - f(x)$.
Consider a uniform mesh with interval $[0, 1]$ in which $0 = x_0 < x_1 < ... < x_N = 1$ where $h = \frac{1}{N}$ and $x_i = ih, i = 0, 1, 2, ..., N$.

For each segment $[x_i, x_{i+1}], i = 1, 2, ..., N-1$ the non-polynomial cubic spline $S(x)$ has the following form

$$S(x) = a_i + b_i(x - x_i) + c_i(e^{w(x-x_i)} + e^{-w(x-x_i)}) + d_i(e^{w(x-x_i)} - e^{-w(x-x_i)}), \qquad (10)$$

where a_i, b_i, c_i and d_i are unknown coefficients, and $w \neq 0$ arbitrary parameter which will be used to increase the accuracy of the method.
To determine the unknown coefficients in Eq. (10) in terms of y_i, y_{i+1}, M_i and M_{i+1}, first we define

$$\begin{cases} S(x_i) = y_i, & S(x_{i+1}) = y_{i+1}, \\ S''(x_i) = M_i, & S''(x_{i+1}) = M_{i+1}. \end{cases} \qquad (11)$$

The coefficients in Eq. (10) are determined as

$$\begin{cases} a_i = & y_i - \frac{M_i}{w^2}, \\ b_i = & \frac{y_{i+1}-y_i}{h} + \frac{M_i - M_{i+1}}{w\theta}, \\ c_i = & \frac{M_{i+1}}{w^2(e^{\theta}-e^{-\theta})} - \frac{M_i(e^{\theta}+e^{-\theta})}{2w^2(e^{\theta}-e^{-\theta})}, \\ d_i = & \frac{M_i}{2w^2}, \end{cases} \qquad (12)$$

where $\theta = wh$.
Using the continuity condition of the first derivative at x_i, $S'_{i-1}(x_i) = S'(x_i)$, we have

$$b_{i-1} + wc_{i-1}(e^{\theta} + e^{-\theta}) + wd_{i-1}(e^{\theta} - e^{-\theta}) = b_i + 2wc_i. \qquad (13)$$

Reducing indices of Eq. (12) by one and substituting into Eq. (13), we obtain

$$\frac{y_i - y_{i-1}}{h} + \frac{M_i - M_{i+1}}{w\theta} + w\left(\frac{2M_i - (e^{\theta}+e^{-\theta})M_{i-1}}{2w^2(e^{\theta}+e^{-\theta})}\right) = \frac{y_{i+1}-y_i}{h} + \frac{M_i - M_{i+1}}{w\theta} + 2w\left(\frac{M_{i+1}}{w^2(e^{\theta}-e^{-\theta})} - \frac{M_i(e^{\theta}+e^{-\theta})}{2w^2(e^{\theta}-e^{-\theta})}\right).$$

$$\Longrightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = \alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1}, \qquad (14)$$

where
$\alpha = \frac{1}{\theta^2}\left(1 - \frac{2\theta}{(e^{\theta}-e^{-\theta})}\right)$, $\beta = \frac{1}{\theta^2}\left(\frac{\theta(e^{\theta}+e^{-\theta})}{(e^{\theta}-e^{-\theta})} - 1\right)$.
If $h \to 0$, then $\theta = wh \to 0$. Thus, using L'Hopitals rule we have

$$\lim_{\theta \to 0} \alpha = \frac{1}{6} \text{ } and \text{ } \lim_{\theta \to 0} \beta = \frac{1}{3}.$$

Using $S''(x_i) = y''_i = M_i$ in to Eq. (9), we get

$$\begin{cases} \varepsilon M_i = R_i - p_i y'_i - q_i y_i, \\ \varepsilon M_{i-1} = R_{i-1} - p_{i-1} y'_{i-1} - q_{i-1} y_{i-1}, \\ \varepsilon M_{i+1} = R_{i+1} - p_{i+1} y'_{i+1} - q_{i+1} y_{i+1}. \end{cases} \qquad (15)$$

Using Taylor's series expansions of $y_{i-1}, y_{i+1}, y'_{i-1}$ and y'_{i+1} simplifying, we have

$$\begin{cases} y'_i = \frac{y_{i+1}-y_{i-1}}{2h} + T_1, \\ y'_{i-1} = \frac{-y_{i+1}+4y_i-3y_{i-1}}{2h} + T_2, \\ y'_{i+1} = \frac{3y_{i+1}-4y_i+y_{i-1}}{2h} + T_2, \end{cases} \qquad (16)$$

where
$T_1 = -\frac{h^2}{6} y'''(\xi)$ and $T_2 = \frac{h^2}{12} y'''(\xi)$, for $\xi \in (x_{i-1}, x_i)$.

Substituting Eq. (16) in to Eq. (15), we get

$$\begin{cases} M_i = \frac{1}{\varepsilon}\left\{R_i - p_i\left(\frac{y_{i+1}-y_{i-1}}{2h} + T_1\right) - q_i y_i\right\}, \\ M_{i-1} = \frac{1}{\varepsilon}\left\{R_{i-1} - p_{i-1}\left(\frac{-y_{i+1}+4y_i-3y_{i-1}}{2h} + T_2\right) - q_{i-1}y_{i-1}\right\}, \\ M_{i+1} = \frac{1}{\varepsilon}\left\{R_{i+1} - p_{i+1}\left(\frac{3y_{i+1}-4y_i+y_{i-1}}{2h} + T_2\right) - q_{i+1}y_{i+1}\right\}. \end{cases} \tag{17}$$

Substituting Eq. (17) into Eq. (14) and rearranging, we get

$$\begin{aligned} &\frac{\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha p_{i-1}}{2h}(-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta p_i}{2h}(y_{i+1} - y_{i-1}) \\ &\quad + \frac{\alpha p_{i+1}}{2h}(3y_{i+1} - 3y_i + y_{i-1}) = \alpha(R_{i-1} - q_{i-1}y_{i-1} + R_{i+1} - q_{i+1}y_{i+1}) \\ &\quad + 2\beta(R_i - q_i y_i) + T, \end{aligned} \tag{18}$$

where $T = (4\beta p_i - \alpha p_{i-1} - \alpha p_{i+1})\frac{h^2}{12}y'''(\xi)$ is the local truncation error.
From the theory of singular perturbations described in [22] and the Taylor's series expansion of $y(x)$ about the point $'0'$ in the asymptotic solution of the problem in Eq.(9), we have

$$y(x_i) \approx y_0(x_i) + (\phi_0 - y_0(0))e^{-p(0)\frac{ih}{\varepsilon}},$$

and letting $\rho = \frac{h}{\varepsilon}$, we get

$$\lim_{h\to 0} y(ih) \approx y_0(0) + (\phi_0 - y_0(0))e^{-p(0)i\rho},$$

since $x_i = x_0 + ih$.

To handle the effect of the perturbation parameter exponentially fitting factor (artificial viscosity) $\sigma(\rho)$ is multiplied on the term containing the perturbation parameter in Eq. (18) and it becomes

$$\begin{aligned} &\frac{\sigma(\rho)\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha p_{i-1}}{2h}(-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta p_i}{2h}(y_{i+1} - y_{i-1}) \\ &\quad + \frac{\alpha p_{i+1}}{2h}(3y_{i+1} - 3y_i + y_{i-1}) = \alpha(R_{i-1} - q_{i-1}y_{i-1} + R_{i+1} - q_{i+1}y_{i+1}) \\ &\quad + 2\beta(R_i - q_i y_i) + T. \end{aligned} \tag{19}$$

Multiplying Eq. (19) by h and taking a limit as $h \to 0$, we get

$$\begin{aligned} &\frac{\sigma}{\rho}\lim_{h\to 0}(y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha p(0)}{h}\lim_{h\to 0}(-y_{i+1} - 4y_i - 3y_{i-1}) \\ &+ \beta p(0)\lim_{h\to 0}(y_{i+1} - y_{i-1}) + \frac{\alpha p(0)}{2}\lim_{h\to 0}(3y_{i+1} - 3y_i + y_{i-1}) = 0. \end{aligned} \tag{20}$$

Thus, we consider two cases of the boundary layers.
Case 1: For $p(x) > 0$ (Left-end boundary layer), we have

$$\begin{cases} \lim_{h\to 0}(y_{i-1} - 2y_i + y_{i+1}) = (\phi_0 - y_0(0))e^{-p(0)i\rho}(e^{p(0)\rho} + e^{-p(0)\rho} - 2), \\ \lim_{h\to 0}(-y_{i+1} - 4y_i - 3y_{i-1}) = (\phi_0 - y_0(0))e^{-p(0)i\rho}(-3e^{p(0)\rho} - e^{-p(0)\rho} + 4), \\ \lim_{h\to 0}(y_{i+1} - y_{i-1}) = (\phi_0 - y_0(0))e^{-p(0)i\rho}(e^{p(0)\rho} + 3e^{-p(0)\rho} - 4), \\ \lim_{h\to 0}(3y_{i+1} - 3y_i + y_{i-1}) = (\phi_0 - y_0(0))e^{-p(0)i\rho}(e^{-p(0)\rho} - e^{p(0)\rho}). \end{cases} \tag{21}$$

Substituting Eq. (21) into Eq. (20) and simplifying, we get

$$\sigma_0 = \rho p(0)(\alpha+\beta)\coth\left(\frac{p(0)\rho}{2}\right). \tag{22}$$

Case 2: For $p(x) < 0$ (Right-end boundary layer), we have

$$\begin{cases} \lim\limits_{h\to 0}(y_{i-1}-2y_i+y_{i+1}) = (\varphi - y_0(1))e^{-p(1)i\rho}(e^{p(1)\rho}+e^{-p(1)\rho}-2), \\ \lim\limits_{h\to 0}(-y_{i+1}-4y_i-3y_{i-1}) = (\varphi - y_0(1))e^{-p(1)i\rho}(-3e^{p(1)\rho}-e^{-p(1)\rho}+4), \\ \lim\limits_{h\to 0}(y_{i+1}-y_{i-1}) = (\varphi - y_0(1))e^{-p(1)i\rho}(e^{p(1)\rho}+3e^{-p(1)\rho}-4), \\ \lim\limits_{h\to 0}(3y_{i+1}-3y_i+y_{i-1}) = (\varphi - y_0(1))e^{-p(1)i\rho}(e^{-p(1)\rho}-e^{p(1)\rho}). \end{cases} \tag{23}$$

Substituting Eq. (23) into Eq. (20) and simplifying, we get

$$\sigma_N = \rho p(1)(\alpha+\beta)\coth\left(\frac{p(1)\rho}{2}\right). \tag{24}$$

In general, we take a variable fitting parameter as

$$\sigma_i = \rho_i p(x_i)(\alpha+\beta)\coth\left(\frac{p(x_i)\rho_i}{2}\right), \tag{25}$$

where, $\rho_i = \frac{h}{\varepsilon}$.
Thus, Eq. (19) can be written as

$$\begin{aligned} &\left\{\frac{\varepsilon\sigma_i}{h^2} - \frac{3\alpha p_{i-1}}{2h} + \alpha q_{i-1} - \frac{\beta p_i}{h} + \frac{\alpha p_{i+1}}{2h}\right\}y_{i-1} - \left\{\frac{2\varepsilon\sigma_i}{h^2} - \frac{2\alpha p_{i-1}}{h} - 2\beta q_i + \frac{2\alpha p_{i+1}}{h}\right\}y_i \\ &\qquad + \left\{\frac{\varepsilon\sigma_i}{h^2} - \frac{\alpha p_{i-1}}{2h} + \alpha q_{i+1} + \frac{\beta p_i}{h} + \frac{3\alpha p_{i+1}}{2h}\right\}y_{i+1} \\ &\qquad = \alpha(R_{i-1}+R_{i+1}) + 2\beta R_i. \end{aligned} \tag{26}$$

Simplifying Eq. (9), for left layer in domain Ω_1, we get the tri-diagonal system of the equation of the form

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \; for \quad i = 1, 2, ..., N-1, \tag{27}$$

where

$$\begin{cases} E_i = \frac{\varepsilon\sigma_i}{h^2} - \frac{3\alpha p_{i-1}}{2h} + \alpha q_{i-1} - \frac{\beta p_i}{h} + \frac{\alpha p_{i+1}}{2h}, \\ F_i = \frac{2\varepsilon\sigma_i}{h^2} - \frac{2\alpha p_{i-1}}{h} - 2\beta q_i + \frac{2\alpha p_{i+1}}{h}, \\ G_i = \frac{\varepsilon\sigma_i}{h^2} - \frac{\alpha p_{i-1}}{2h} + \alpha q_{i+1} + \frac{\beta p_i}{h} + \frac{3\alpha p_{i+1}}{2h}, \\ H_i = \alpha(R_{i-1}+R_{i+1}) + 2\beta R_i. \end{cases}$$

Similarly, if we consider $\Omega_2 = (1, 2)$ from Eqs. (4)–(6), we will obtain the differential equation

$$\begin{cases} \varepsilon y''(x) + p(x)y'(x) + q(x)y(x) + r(x)y(x-1) = S(x), \qquad x \in \Omega_2, \\ y(1) = \theta, \; y(2) = l, \end{cases} \tag{28}$$

where $p(x) = -a(x), q(x) = -b(x), r(x) = -c(x), S(x) = -f(x)$.
Substituting $S''(x_i) = y_i'' = M_i$ in to Eq. (28), we get

$$\begin{cases} \varepsilon M_i = S_i - p_i y_i' - q_i y_i - r_i y(x_i - 1), \\ \varepsilon M_{i-1} = S_{i-1} - p_{i-1} y_{i-1}' - q_{i-1} y_{i-1} - r_{i-1} y(x_{i-1} - 1), \\ \varepsilon M_{i+1} = S_{i+1} - p_{i+1} y_{i+1}' - q_{i+1} y_{i+1} - r_{i+1} y(x_{i+1} - 1). \end{cases} \tag{29}$$

Substituting Eq. (16) in to Eq. (29), we get

$$\begin{cases} M_i = \frac{1}{\varepsilon}\left\{S_i - p_i\left(\frac{y_{i+1}-y_{i-1}}{2h} + T_1\right) - q_i y_i - r_i y(x_i - 1)\right\}, \\ M_{i-1} = \frac{1}{\varepsilon}\left\{S_{i-1} - p_{i-1}\left(\frac{-y_{i+1}+4y_i-3y_{i-1}}{2h} + T_2\right) - q_{i-1}y_{i-1} - r_{i-1}y(x_{i-1} - 1)\right\}, \\ M_{i+1} = \frac{1}{\varepsilon}\left\{S_{i+1} - p_{i+1}\left(\frac{3y_{i+1}-4y_i+y_{i-1}}{2h} + T_2\right) - q_{i+1}y_{i+1} - r_{i+1}y(x_{i+1} - 1)\right\}. \end{cases} \tag{30}$$

Substituting Eq.(30) in to Eq.(14), introducing fitting factor and rearranging, we get

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} + T_i = H_i, \;\; i = N+1, N+2, ..., 2N-1. \tag{31}$$

where

$$\begin{cases} E_i = \frac{\varepsilon\sigma_i}{h^2} - \frac{3\alpha p_{i-1}}{2h} + \alpha q_{i-1} - \frac{\beta p_i}{h} + \frac{\alpha p_{i+1}}{2h}, \\ F_i = \frac{2\varepsilon\sigma_i}{h^2} - \frac{2\alpha p_{i-1}}{h} - 2\beta q_i + \frac{2\alpha p_{i+1}}{h}, \\ G_i = \frac{\varepsilon\sigma_i}{h^2} - \frac{\alpha p_{i-1}}{2h} + \alpha q_{i+1} + \frac{\beta p_i}{h} + \frac{3\alpha p_{i+1}}{2h}, \\ H_i = \alpha(S_{i-1} + S_{i+1}) + 2\beta S_i, \\ T_i = \alpha\{r_{i-1}y(x_{i-1} - 1) + r_{i+1}y(x_{i+1} - 1)\} + 2\beta r_i y(x_i - 1). \end{cases}$$

To treat the integral boundary for $i = 2N$, the composite Simpson's rule approximates the integral of $g(x)y(x)$ by

$$\begin{aligned} \int_0^2 g(x)y(x)dx = \frac{h}{3}\Bigg(g(0)y(0) + g(2)y(2) + 2\sum_{i=1}^{2N-1} g(x_{2i})y(x_{2i}) \\ +4\sum_{i=1}^{2N} g(x_{2i-1})y(x_{2i-1})\Bigg). \end{aligned} \tag{32}$$

Substituting Eq. (32) into Eq. (3) gives

$$y(2) - \frac{\varepsilon h}{3}\left(g(0)y(0) + g(2)y(2) + 2\sum_{i=1}^{2N-1} g(x_{2i})y(x_{2i}) + 4\sum_{i=1}^{2N} g(x_{2i-1})y(x_{2i-1})\right) = l \tag{33}$$

Since $y(0) = \phi(0)$ in Eq. (2), Eq. (33) can be re-written as

$$\begin{aligned} -\frac{4\varepsilon h}{3}\sum_{i=1}^{2N} g(x_{2i-1})y(x_{2i-1})\Bigg) - \frac{2\varepsilon h}{3}\sum_{i=1}^{2N-1} g(x_{2i})y(x_{2i}) + \left(1 - \frac{\varepsilon h}{3}g(2)\right)y(2) \\ = l + \frac{\varepsilon h}{3}g(0)\phi(0). \end{aligned} \tag{34}$$

Therefore, on the whole domain $\bar{\Omega} = [0, 2]$, the basic schemes to solve Eqs. (1)-(2) are the schemes given in Eq. (27), Eq. (31) and Eq. (34).

5. Stability and Convergence Analysis

Theorem 5.1. *(Stability) Let B be a coefficient matrix of the tri-diagonal system, Eq.(27). Then, for all $\varepsilon > 0$ and sufficiently small h, the matrix B is an irreducible and diagonally dominant matrix and hence the scheme is stable.*

Proof. Substituting Eq.(25) in Eq.(26) and multiplying both sides of the equation by h we get the equivalent tri-diagonal scheme:

$$\begin{aligned}\left\{\frac{p_i}{2}\coth\left(\frac{p_i\rho_i}{2}\right)-\frac{3\alpha p_{i-1}}{2}+h\alpha q_{i-1}-\beta p_i+\frac{\alpha p_{i+1}}{2}\right\}y_{i-1}\\ -\left\{p_i\coth\left(\frac{p_i\rho_i}{2}\right)-2\alpha p_{i-1}-2h\beta q_i+2\alpha p_{i+1}\right\}y_i\\ +\left\{\frac{p_i}{2}\coth\left(\frac{p_i\rho_i}{2}\right)+\frac{3\alpha p_{i+1}}{2}+h\alpha q_{i+1}+\beta p_i-\frac{\alpha p_{i-1}}{2}\right\}y_{i+1}\\ =h(\alpha(R_{i-1}+R_{i+1})+2\beta R_i).\end{aligned} \tag{35}$$

This can be written as

$$E_i^*y_{i-1}-F_i^*y_i+G_i^*y_{i+1}=H_i^*,\quad i=1,2,...,N-1. \tag{36}$$

where

$$\begin{cases}E_i^*=\frac{p_i}{2}\coth\left(\frac{p_i\rho_i}{2}\right)-\frac{3\alpha p_{i-1}}{2}+h\alpha q_{i-1}-\beta p_i+\frac{\alpha p_{i+1}}{2},\\ F_i^*=p_i\coth\left(\frac{p_i\rho_i}{2}\right)-2\alpha p_{i-1}-2h\beta q_i+2\alpha p_{i+1},\\ G_i^*=\frac{p_i}{2}\coth\left(\frac{p_i\rho_i}{2}\right)+\frac{3\alpha p_{i+1}}{2}+h\alpha q_{i+1}+\beta p_i-\frac{\alpha p_{i-1}}{2},\\ H_i^*=h(\alpha(R_{i-1}+R_{i+1})+2\beta R_i).\end{cases}$$

Rewriting Eq. (36) in a matrix vector form, we obtain

$$BY=C \tag{37}$$

where, B is a coefficient matrix, $Y=(y_1,y_2,...,y_{N-1})^T$ and $C=(H_1^*-E_1^*\phi_0,H_2^*-E_2^*\phi_1,...,H_{N-1}^*-E_{N-1}^*\varphi)^T$.
The matrix B is tri-diagonal matrix and its off-diagonal elements are E_i^* and G_i^*.
Now,

$$\begin{aligned}|E_i^*+G_i^*|=&|p_i\coth\left(\frac{p_i\rho_i}{2}\right)+2\alpha p_{i+1}-2\alpha p_{i-1})|\\ =&|p_i\coth\left(\frac{p_i\rho_i}{2}\right)+2\alpha(p_{i+1}-p_{i-1})|\\ \le&|F_i^*|\end{aligned}$$

This implies that for each row of B, the sum of the two off-diagonal elements is less than the modulus of the diagonal element. Therefore, B is diagonally dominant.
Further, for sufficiently small $h(i.e, h\to 0)$, we have, $E_i^*\neq 0$ and $G_i^*\neq 0$, $\forall i=1,2,...,N-1$. Hence, B is irreducible [27]. Therefore, from these two conditions, the scheme in (27) is stable [16]. ☐

Theorem 5.2. *(Convergence) Let $y(x)$ be the analytical solution of the problem in Eq. (8) and y^N be the numerical solution of the discretized problem of Eq. (27). Then, $||y-y^N||\le Ch^2$ for sufficiently small h and C is positive constant.*

Proof. Multiplying both sides of Eq. (26) by $\frac{-h^2}{\varepsilon\sigma_i}$ and simplifying, we obtain

$$(-1+u_i)y_{i-1}+(2+v_i)y_i+(-1+w_i)y_{i+1}+g_i+T_i=0, \tag{38}$$

where

$$\begin{cases} u_i = \frac{1}{\varepsilon\sigma_i}\left(\frac{3\alpha h p_{i-1}}{2} - \alpha h^2 q_{i-1} + \beta p_i h - \frac{\alpha h p_{i+1}}{2}\right), \\ v_i = \frac{2}{\varepsilon\sigma_i}\left(\alpha h p_{i+1} - \alpha h p_{i-1} - \beta h^2 q_i\right), \\ w_i = \frac{1}{\varepsilon\sigma_i}\left(\frac{\alpha h p_{i-1}}{2} - \beta p_i h - \frac{3\alpha h p_{i+1}}{2} - \alpha h^2 q_{i+1}\right), \\ g_i = \frac{-h^2}{\varepsilon\sigma_i}\{\alpha(R_{i-1} + R_{i+1}) + 2\beta R_i\}, \end{cases}$$

and $T_i = \frac{\alpha(p_{i-1}+p_{i+1})-4\beta p_i}{12\varepsilon\sigma_i} h^4 y'''(\xi)$ is a local truncation error for $i = 1, 2, ..., N-1$. Incorporating the boundary condition $y_0 = \phi(x_0) = \phi_0, \quad y_N = \phi(1) = \varphi$ in Eq. (38), we get the system of equation of the form

$$(D + P)y + M + T(h) = \bar{0}, \tag{39}$$

where
$D = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & - & - & & 0 \\ \vdots & & & & -1 \\ 0 & - & - & -1 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} v_1 & w_1 & 0 & \dots & 0 \\ u_2 & v_2 & w_2 & \dots & 0 \\ 0 & - & - & & 0 \\ \vdots & & & & w_{N-2} \\ 0 & - & - & u_{N-1} & u_{N-1} \end{pmatrix}$ are tri-diagonal matrices of order $N-1$, $M = [(g_1 + (-1 + u_1)\phi_0, g_2, g_3, ..., (g_{N-1} + (-1 + w_{N-1})\varphi]^T$, $T(h) = O(h^4)$ and $y = [y_1, y_2, ..., y_{N-1}]^T$, $T(h) = [T_1, T_2, ..., T_{N-1}]^T$, $\bar{0} = [0, 0, ..., 0]^T$ are associated vectors of Eq. (39).
Let $y^N = [y_1^N, y_2^N, ..., y_{N-1}^N]^T \cong y$ be the solution which satisfies Eq. (39), we have

$$(D + P)y^N + M = \bar{0}, \tag{40}$$

Let $e_i = y_i - y_i^N$ for $i = 1, 2, ..., N-1$ be the discretization error then $y - y^N = [e_1, e_2, ..., e_{N-1}]^T$.
Subtracting Eq. (39) from Eq.(40), we get

$$(D + P)(y^N - y) = T(h). \tag{41}$$

Let $|p_{i-1}| \le c_1, |p_i| \le c_2, |p_{i+1}| \le c_3, |q_{i-1}| \le k_1, |q_i| \le k_2, |q_{i+1}| \le k_3$ and $t_{i,j}$ be the $(i,j)^{th}$ element of the matrix P, then

$$\begin{cases} |t_{i,i+1}| = |w_i| \le \frac{h}{\varepsilon\sigma_i}\left(\frac{3\alpha c_1}{2} + \alpha h c_2 + \frac{3\alpha c_3}{2} + \alpha k_3\right), & i = 1, 2, ..., N-2, \\ |t_{i,i-1}| = |u_i| \le \frac{h}{\varepsilon\sigma_i}\left(\frac{3\alpha c_1}{2} + \alpha h k_2 + \beta c_2 + \frac{\alpha c_3}{2}\right), & i = 2, 3, ..., N-1. \end{cases}$$

Thus, for sufficiently small h, we have

$$\begin{cases} -1 + |t_{i,i+1}| \neq 0, & i = 1, 2, ..., N-2, \\ -1 + |t_{i,i-1}| \neq 0, & i = 2, 3, ..., N-1. \end{cases}$$

Hence, the matrix $(D+P)$ is irreducible [27].
Let A_i be the sum of the elements of the i^{th} row of the matrix $(D+P)$, then

$$\begin{cases} A_i = 1 + v_i + w_i \\ \quad = 1 + \frac{2h}{\varepsilon\sigma_i}\Bigg(\alpha p_{i+1} - \alpha p_{i-1} + \frac{\alpha p_{i-1}}{4} - \frac{\beta p_i}{2} - \frac{3\alpha p_{i+1}}{4}\Bigg) + O(h^2), & i = 1, \\ A_i = u_i + v_i + w_i \\ \quad = \frac{h^2}{\varepsilon\sigma_i}\Bigg(- \alpha q_{i-1} - \beta q_i - \alpha q_{i+1}\Bigg), & i = 2, 3, ...N-2, \\ A_i = 1 + u_i + v_i \\ \quad = 1 + \frac{2h}{\varepsilon\sigma_i}\Bigg(\frac{3\alpha p_{i+1}}{4} - \frac{\alpha p_{i-1}}{4} + \frac{\beta p_i}{2}\Bigg) + O(h^2), & i = N-1. \end{cases}$$

Let

$$\begin{cases} d_1 = \min\limits_{1\le i\le N-1} \frac{1}{\varepsilon\sigma_i}(-\alpha q_{i-1} - 2\beta q_i - \alpha q_{i+1}), \\ d_2 = \max\limits_{1\le i\le N-1} \frac{1}{\varepsilon\sigma_i}(-\alpha q_{i-1} - 2\beta q_i - \alpha q_{i+1}), \end{cases}$$

then, $0 \le d_1 \le d_2$.
For sufficiently small h, $(D+P)$ is monotone [27] and [28]. Hence, $(D+P)^{-1}$ exists and $(D+P)^{-1} \ge 0$.
From the error in Eq. (41), we have

$$||y - y^N|| \le ||(D+P)^{-1}||\ ||T(h)||. \tag{42}$$

For sufficiently small h, we have $A_i > h^2 d_1$, for $i = 1, ..., N-1$,
where $d_1 = \min_{1\le i\le N-1}\left(\frac{1}{\varepsilon\sigma_i}(-\alpha q_{i-1} - 2\beta q_i - \alpha q_{i+1})\right)$.
Let $(D+P)^{-1}_{i,k}$ be the$(i,k)^{th}$ elements of $(D+P)^{-1}$ and we define

$$||(D+P)^{-1}|| = \max_{1\le i\le N-1} \Sigma_{k=1}^{N-1}(D+P)^{-1}_{i,k}\ and\ ||T(h)|| = \max_{1\le i\le N-1}|T_i| \tag{43}$$

Since $(D+P)^{-1}_{i,k} \ge 0$ from the theory of matrices, we have

$$\Sigma_{k=1}^{N-1}(D+P)^{-1}_{i,k}.A_k = 1 \quad for \quad i = 1, 2, ..., N-1.$$

Hence,

$$\Sigma_{k=1}^{N-1}(D+P)^{-1}_{i,k}A_k \le \frac{1}{\min\limits_{1\le i\le N-1} A_k} \le \frac{1}{h^2 d_1}. \tag{44}$$

Now, from Eqs. (42)–(44), we get

$$\begin{aligned} ||y - y^N|| &\le \frac{1}{h^2 d_1}\left|\left(\frac{\alpha(p_{i-1} + p_{i+1}) - 4\beta p_i}{\varepsilon\sigma_i}\right)\frac{1}{12}h^4 y'''(\xi)\right|, \\ &\le \left(\frac{y'''(\xi)(4\beta p_i + \alpha(p_{i-1} + p_{i+1})}{12 d_1 \sigma_i}\right)h^2, \\ &= Ch^2, \end{aligned}$$

where $C = \frac{y'''(\xi)(4\beta p_i + \alpha(p_{i-1} + p_{i+1}))}{12 d_1 \sigma_i}$ which is independent of mesh size h. This establishes that the method is of second order uniform convergent. $\square$

6. Numerical Examples and Results

In this section, two examples are given to illustrate the numerical method discussed above. The exact solutions of the test problems are not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. For this we put

$$E_\varepsilon^h = \max_{0\le i\le 2N} |Y_i^N - Y_{2i}^{2N}|, \tag{45}$$

where Y_i^N and Y_{2i}^{2N} are the i^{th} components of the numerical solutions on meshes of N and $2N$ respectively. We compute the uniform error and the rate of convergence as

$$E^N = \max_\varepsilon E_\varepsilon^h \quad \textit{and} \quad R^N = \log_2\left(\frac{E^N}{E^{2N}}\right). \tag{46}$$

The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{\ 2^{-4}, 2^{-8}, ..., 2^{-32}\}$.

Example 6.1. *Consider the model singularly perturbed boundary value problem*

$$-\varepsilon y''(x) + 3y'(x) + y(x) - y(x-1) = 1 \quad x \in (0,1)\cup(1,2),$$

subject to the boundary conditions

$$y(x) = 1, \quad x \in [-1,0], \quad y(2) = 2 + \frac{\varepsilon}{3}\int_0^2 xy(x)dx.$$

Example 6.2. *Consider the model singularly perturbed boundary value problem*

$$-\varepsilon y''(x) + (1+x)y'(x) + (x+10)y(x) - e^x y(x-1) = \frac{4}{\pi^2}x(1-x), \quad x \in (0,1)\cup(1,2),$$

subject to the boundary conditions

$$y(x) = 2 + x, \quad x \in [-1,0], \quad y(2) = 2 + \frac{\varepsilon}{3}\int_0^2 xe^x \sin(x)y(x)dx.$$

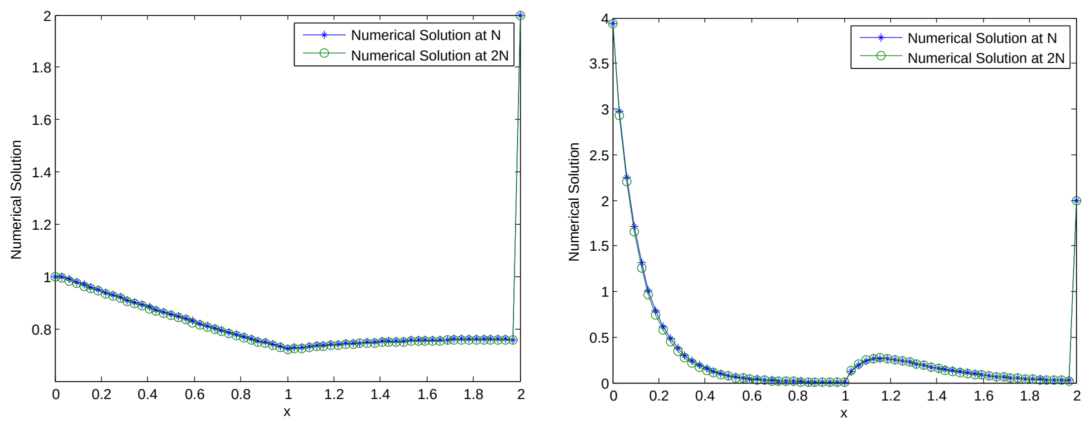
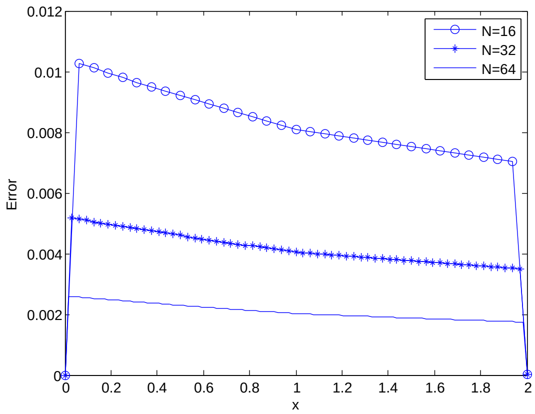


Figure 1. The behavior of the Numerical Solution for Example 6.1 and Example 6.2 at $\varepsilon = 10^{-12}$ and $N = 32$ respectively.

TABLE 1. Maximum absolute errors and rate of convergence for Example 6.1 at different number of mesh points.

ε	N=32	N=64	N=128	N=256	N=512
	Present Method				
2^{-4}	3.8313e-03	1.1778e-03	3.2791e-04	8.6603e-05	2.2259e-05
2^{-8}	5.1839e-03	2.7107e-03	1.4599e-04	6.6728e-04	2.4157e-05
2^{-12}	5.1713e-03	2.5949e-03	1.2998e-03	6.5046e-04	3.2618e-04
2^{-16}	5.1713e-03	2.5949e-03	1.2998e-03	6.5046e-04	3.2538e-04
2^{-20}	5.1713e-03	2.5949e-03	1.2998e-03	6.5046e-04	3.2538e-04
2^{-24}	5.1713e-03	2.5949e-03	1.2998e-03	6.5046e-04	3.2538e-04
2^{-28}	5.1713e-03	2.5949e-03	1.2998e-03	6.5046e-04	3.2538e-04
2^{-32}	5.1713e-03	2.5949e-03	1.2998e-03	6.5046e-04	3.2538e-04
E^N	5.1713e-03	2.5949e-03	1.2998e-03	6.5046e-04	3.2538e-04
R^N	0.9948	0.9974	0.9988	0.9993	
	Result in [17]				
2^{-4}	4.0400e-01	1.1900e-01	2.6200e-02	6.8300e-03	1.9400e-03
2^{-8}	6.7100e-01	3.1700e-01	1.1800e-01	3.5100e-02	1.0000e-02
2^{-12}	6.7000e-01	3.1800e-01	1.2100e-01	3.7300e-02	1.1300e-02
2^{-16}	6.7000e-01	3.1700e-01	1.2000e-01	3.7300e-02	1.2500e-02
2^{-20}	6.7000e-01	3.1700e-01	1.2000e-01	3.6900e-02	1.1700e-02
2^{-24}	6.7000e-01	3.1700e-01	1.2000e-01	3.6900e-02	1.1600e-02
2^{-28}	6.7000e-01	3.1700e-01	1.2000e-01	3.6900e-02	1.1600e-02
2^{-32}	6.7000e-01	3.1700e-01	1.2000e-01	3.6900e-02	1.1600e-02
E^N	6.7000e-01	3.1700e-01	1.2100e-01	3.7300e-02	1.2500e-02
R^N	1.0818	1.3895	1.6978	1.5429	



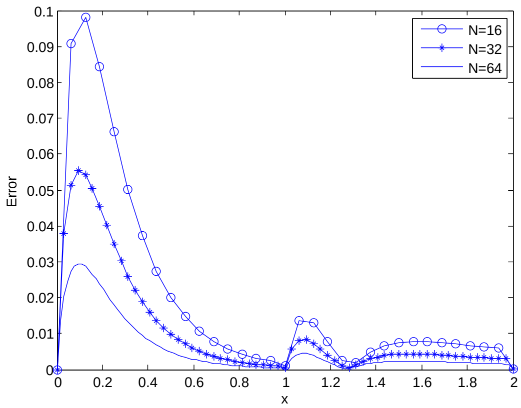


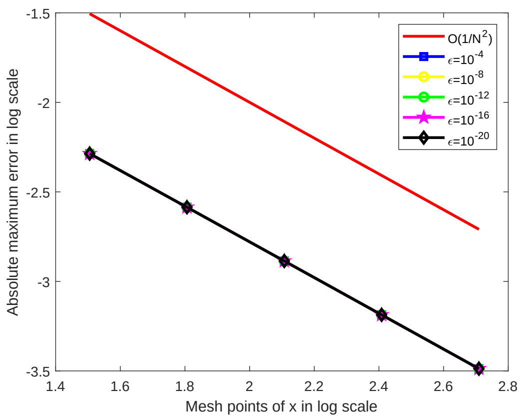
FIGURE 2. Point wise absolute error of Example 6.1 and Example6.2 at $\varepsilon = 10^{-12}$ with different mesh point N respectively.

7. DISCUSSION AND CONCLUSION

This study introduces a fitted non-polynomial cubic spline method for singularly perturbed differential-difference equations with Integral boundary condition. The behavior of

TABLE 2. Maximum absolute errors and rate of convergence for Example 6.2 at different number of mesh points.

ε	N=32	N=64	N=128	N=256	N=512
	Present Method				
2^{-4}	1.4188e-02	7.8640e-03	4.1253e-03	2.1109e-03	1.0675e-03
2^{-8}	4.4111e-02	1.4909e-02	4.2638e-03	1.9668e-03	1.1364e-03
2^{-12}	5.5346e-02	2.9454e-02	1.5214e-02	7.5267e-03	3.1077e-03
2^{-16}	5.5346e-02	2.9454e-02	1.5221e-02	7.7407e-03	3.9043e-03
2^{-20}	5.5346e-02	2.9454e-02	1.5221e-02	7.7407e-03	3.9043e-03
2^{-24}	5.5346e-02	2.9454e-02	1.5221e-02	7.7407e-03	3.9043e-03
2^{-28}	5.5346e-02	2.9454e-02	1.5221e-02	7.7407e-03	3.9043e-03
2^{-32}	5.5346e-02	2.9454e-02	1.5221e-02	7.7407e-03	3.9043e-03
E^N	5.5346e-02	2.9454e-02	1.5221e-02	7.7407e-03	3.9043e-03
R^N	0.9100	0.9524	0.9755	0.9874	
	Result in [17]				
2^{-4}	6.6500e-01	1.7600e-01	3.5500e-02	7.7000e-03	1.5500e-03
2^{-8}	8.7700e-01	4.1500e-01	1.5100e-01	4.1900e-02	1.2700e-02
2^{-12}	8.8000e-01	4.1100e-01	1.5100e-01	4.3600e-02	1.2500e-02
2^{-16}	8.8000e-01	4.1100e-01	1.5100e-01	4.3900e-02	1.2900e-02
2^{-20}	8.8000e-01	4.1100e-01	1.5100e-01	4.3900e-02	1.2900e-02
2^{-24}	8.8000e-01	4.1100e-01	1.5100e-01	4.3900e-02	1.2900e-02
2^{-28}	8.8000e-01	4.1100e-01	1.5100e-01	4.3900e-02	1.2900e-02
2^{-32}	8.8000e-01	4.1100e-01	1.5100e-01	4.3900e-02	1.2900e-02
E^N	8.8000e-01	4.1100e-01	1.5100e-01	4.3900e-02	1.2900e-02
R^N	1.0984	1.4446	1.7823	1.7668	



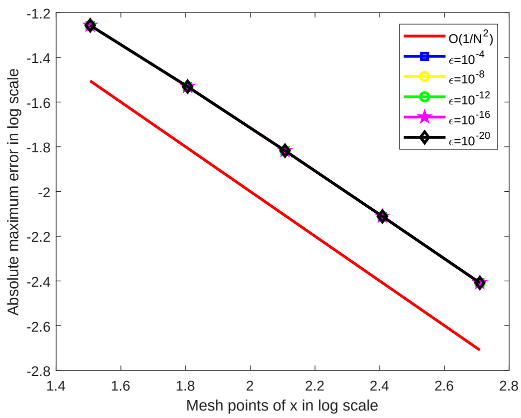


FIGURE 3. Uniform convergence with fitted operator in log-log scale for Example 6.1 and Example 6.2 respectively.

the continuous solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme is developed on uniform mesh using exponential fitted operator in the given differential equation. The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, two model problems of (one variable coefficient and one constant coefficient) are considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see Tables 1-2). Further, behavior of the numerical solution (Figure 1), point-wise absolute errors (Figure 2) and the uniform convergence of the method is shown by the log-log plot (Figure 3). The method is shown to be uniformly convergent with order of convergence $O(h^2)$. The performance of the proposed scheme is investigated by comparing with prior study (see Table 1 and 2). The proposed method gives more accurate, stable and uniform numerical result.

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