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CERTAIN SUBCLASSES OF ANALYTIC FUNCTION BY SĂLĂGEAN q-DIFFERENTIAL OPERATOR

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ABSTRACT. The theory of q-analysis has many applications in various sub-fields of mathematics and quantum physics. In the present article, we define the class $\mathcal{T}_n(\alpha, \lambda; q)$ using the Sălăgean q-differential operator. For functions belonging to this class we obtain coefficient estimates, extreme points and integral preserving properties .

Keywords: Univalent functions, Sălăgean, q-derivative, coefficient estimate.

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1. Introduction

The class of all analytic univalent functions denoted by $\mathcal{A}$ is of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1}$$

defined in the unit disc $\mathbb{U} = \{z : |z| < 1\}$.
Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathbb{U}$, consisting of analytic functions whose non-zero coefficients from the second onwards are negative. That is, an analytic function $f \in \mathcal{T}$ if it has a Taylor expansion of the form

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m \quad (a_m \geq 0) \tag{2}$$

which are univalent in the open disc $\mathbb{U}$.
For functions $f \in \mathcal{A}$ of the form (1), Govindaraj and S Sivasubramanian [2] introduced the following operator $\mathcal{S}_q^n$ which is called as Sălăgean q-differential operator.

$$\mathcal{S}_q^0 f(z) = f(z), \qquad \mathcal{S}_q^1 f(z) = z\partial_q f(z), \quad \cdots, \qquad \mathcal{S}_q^n f(z) = z\partial_q(\mathcal{S}_q^{n-1} f(z)).$$

A simple calculation implies

$$\mathcal{S}_q^n f(z) = f(z) * G_{q,n}(z),$$

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where

$$G_{q,n}(z) = z + \sum_{m=2}^{\infty} [m]_q^n z^m, \quad (n \in \mathbb{N}),$$

where $[m]_q = \dfrac{1-q^m}{1-q}$.

The power series of $\mathcal{S}_q^n f(z)$ for functions $f \in \mathcal{A}$ of the form (1), is given by

$$\mathcal{S}_q^n f(z) = z + \sum_{m=2}^{\infty} [m]_q^n a_m z^m. \tag{3}$$

Note that

$$\lim_{q \to 1-} G_{q,n}(z) = z + \sum_{m=2}^{\infty} m^n z^m$$

and

$$\lim_{q \to 1-} \mathcal{S}_q^n f(z) = f(z) * \left(z + \sum_{m=2}^{\infty} m^n z^m \right)$$

which is the familiar Sălăgean derivative [5].

Now we define the following subclass of $\mathcal{T}$ by using Sălăgean q-differential operator.

Let $\mathcal{T}_n(\alpha, \lambda; q)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the conditions

$$\Re \left\{ \frac{z(\mathcal{S}_q^n f)'}{\lambda z(\mathcal{S}_q^n f)' + (1-\lambda)\mathcal{S}_q^n f} \right\} > \alpha, \tag{4}$$

for some $\alpha, \ \lambda \quad (0 \leq \alpha, \lambda < 1)$ and $n \in \mathbf{N}_0$.

If $q \to 1$, we get the class studied by Dileep L and Latha S [1]. For different parametric values of n and $q \to 1$ we get the classes studied by Mostafa [3].

2. Main Results

Theorem 2.1. *A function f defined by (1.2) is in the class $\mathcal{T}_n(\alpha, \lambda; q)$ if and only if*

$$\sum_{m=2}^{\infty} [m]_q^n a_m [m - \alpha + \alpha\lambda - \alpha\lambda m] < 1 - \alpha, \tag{5}$$

where $\alpha, \ \lambda \quad (0 \leq \alpha, \lambda < 1)$ and $n \in \mathbf{N}_0$.

Proof. Suppose $f \in \mathcal{T}_n(\alpha, \lambda; q)$. Then

$$\Re \left\{ \frac{z(\mathcal{S}_q^n f)'}{\lambda z(\mathcal{S}_q^n f)' + (1-\lambda)\mathcal{S}_q^n f} \right\} > \alpha,$$

$$\Re \left\{ \frac{z - \sum_{m=2}^{\infty} m[m]_q^n a_m z^m}{\lambda \left[z - \sum_{m=2}^{\infty} [m]_q^n m a_m z^m \right] + (1-\lambda) \left[z - \sum_{m=2}^{\infty} [m]_q^n a_m z^m \right]} \right\} > \alpha.$$

$$\Re \left\{ \frac{z - \sum_{m=2}^{\infty} m[m]_q^n a_m z^m}{z - \sum_{m=2}^{\infty} [m]_q^n a_m z^m \left[\lambda(m-1) + 1 \right]} \right\} > \alpha.$$

Letting $z \to 1$, then we get,

$$1-\sum_{m=2}^{\infty}[m]_q^n a_m m > \alpha\left\{1-\sum_{m=2}^{\infty}[m]_q^n a_m\left[\lambda(m-1)+1\right]\right\}$$

$$\sum_{m=2}^{\infty}[m]_q^n a_m m-\alpha\sum_{m=2}^{\infty}[m]_q^n a_m\left[\lambda(m-1)+1\right]<(1-\alpha)$$

$$\sum_{m=2}^{\infty}[m]_q^n a_m\left[m-\alpha\lambda m+\alpha\lambda-\alpha\right]<(1-\alpha).$$

Conversely, assume that (5) be true. We have to show that (4) is satisfied or equivalently

$$\left|\left\{\frac{z(\mathcal{S}_q^n f)'}{\lambda z(\mathcal{S}_q^n f)'+(1-\lambda)\mathcal{S}_q^n f}\right\}-1\right|<1-\alpha.$$

But

$$\left|\left\{\frac{z-\sum_{m=2}^{\infty} m[m]_q^n a_m z^m}{z-\sum_{m=2}^{\infty}[m]_q^n a_m z^m\left[\lambda(m-1)+1\right]}\right\}-1\right|=$$

$$\left|\frac{\sum_{m=2}^{\infty}[m]_q^n a_m(m-1)(\lambda-1)z^m}{z-\sum_{m=2}^{\infty}[m]_q^n a_m[\lambda(m-1)+1]z^m}\right|$$

$$\leq\frac{\sum_{m=2}^{\infty}[m]_q^n a_m(m-1)(\lambda-1)|z^m|}{|z|-\sum_{m=2}^{\infty}[m]_q^n a_m[\lambda(m-1)+1]|z^m|}$$

$$\leq\frac{\sum_{m=2}^{\infty}[m]_q^n a_m(m-1)(\lambda-1)}{1-\sum_{m=2}^{\infty}[m]_q^n a_m[\lambda(m-1)+1]}.$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{m=2}^{\infty}[m]_q^n a_m(m-1)(\lambda-1)\leq(1-\alpha)\left(1-\sum_{m=2}^{\infty}[m]_q^n a_m[\lambda(m-1)+1]\right),$$

or

$$\sum_{m=2}^{\infty}[m]_q^n a_m[m-\alpha+\alpha\,\lambda-\alpha\,\lambda m]<1-\alpha,$$

which is true by hypothesis. This completes the assertion of Theorem 2.1.

□

Corollary 2.1. *If* $f\in\mathcal{T}_n(\alpha,\lambda;q)$ *then*

$$|a_m|\leq\frac{1-\alpha}{[m]_q^n\left[m-\alpha\lambda m+\alpha\lambda-\alpha\right]}.$$

Theorem 2.2. *Let* $0 \leq \alpha < 1, \quad 0 \leq \lambda_1 \leq \lambda_2 < 1, \quad n \in \mathbb{N}_0,$ *then* $\mathcal{T}_n(\alpha, \lambda_2; q) \subset \mathcal{T}_n(\alpha, \lambda_1; q)$.

Proof. From Theorem 2.1,

$$\begin{aligned}\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda_2 m + \alpha\lambda_2 - \alpha\right] a_m \\ \leq \sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda_1 m + \alpha\lambda_1 - \alpha\right] a_m \\ \leq (1-\alpha).\end{aligned}$$

For $f(z) \in \mathcal{T}_n(\alpha, \lambda_2; q)$. Hence $f(z) \in \mathcal{T}_n(\alpha, \lambda_1; q)$. □

Theorem 2.3. *Let* $f(z) \in \mathcal{T}_n(\alpha, \lambda; q)$. *Define* $f_1(z) = z$ *and*

$$f_m(z) = z + \frac{1-\alpha}{[m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right]} z^m, \quad m = 2, 3, \cdots,$$

for some $\alpha, \ \lambda\,(0 \leq \lambda < 1), n \in \mathbb{N}_0$ *and* $z \in \mathbb{U}$. *Then* $f \in \mathcal{T}_n(\alpha, \lambda; q)$ *if and only if* f *can be expressed as* $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ *where* $\mu_m \geq 0$ *and* $\sum_{m=1}^{\infty} \mu_m = 1$.

Proof. If $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ with $\sum_{m=1}^{\infty} \mu_m = 1, \quad \mu_m \geq 0$, then

$$\begin{aligned}&\sum_{m=2}^{\infty} \frac{[m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] \mu_m}{[m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right]} (1-\alpha) \\ &= \sum_{m=2}^{\infty} \mu_m (1-\alpha) = (1-\mu_1)(1-\alpha) \\ &\leq (1-\alpha).\end{aligned}$$

Hence $f \in \mathcal{T}_n(\alpha, \lambda; q)$.

Conversely, let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in \mathcal{T}_n(\alpha, \lambda; q)$, define

$$\mu_m = \frac{[m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] |a_m|}{(1-\alpha)}, \quad m = 2, 3, \cdots,$$

and define $\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m$. From Theorem 2.1, $\sum_{m=2}^{\infty} \mu_m \leq 1$ and so $\mu_1 \geq 0$.
Since $\mu_m f_m(z) = \mu_m f + a_m z^m$,
$\sum_{m=1}^{\infty} \mu_m f_m(z) = z - \sum_{m=2}^{\infty} a_m z^m = f(z)$. □

Theorem 2.4. *The class* $\mathcal{T}_n(\alpha, \lambda; q)$ *is closed under convex linear combination.*

Proof. Let $f, \ g \in \mathcal{T}_n(\alpha, \lambda; q)$ and let

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = z - \sum_{m=2}^{\infty} b_m z^m.$$

For η such that $0 \leq \eta \leq 1$, it suffices to show that the function defined by $h(z) = (1-\eta)f(z) + \eta g(z), \quad z \in \mathbb{U}$ belongs to $\mathcal{T}_n(\alpha, \lambda; q)$. Now

$$h(z) = z - \sum_{m=2}^{\infty} [(1-\eta)a_m + \eta b_m] z^m.$$

Applying Theorem 2.1, to $f,\ g \in \mathcal{T}_n(\alpha, \lambda; q)$, we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] \left[(1-\eta)a_m + \eta b_m\right] \\
&\quad = (1-\eta) \sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] a_m + \eta \sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] b_m \\
&\quad \leq (1-\eta)(1-\alpha) + \eta(1-\alpha) = (1-\alpha).
\end{aligned}$$

This implies that $h \in \mathcal{T}_n(\alpha, \lambda)$. □

Corollary 2.2. *If* $f_1(z),\ f_2(z)$ *are in* $\mathcal{T}_n(\alpha, \lambda; q)$ *then the function defined by* $g(z) = \frac{1}{2}[f_1(z) + f_2(z)]$ *is also in* $\mathcal{T}_n(\alpha, \lambda; q)$.

Theorem 2.5. *Let for* $j = 1, 2, \cdots, m, \quad f_j(z) = z - \sum_{m=2}^{\infty} a_{m,j} z^m \in \mathcal{T}_n(\alpha, \lambda; q)$ *and* $0 < \lambda_j < 1$ *such that* $\sum_{j=1}^{m} \lambda_j = 1$, *then the function* $F(z)$ *defined by* $F(z) = \sum_{j=1}^{m} \lambda_j f_j(z)$ *is also in* $\mathcal{T}_n(\alpha, \lambda; q)$.

Proof. For each $j \in \{1, 2, 3, \cdots, m\}$ we obtain

$$\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] |a_m| < (1-\alpha).$$

Since

$$\begin{aligned}
F(z) &= \sum_{j=1}^{m} \lambda_j (z - \sum_{m=2}^{\infty} a_{m,j} z^m) \\
&= z - \sum_{m=2}^{\infty} (\sum_{j=1}^{m} \lambda_j a_{m,j}) z^m.
\end{aligned}$$

$$\begin{aligned}
&\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] \left[\sum_{j=1}^{m} \lambda_j a_{m,j}\right] \\
&\quad = \sum_{j=1}^{m} \lambda_j \left[\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right]\right] \\
&\quad < \sum_{j=1}^{m} \lambda_j (1-\alpha) < (1-\alpha).
\end{aligned}$$

Therefore $F(z) \in \mathcal{T}_n(\alpha, \lambda; q)$. □

Theorem 2.6. *Let* $f(z) \in \mathcal{T}_n(\alpha, \lambda; q)$. *Komato operator of* f *is defined by*

$$k(z) = \int_0^1 \frac{(c+1)^\gamma}{\Gamma(\gamma)} t^c \left(\log \frac{1}{t}\right)^{\gamma-1} \frac{f(tz)}{t} dt,$$

$c > -1, \quad \gamma \geq 0$ *then* $k(z) \in \mathcal{T}_n(\alpha, \lambda; q)$.

Proof. We have

$$\int_0^1 t^c \left(\log \frac{1}{t}\right)^{\gamma-1} dt = \frac{\Gamma(\gamma)}{(c+1)^\gamma}$$
$$\int_0^1 t^{m+c-1} \left(\log \frac{1}{t}\right)^{\gamma-1} dt = \frac{\Gamma(\gamma)}{(c+1)^\gamma}, \quad m = 2, 3, \cdots,$$

$$\begin{aligned} k(z) &= \frac{(c+1)^\gamma}{\Gamma(\gamma)} \left[\int_0^1 t^c \left(\log \frac{1}{t}\right)^{\gamma-1} z dt - \sum_{m=2}^{\infty} z^m \int_0^1 a_m t^{m+c-1} \left(\log \frac{1}{t}\right)^{\gamma-1} dt\right] \\ &= z - \sum_{m=2}^{\infty} \left(\frac{c+1}{c+m}\right)^\gamma a_m z^m. \end{aligned}$$

Since $f \in \mathcal{T}_n(\alpha, \lambda; q)$ and since $\left(\frac{c+1}{c+m}\right)^\gamma < 1$, we have

$$\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] \left(\frac{c+1}{c+m}\right)^\gamma a_m < (1-\alpha).$$

□

Theorem 2.7. *Let* $f \in \mathcal{T}_n(\alpha, \lambda; q)$, *then for every* $0 \leq \delta < 1$ *the function*

$$\mathcal{H}_\delta(z) = (1-\delta) f(z) + \delta \int_0^z \frac{f(t)}{t} dt.$$

Proof. We have $\mathcal{H}_\delta(z) = z - \sum_{m=2}^{\infty} \left(1 + \frac{\delta}{m} - \delta\right) a_m z^m$.

Since $\left(1 + \frac{\delta}{m} - \delta\right) < 1, \quad m \geq 2$, so by Theorem 2.1,

$$\begin{aligned} &\sum_{m=2}^{\infty} \left(1 + \frac{\delta}{m} - \delta\right) [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] a_m \\ &< \sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] a_m \\ &< (1-\alpha). \end{aligned}$$

Therefore $\mathcal{H}_\delta(z) \in \mathcal{T}_n(\alpha, \lambda; q)$. □

3. Conclusions

Here, in our present investigation, we have successfully introduced a new subclass of analytic functions $\mathcal{T}_n(\alpha, \lambda; q)$ using the Sălăgean q- differential operator. Many properties and characteristics of this newly-defined function class such as coefficient estimates, extreme points, integral theorem have been studied.

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