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BAYESIAN ESTIMATION USING LINDLEY'S APPROXIMATION OF INVERTED KUMARASWAMY DISTRIBUTION BASED ON LOWER RECORD VALUES

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Abstract. In this paper, we have considered estimation of unknown parameters based on lower record values for Inverted Kumaraswamy distribution. Maximum likelihood and approximate Bayes estimators based on lower record values for unknown parameters of this distribution are obtained. Lindley's approximation (L-approximation) is used to obtain approximate Bayes estimators under DeGroot loss function based on lower record values. A Simulation study and a real data analysis are presented to illustrate the results.

Keywords: Bayesian estimation, Maximum likelihood estimation, Lindley's approximation, Inverted Kumaraswamy distribution, DeGroot loss function.

AMS Subject Classification: 83-02, 99A00.

1. Introduction

Chandler (1952) introduced the study of record values and documented many of the basic properties of records. Record values can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of observations. Although the conception of record values was not introduced by a long time as other statistical conceptions like common order statistic, there are a considerable stack of publications on record values. In a little over forty years, a large number of publications devoted to records have appeared. This is possibly due to the fact that we encounter this notion frequently in daily life, especially in singling out record values from a set of others and in registering and recalling record values. Record data arise in a wide variety of practical situations. Examples include industrial stress testing, meteorological analysis, hydrology, seismology, sporting, athletic events, economics and life testing.
Some work has been done on statistical inference based on record values. See for instance, Jaheen (2003), Raqab *et al.* (2007), Doostparast (2009), Nadar *et al.* (2013), Danish

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and Aslam (2013), Dey *et al.* (2013), Hussian and Amin (2014), Mahmoud *et al.* (2016), Faizan and Sana (2018) and Sana and Faizan (2019).
Let $X_1, X_2, ...$ be a sequence of independent and identically distributed *(iid)* random variables (rv's) with a cumulative distribution function *(cdf)* $F(x)$ and a probability density function *(pdf)* $f(x)$. Let $Y_n = min(X_1, X_2, ..., X_n)$, $n \geq 1$; then, the observation X_j is a lower record value of $(X_n, n \geq 1)$, if it is smaller than all the preceding observations, in other words, if $Y_j < Y_{j-1}, j > 1$. By definition X_1 is a lower, as well as upper record value, called the base record value. For more details on record values [see, Ahsanullah (2004)].
The Inverted Kumaraswamy distribution with shape parameters α and $\lambda > 0$, denoted by $IKum(\alpha, \lambda)$, then the probability density function *(pdf)* and cumulative distribution function *(cdf)*, respectively, are given by [see, AL-Fattah *et al.* (2017)]

$$f(x) = \alpha\lambda(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\lambda-1}, \quad x, \alpha, \lambda > 0 \tag{1}$$

and

$$F(x) = (1-(1+x)^{-\alpha})^{\lambda}, \quad x, \alpha, \lambda > 0. \tag{2}$$

Bayesian estimation under the loss function is not frequently discussed, perhaps, because the estimators under asymmetric loss function involve integral expressions, which are not analytically solvable. Therefore, one has to use the numerical techniques or certain approximation methods for the solution. Lindley's approximation technique is one of the methods suitable for solving such problems.
In this paper, the maximum likelihood estimators of the parameters are calculated and approximate Bayes estimators of the parameters of the *IKum* distribution based on lower record values are obtained under DeGroot loss function using Lindley's approximation technique with non-informative prior information. Moreover, a simulation study and a real data analysis is taken place in section 3 and 4. Finally the conclusions are presented in section 11.

2. Estimation of the parameters

In this section, we shall be concerned with estimation of the two unknown parameters α and λ of the *IKum* distribution based on record values.

2.1. **Maximum likelihood estimator.** Let $X_{L(1)} = x_1, X_{L(2)} = x_2, ..., X_{L(m)} = x_m$, are m lower record values taken from $IKum(\alpha, \lambda)$ distribution with *(pdf)* eq(1). In this case, Arnold *et al.* (1998) gives the likelihood function as

$$l(\alpha, \lambda|\underline{x}) = f(x_{L(m)}; \alpha, \lambda)\prod_{i=1}^{m-1}\frac{f(x_{L(i)}; \alpha, \lambda)}{F(x_{L(m)}; \alpha, \lambda)}. \tag{3}$$

Substituting eq(1) and eq(2) in eq(3), we get

$$l(\alpha, \lambda|\underline{x}) = \alpha^m\lambda^m(1-(1+x_m)^{-\alpha})^{\lambda}t(\underline{x}), \tag{4}$$

where

$$t(\underline{x}) = \prod_{i=1}^{m}\frac{(1+x_i)^{-(\alpha+1)}}{(1-(1+x_i)^{-\alpha})} \quad \text{and} \quad \underline{x} = (x_1, x_2, ..., x_m).$$

The log-likelihood function is

$$L(\alpha, \lambda|\underline{x}) = m(log\alpha + log\lambda) + \lambda log(1-(1+x_m)^{-\alpha})$$

$$-(\alpha+1)\sum_{i=1}^{m} log(1+x_i) - \sum_{i=1}^{m} log(1-(1+x_i)^{-\alpha}). \tag{5}$$

By differentiating the equation eq(5) with respect to α and λ and equating to zero, we get

$$\frac{\partial L(\alpha,\lambda|\underline{x})}{\partial\alpha} = \frac{m}{\alpha} + \lambda\frac{(1+x_m)^{-\alpha}log(1+x_m)}{(1-(1+x_m)^{-\alpha})} - \sum_{i=1}^{m} log(1+x_i)$$

$$-\sum_{i=1}^{m}\frac{(1+x_i)^{-\alpha}log(1+x_i)}{(1-(1+x_i)^{-\alpha})} = 0 \tag{6}$$

and

$$\frac{\partial L(\alpha,\lambda|\underline{x})}{\partial\lambda} = \frac{m}{\lambda} + log(1-(1+x_m)^{-\alpha}) = 0. \tag{7}$$

From equation eq(7), we get

$$\hat{\lambda} = -\frac{m}{log(1-(1+x_m)^{-\alpha})}. \tag{8}$$

It should be noted that the equation eq(6) is complicated to solve mathematically because it is a non linear equation, so a numerical technique, such as Newton-Raphson method, can be used to obtain the *MLE's* of the unknown parameters α and λ.

2.2. **Loss Function.** A loss function represents losses incurred when we estimate the parameter θ by $\hat{\theta}$. A asymmetric loss function are proposed for use, among this, we use the following loss function.

2.2.1. *Degroot Loss Function (DLF).* DeGroot(1970) discussed different types of loss functions and obtained the Bayes estimates under these loss functions. Here is one example of the asymmetric loss function defined for the positive values of the parameter. If $\hat{\theta}$ is an estimate of θ then the DeGroot loss function is defined as:

$$L_{DeGroot}(\hat{\theta},\theta) = \left(\frac{\theta-\hat{\theta}}{\hat{\theta}}\right)^2.$$

The Bayes estimator under *DLF* can be derived by using following formulae:

$$\hat{\theta}_{DeGroot} = \frac{E_{\theta|x}(\theta^2)}{E_{\theta|x}(\theta)}.$$

2.3. **Bayes Estimator.** In this subsection, we investigate the Bayes estimators for parameters α and λ. Assuming that both of the parameters α and λ are unknown and independent distributions, the joint non-informative prior of α and λ is

$$g(\alpha,\lambda) = \frac{1}{\alpha\lambda}, \quad \alpha,\lambda > 0. \tag{9}$$

Applying Bayes Theorem, The joint posterior distribution of α and λ can be obtained using eq(4) and eq(9) as follows

$$\pi(\alpha,\lambda|\underline{x}) \propto l(\alpha,\lambda|\underline{x})g(\alpha,\lambda), \tag{10}$$

$$\propto \alpha^{m-1}\lambda^{m-1}(1-(1+x_m)^{-\alpha})^{\lambda}t(\underline{x}),$$

$$= C\alpha^{m-1}\lambda^{m-1}(1-(1+x_m)^{-\alpha})^{\lambda}t(\underline{x}), \tag{11}$$

where
$C^{-1} = \int_o^\infty \int_o^\infty \alpha^{m-1}\lambda^{m-1}(1-(1+x_m)^{-\alpha})^\lambda t(\underline{x})d\alpha d\lambda$,
which is a normalizing constant.
Now, from eq(11), we get

$$\pi(\alpha,\lambda|\underline{x}) = \frac{\alpha^{m-1}\lambda^{m-1}(1-(1+x_m)^{-\alpha})^\lambda t(\underline{x})}{\int_o^\infty \int_o^\infty \alpha^{m-1}\lambda^{m-1}(1-(1+x_m)^{-\alpha})^\lambda t(\underline{x})d\alpha d\lambda}. \tag{12}$$

It may be noted here that the posterior distribution of α and λ takes a ratio form that involves an integration in the denominator and cannot be reduced to a closed form. Hence, the evaluation of the posterior expectation for obtaining the Bayes estimator of α and λ will be tedious. Among the various methods suggested to approximate the ratio of integrals of the above form, perhaps the simplest one is Lindley (1980) approximation method, which approaches the ratio of the integrals as a whole and produces a single numerical result. Thus, we propose the use of Lindley's approximation for obtaining the Bayes estimator of α and λ . Many authors have used this approximation for obtaining the Bayes estimators for various distributions; see among others, Howlader and Hossain (2002), Nassar and Eissa (2005), Singh *et al.* (2008), Xu and Tang (2010), Kim *et al.* (2011) and Dey *et al.* (2016).
Lindley (1980), proposed an approximation procedure to evaluate the ratio of two integrals such that

$$I(x) = E[u(\alpha,\lambda|\underline{x})] = \frac{\int_{(\alpha,\lambda)} u(\alpha,\lambda)e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)}{\int_{(\alpha,\lambda)} e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)}, \tag{13}$$

where
$u(\alpha,\lambda)$ = function of α and λ only;
$L(\alpha,\lambda)$= log of likelihood;
$G(\alpha,\lambda)$ = log of joint prior of α and λ
can be evaluated as

$$I(x) = u(\hat{\alpha},\hat{\lambda}) + \frac{1}{2}[(\hat{u}_{\lambda\lambda} + 2\hat{u}_\lambda\hat{p}_\lambda)\hat{\sigma}_{\lambda\lambda} + (\hat{u}_{\alpha\lambda} + 2\hat{u}_\alpha\hat{p}_\lambda)\hat{\sigma}_{\alpha\lambda}$$

$$+(\hat{u}_{\lambda\alpha} + 2\hat{u}_\lambda\hat{p}_\alpha)\hat{\sigma}_{\lambda\alpha} + (\hat{u}_{\alpha\alpha} + 2\hat{u}_\alpha\hat{p}_\alpha)\hat{\sigma}_{\alpha\alpha}]$$

$$+\frac{1}{2}[(\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda} + \hat{u}_\alpha\hat{\sigma}_{\lambda\alpha})(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha})$$

$$+(\hat{u}_\lambda\hat{\sigma}_{\alpha\lambda} + \hat{u}_\alpha\hat{\sigma}_{\alpha\alpha})(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})], \tag{14}$$

where
$\hat{\alpha}$= MLE of α;
$\hat{\lambda}$= MLE of λ;
$\hat{u}_\lambda = \frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}}$; $\hat{u}_\alpha = \frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}}$; $\hat{u}_{\lambda\alpha} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\alpha}}$; $\hat{u}_{\alpha\lambda} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\lambda}}$;
$\hat{u}_{\lambda\lambda} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}^2}$; $\hat{u}_{\alpha\alpha} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}^2}$; $\hat{L}_{\lambda\lambda\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\lambda}\partial\hat{\alpha}}$; $\hat{L}_{\lambda\lambda\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\lambda}\partial\hat{\lambda}}$;
$\hat{L}_{\lambda\alpha\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\alpha}\partial\hat{\lambda}}$; $\hat{L}_{\alpha\alpha\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\alpha}\partial\hat{\lambda}}$; $\hat{L}_{\alpha\lambda\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\lambda}\partial\hat{\lambda}}$; $\hat{L}_{\lambda\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\delta\hat{\lambda}\partial\hat{\alpha}\partial\hat{\alpha}}$;
$\hat{L}_{\alpha\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\alpha}\partial\hat{\alpha}}$; $\hat{L}_{\alpha\lambda\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\lambda}\partial\hat{\alpha}}$; $\hat{p}_\alpha = \frac{\partial G(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}}$; $\hat{p}_\lambda = \frac{\partial G(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}}$;

2.4. **Bayes estimator of α under DeGroot loss function.** To estimate the Bayes estimator of α, the following are considered
$u(\hat{\alpha},\hat{\lambda})=\alpha$ and $u(\hat{\alpha},\hat{\lambda})=\alpha^2$,

$$L(\alpha,\lambda|\underline{x})=m(log\alpha+log\lambda)+\lambda log(1-(1+x_m)^{-\alpha})$$

$$-(\alpha+1)\sum_{i=1}^{m}log(1+x_i)-\sum_{i=1}^{m}log(1-(1+x_i)^{-\alpha}).$$

$$G(\hat{\alpha},\hat{\lambda})=ln(g(\alpha,\lambda))=-ln(\alpha)-ln(\lambda).$$

It can easily be verified that:
If $u(\hat{\alpha},\hat{\lambda})=\alpha$, then
$\hat{u}_\alpha=1$ and $\hat{u}_\lambda=\hat{u}_{\lambda\alpha}=\hat{u}_{\lambda\lambda}=\hat{u}_{\alpha\alpha}=0.$
If $u(\hat{\alpha},\hat{\lambda})=\alpha^2$, then
$\hat{u}_\alpha=2\alpha,$ $\hat{u}_{\alpha\alpha}=2,$ and $\hat{u}_\lambda=\hat{u}_{\lambda\alpha}=\hat{u}_{\lambda\lambda}=0.$
Now,
$\hat{p}_\lambda=-\frac{1}{\lambda},$ $\hat{p}_\alpha=-\frac{1}{\alpha},$ $\hat{L}_\lambda=\frac{m}{\lambda}+log(1-(1+x_m)^{-\alpha}),$ $\hat{L}_{\lambda\lambda}=-\frac{m}{\lambda^2},$ $\hat{L}_\alpha=\frac{m}{\alpha}+\lambda\frac{(1+x_m)^{-\alpha}log(1+x_m)}{(1-(1+x_m)^{-\alpha})}-\sum_{i=1}^{m}log(1+x_i)-\sum_{i=1}^{m}\frac{(1+x_i)^{-\alpha}log(1+x_i)}{(1-(1+x_i)^{-\alpha})},$
$\hat{L}_{\lambda\alpha}=\frac{(1+x_m)^{-\alpha}log(1+x_m)}{(1-(1+x_m)^{-\alpha})},$ $\hat{L}_{\alpha\alpha}=-\frac{m}{\alpha^2}-\lambda a(x_m)+\sum_{i=1}^{m}a(x_i),$
$\hat{L}_{\lambda\lambda\alpha}=\hat{L}_{\alpha\lambda\lambda}=0,$ $\hat{L}_{\alpha\alpha\alpha}=\frac{2m}{\alpha^3}+\lambda c(x_m)-\sum_{i=1}^{m}c(x_i),$
$\hat{L}_{\lambda\alpha\alpha}=\hat{L}_{\alpha\alpha\lambda}=-a(x_m)$ and $\hat{L}_{\lambda\lambda\lambda}=\frac{2m}{\lambda^3}.$
where
$a(x_m)=(1+x_m)^{-\alpha}(log(1+x_m))^2[\frac{1}{(1-(1+x_m)^{-\alpha})^2}],$
$a(x_i)=(1+x_i)^{-\alpha}(log(1+x_i))^2[\frac{1}{(1-(1+x_i)^{-\alpha})^2}],$
$b(x_m)=[\frac{2(1-(1+x_m)^{-\alpha})^2(1+x_m)^{-\alpha}+2(1-(1+x_m)^{-\alpha})((1+x_m)^{-\alpha})^2}{(1-(1+x_m)^{-\alpha})^2}],$
$b(x_i)=[\frac{2(1-(1+x_i)^{-\alpha})^2(1+x_i)^{-\alpha}+2(1-(1+x_i)^{-\alpha})((1+x_i)^{-\alpha})^2}{(1-(1+x_i)^{-\alpha})^2}],$
$c(x_m)=a(x_m)log(1+x_m)[1+b(x_m)]$
and
$c(x_i)=a(x_i)log(1+x_i)[1+b(x_i)].$
Again, because α and λ are independent,
$\hat{\sigma}_{\lambda\alpha}=0;$ $\hat{\sigma}_{\lambda\lambda}=-\frac{1}{\hat{L}_{\lambda\lambda}}$ *and* $\hat{\sigma}_{\alpha\alpha}=-\frac{1}{\hat{L}_{\alpha\alpha}}.$
Evaluating u-terms, L-terms and p-terms mentioned above at point $(\hat{\alpha},\hat{\lambda})$ and using eq(14) we get, the Bayes estimator of α under the DeGroot loss function is

$$\hat{\alpha}_{DeGroot}=\frac{\alpha^2+[\frac{1}{-\frac{m}{\alpha^2}-\lambda a(x_m)+\sum_{i=1}^{m}a(x_i)}]+\alpha[\frac{\frac{2m}{\alpha^3}+\lambda c(x_m)-\sum_{i=1}^{m}c(x_i)}{(-\frac{m}{\alpha^2}-\lambda a(x_m)+\sum_{i=1}^{m}a(x_i))^2}]}{[\alpha+\frac{1}{\alpha}[\frac{1}{-\frac{m}{\alpha^2}-\lambda a(x_m)+\sum_{i=1}^{m}a(x_i)}]+\frac{1}{2}[\frac{\frac{2m}{\alpha^3}+\lambda c(x_m)-\sum_{i=1}^{m}c(x_i)}{(-\frac{m}{\alpha^2}-\lambda a(x_m)+\sum_{i=1}^{m}a(x_i))^2}]]}.$$

2.4.1. *Bayes estimator of λ under DeGroot loss function.* To estimate the Bayes estimator of λ, the following are considered
$u(\hat{\alpha},\hat{\lambda})=\lambda$ and $u(\hat{\alpha},\hat{\lambda})=\lambda^2$,
$L(\alpha,\lambda|\underline{x})$ and $G(\hat{\alpha},\hat{\lambda})$ are the same as those given in section (7).
It can easily be verified that:
If $u(\hat{\alpha},\hat{\lambda})=\lambda$, then
$\hat{u}_\lambda=1$ and $\hat{u}_\alpha=\hat{u}_{\lambda\alpha}=\hat{u}_{\lambda\lambda}=\hat{u}_{\alpha\alpha}=0.$
If $u(\hat{\alpha},\hat{\lambda})=\lambda^2$, then

$\hat{u}_\lambda = 2\lambda, \quad \hat{u}_{\lambda\lambda} = 2 \quad$ and $\quad \hat{u}_\alpha = \hat{u}_{\lambda\alpha} = \hat{u}_{\alpha\alpha} = 0.$

Following the procedure as discussed in section (7), we get after simplification, the Bayes estimator of λ under the DeGroot loss function is

$$\hat{\lambda}_{DeGroot} = \frac{\lambda^2 - \frac{\lambda^2}{m} - [\frac{\lambda^3}{m}(-\frac{2}{\lambda} - \frac{a(x_m)}{-\frac{m}{\alpha^2} - \lambda a(x_m) + \sum_{i=1}^{m} a(x_i)})]}{[\lambda - \frac{\lambda}{m} + \frac{1}{2}(-\frac{\lambda^2}{m}(-\frac{2}{\lambda} - \frac{a(x_m)}{-\frac{m}{\alpha^2} - \lambda a(x_m) + \sum_{i=1}^{m} a(x_i)}))]}.$$

3. A Simulation Study

Simulation study is conducted to illustrate all the results described in the previous Sections.

The MLE's and Bayes estimators are obtained according to following steps:

(i) Samples of lower records with different sizes of $m \in \{8, 9, 10\}$ are generated from the $IKum(\alpha, \lambda)$ distribution for $\alpha = 3$ and $\lambda = 2$.

(ii) Estimates of α and λ are obtained.

(iii) Above steps are repeated 1,000 times to evaluate these estimates.

All these results are presented in Table [1].

Table 1. MLE's and Bayes estimates based on generated lower record values of sample size n=2000, when the parameters are $\alpha = 3$ and $\lambda = 2$

Number of Records m	MLEs		Bayes	
	$\hat{\alpha}_{MLE}$	$\hat{\lambda}_{MLE}$	$\hat{\alpha}_{DeGroot}$	$\hat{\lambda}_{DeGroot}$
8	3.0006	2.0017	2.9996	2.2500
9	3.0020	2.0027	2.9996	2.2222
10	3.0024	2.0052	2.9997	2.2000

4. An Illustrative Example

In this Section, we provide a real data analysis in order to indicate fit to the $IKum(\alpha, \lambda)$ distribution.

(i) The vinyl chloride data obtained from clean upgrading, monitoring wells in mg/L; this data set was used by Bhaumik *et al.* (2009). The data is
5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

(ii) The data consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul; this real data set is given by Hinkley (1977). The data is
0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

(iii) The data refers to the time between failures for repairable items; this data set is given by Murthy *et al.* (2004). The data is
1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

Firstly, we have examined whether these data set fit to the $IKum(\alpha, \lambda)$ distribution. The parameters of this distribution are estimated by maximum likelihood method. The estimated parameters and Kolmogorov-Smirnov (K-S) distance between the theoretical

TABLE 2. MLE's and Kolmogorov-Smirnov test statistics summary of $IKum(\alpha, \lambda)$ distribution for the real data set

Real data set	$\hat{\alpha}_{MLE}$	$\hat{\lambda}_{MLE}$	$K-S$ distance $D_{(30,0,1)}$	$p-value$
First	3.0006	2.0017	2.9996	2.2501
Second	3.0020	2.0027	2.9996	2.2222
Third	3.0024	2.0052	2.9997	2.2000

and empirical distribution functions and p-values are given in Table [2]. From this table, it is clear that the data sets fit to the $IKum(\alpha, \lambda)$ distribution.
From the data sets, the observed lower record values are obtained as follows;
$M_{(1)} : 5.1, 1.2, 0.6, 0.5, 0.4, 0.2, 0.1,$
$M_{(2)} : 0.77, 0.47, 0.32$
and
$M_{(3)} : 1.43, 0.11.$
The MLE's and Bayes estimates for unknown parameters of $IKum(\alpha, \lambda)$ distribution based on observed lower record values are given in Table [3].

TABLE 3. MLE's and Bayes estimates for unknown parameters based on observed lower record values

Sample size n	Observed Lower Records	MLEs		Bayes	
		$\hat{\alpha}_{MLE}$	$\hat{\lambda}_{MLE}$	$\hat{\alpha}_{DeGroot}$	$\hat{\lambda}_{DeGroot}$
34	5.1, 1.2, 0.6, 0.5, 0.4, 0.2, 0.1	0.9973	2.9161	1.3154	3.2775
30	0.77, 0.47, 0.32	1.2838	2.4907	1.2117	3.4969
30	1.43, 0.11	1.1076	0.9029	1.5628	4.3423

5. CONCLUSIONS

In this chapter, theoretical results of the study are explained numerically. In Section 3, we apply the simulation algorithm with 1,000 replication to obtain the MLE's and Bayes estimators. The simulated estimators are presented in Table [1]. From Table [1], we see that if the no. of record values increases, the MLE's and Bayes estimators of α increases regularly, while MLE's of λ increases and Bayes estimators of λ decreases. In Section 4, MLE's and Bayes estimators based on lower record values generated from the real data set are calculated. Firstly from the real data, we fit the $IKum(\alpha, \lambda)$ distribution defined in equation (1). From Table [2], we see that the data sets fit to the $IKum(\alpha, \lambda)$ distribution because the p-value is higher enough than significance level usually referred in statistical literature. In Table [3], from the first real data set, 7 lower record values have been observed, from the second real data set, 3 lower record values have been observed and from the third real data set, 2 lower record values have been observed. Now we see that the no. of record values decreases at different real data sets. It is noted that as no. of record values decreases, the MLE's and Bayes estimators of α increases and decreases irregularly and MLE's of λ decreases and Bayes estimators of λ increases.

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