TWMS J. App. and Eng. Math. V.13, N.1, 2023, pp. 143-151

REGULARIZED TRACE ON SEPARABLE BANACH SPACES

E. GÜL[1*], T. L. GILL[2], §

ABSTRACT. If $\mathcal{H}$ is a separable Hilbert space, Gül (2008) has shown that a regularized trace formula can be computed on $L^2(H, [0, \pi])$ for a second order differential operator with bounded operator-valued coefficients, where $\mathcal{H}$ is a separable Hilbert space. Kuelbs (1970) has shown that every separable Banach space can be continuously and densely embedded into a separable Hilbert space, while Gill (2016) has used Kuelbs result to show that the dual of a Banach space does not have a unique representation. In this paper, we use the results of Kuelbs and Gill to study the regularized trace formula on $L^2(\mathcal{B}, [0, \pi])$, where $\mathcal{B}$ is an arbitrary separable Banach space.

Keywords: Dual space, adjoint operator, Schatten classes, regularized trace formula.

AMS Subject Classification: 47B37, 46B10, 46C99.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and let $S_1[\mathcal{H}]$ be the trace class operators on $\mathcal{H}$. Let $\mathcal{H}_1 = L^2(\mathcal{H};\ [0, \pi])$ and define an inner product on $\mathcal{H}_1$ by:

$$(f, g)_{\mathcal{H}_1} = \int_0^{\pi} (f(t), g(t))_{\mathcal{H}} dt$$

for all $f, g \in \mathcal{H}_1$. It is easy to see that, with this inner product $\mathcal{H}_1$ is a separable Hilbert space. In [3], we defined operators L_0 and L on $\mathcal{H}_1$ by:,

$$L_0(y) = -y''(t) \qquad \text{and} \qquad L(y) = -y''(t) + Q(t)y(t)$$

with the same boundary conditions $y'(0) = y(\pi) = 0$. We assumed that the operator valued function $Q(t)$ has the following properties:

(1) $Q(t)$ has a weak second-order derivative in $[0, \pi]$ and for $t \in [0, \pi]$, $Q^{(i)}(t)$ $(i = 0, 1, 2)$ is a self adjoint trace class operator on $\mathcal{H}$.
(2) $\|Q\|_{\mathcal{H}_1} < 1$.
(3) $\mathcal{H}_1$ has an o.n.b. (orthonormal basis) $\{\varphi_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \|Q\varphi_n\|_{\mathcal{H}_1} < \infty$.

[1] Yıldız Technical University, Faculty of Arts and Science, Department of Mathematics, 34210, Davutpasa, İstanbul, Turkey.
e-mail: gul@yildiz.edu.tr; ORCID: https://orcid.org/0000-0003-0626-0148.
[*] Corresponding author.
[2] Howard University, Department of EECS and Mathematics, Washington DC, 20059, USA.
e-mail: tgill@howard.edu; ORCID: https://orcid.org/0000-0002-7477-2076.
§ Manuscript received: October 20, 2020; accepted: December 28, 2020.

(4) $\left\|Q^i(t)\right\|_{S_1[\mathcal{H}]}$ $(i=0,1,2)$ is a bounded measurable function on $[0,\pi]$.

1.1. **Purpose.** The purpose of this paper is to study the above problem, with $\mathcal{H}$ replaced by a arbitrary separable Banach space $\mathcal{B}$, under the following conditions:

(1) $Q(t)$ has a weak second-order derivative in $[0,\pi]$ and for $t \in [0,\pi]$, $Q^{(i)}(t)$ $(i = 0,1,2)$ is a self adjoint trace class operator on $\mathcal{B}$.
(2) $\|Q\|_{\mathcal{H}_1} < 1$.
(3) $\mathcal{H}_1$ has an o.n.b. $\{\varphi_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \|Q\varphi_n\|_{\mathcal{H}_1} < \infty$.
(4) $\left\|Q^i(t)\right\|_{S_1[\mathcal{B}]}$ $(i=0,1,2)$ is a bounded measurable function on $[0,\pi]$.

It is clear from (4), that this is a nontrivial problem since, among other things, in the standard approach, there are a number of possible definitions of $S_1[\mathcal{B}]$ (see [2] and Pietsch [8]).

1.2. **Preliminaries.** Let $\mathcal{B}$ be a separable Banach space with dual space $\mathcal{B}^*$, let $\mathcal{C}[\mathcal{B}]$ be the closed densely defined linear operators and $\mathcal{L}[\mathcal{B}]$ be the bounded linear operators on $\mathcal{B}$.

In 1965, Gross [1] proved that every separable Banach space contains a separable Hilbert space as a continuous dense embedding.Then, in 1970, Kuelbs [5] gave an extension of Gross' theorem. The following lemma is the important part of this extension due to Kuelbs [5].

Lemma 1.1. (Kuelbs Lemma) *Let $\mathcal{B}$ be a separable Banach space. Then, there exist a separable Hilbert space $\mathcal{H}$ such that $\mathcal{B} \subset \mathcal{H}$ as continuous dense embedding.*

2. Operator Theory

If T is an operator, we let $\sigma(T)$ denote the spectrum of T and $\sigma_p(T) \subset \sigma(T)$ denote the point spectrum of T. The following theorem is due to Lax [6].

Theorem 2.1. (Lax's Theorem) *Let $\mathcal{B}$ be a separable Banach space that is continuously and densely embedded in a Hilbert space $\mathcal{H}$, and let T be a bounded linear operator on $\mathcal{B}$ that is symmetric with respect to the inner product of $\mathcal{H}$ (i.e., $(Tu,v)_{\mathcal{H}} = (u,Tv)_{\mathcal{H}}$ for all $u,v \in \mathcal{B}$). Then,*

(1) *T is bounded with respect to the $\mathcal{H}$ norm, and*
$$\|T^*T\|_{\mathcal{H}} = \|T\|_{\mathcal{H}}^2 \leqslant k\,\|T\|_{\mathcal{B}}^2,$$
where k is a positive constant.
(2) *$\sigma(T)$ relative to $\mathcal{H}$ is a subset of $\sigma(T)$ relative to $\mathcal{B}$.*
(3) *$\sigma_p(T)$ relative to $\mathcal{H}$ is equal to $\sigma_p(T)$ relative to $\mathcal{B}$.*

Let $\mathbf{J}$ be the (conjugate) isometric isomorphism of $\mathcal{H} \to \mathcal{H}^*$, and let $\mathbf{J}_{\mathcal{B}} = \mathbf{J}|_{\mathcal{B}}$ (restriction). Since $\mathcal{B}$ is a continuous dense embedding in $\mathcal{H}$, $\mathbf{J}_{\mathcal{B}}$ is a (conjugate) isometric isomorphism of $\mathcal{B}$ onto $\mathbf{J}_{\mathcal{B}}(\mathcal{B}) \subset \mathcal{H}^*$ as a continuous dense embedding.

Definition 2.2. *Let $u \in \mathcal{B}$. We define $u_h^* = \mathbf{J}_{\mathcal{B}}(u)$ and $\mathcal{B}_h^* = \{u_h^* \in \mathcal{B}^* : u \in \mathcal{B}\}$, so that $\langle u, u_h\rangle = (u,u)_{\mathcal{H}} = \|u\|_{\mathcal{H}}^2$. We call $\mathcal{B}_h^*$ the canonical Hilbert representation for $\mathcal{B}$ in $\mathcal{B}^*$.*

A proof of the next two results can be found in [2].

Theorem 2.3. *If $A \in \mathcal{C}[\mathcal{B}]$, then there is a unique operator $A^* \in \mathcal{C}[\mathcal{B}]$ that satisfies the following:*

(1) *$(aA)^* = \bar{a}A^*$;*

(2) $A^{**} = A$;
(3) $(A+B)^* = A^* + B^*$;
(4) $(AB)^* = B^*A^*$ *on* $D(A^*) \bigcap D(B^*)$;
(5) *if* $A \in \mathcal{L}[\mathcal{B}]$, *then* $\|A^*A\|_{\mathcal{B}} \leq M \|A\|_{\mathcal{B}}^2$, *for some constant* M *and it has a bounded extension to* $\mathcal{L}[\mathcal{H}]$.

Theorem 2.4. *For every* $\phi \in \mathcal{B}$, *there exists a* $\varphi^* \in \mathcal{B}^*$ *and a constant* $c_\phi > 0$ *depending on* ϕ *such that* $(f,\phi)_{\mathcal{H}} = c_\phi^{-1}\langle f, \varphi^*\rangle_{\mathcal{B}^*}$ *for all* $f \in \mathcal{B}$.

Let $\mathbb{S}_\infty[\mathcal{B}]$ be the set of compact operators on $\mathcal{B}$. If $A = U[A^*A]^{1/2} \in \mathbb{S}_\infty[\mathcal{B}]$, let $\bar{A} = \bar{U}[\bar{A}^*\bar{A}]^{1/2}$ be its extension to $\mathcal{H}$. For each compact operator $\bar{A}$, an orthonormal family $\{\phi_n \, |n \geqslant 1\}$ exists such that

$$\bar{A} = \sum_{n=1}^{\infty} s_n(\bar{A})\,(\cdot\,,\phi_n)_{\mathcal{H}}\,\bar{U}\phi_n. \tag{2.1}$$

Here, the $s_n(\bar{A})$ are the eigenvalues of $[\bar{A}^*\bar{A}]^{1/2} = |\bar{A}|$, counted by multiplicity and in decreasing order (s-numbers). Without loss of generality, we can assume that $\{\phi_n \, |n \geqslant 1\} \subset \mathcal{B}$. From Theorem 2.4 and the fact that $s_n(\bar{A}) = s_n(A)$ by Lax's theorem, we can write A as follows:

$$A = \sum_{n=1}^{\infty} s_n(A) c_n^{-1} \langle \cdot, \varphi_n^* \rangle_{\mathcal{B}^*} U\phi_n. \tag{2.2}$$

If $\bar{A} \in \mathbb{S}_p[\mathcal{H}]$ (the Schatten class of order p in $\mathcal{L}[\mathcal{H}]$), its norm can be represented as follows:

$$\begin{aligned}\|\bar{A}\|_p^{\mathcal{H}} &= \left\{Tr\left[\bar{A}^*\bar{A}\right]^{p/2}\right\}^{1/p} = \left\{\sum_{n=1}^{\infty} \left(\bar{A}^*\bar{A}\phi_n, \phi_n\right)_{\mathcal{H}}^{p/2}\right\}^{1/p} \\ &= \left\{\sum_{n=1}^{\infty} \left|s_n\left(\bar{A}\right)\right|^p\right\}^{1/p}.\end{aligned}$$

Definition 2.5. *We define* $\mathbb{S}_p[\mathcal{B}]$, *the Schatten class of order* p *in* $\mathcal{L}[\mathcal{B}]$, *as follows:*

$$\mathbb{S}_p[\mathcal{B}] = \left\{ A \in \mathbb{S}_\infty[\mathcal{B}] : \|A\|_p^{\mathcal{B}} = \left\{\sum_{n=1}^{\infty} |s_n(A)|^p\right\}^{1/p} < \infty \right\}.$$

Since $s_n(A) = s_n(\bar{A})$, we have the following:

Corollary 2.6. *If* $A \in \mathbb{S}_p[\mathcal{B}]$, *then* $\bar{A} \in \mathbb{S}_p[\mathcal{H}]$ *and* $\|A\|_p^{\mathcal{B}} = \|\bar{A}\|_p^{\mathcal{H}}$.

Note that more detailed explanations and results on the subject can be found in [4].

3. Regularized Trace on $\mathcal{B}$

In this section, we assume that $\mathcal{B}$ is a continuous dense embedding in a separable Hilbert space $\mathcal{H}$ and for each $f, g \in \mathcal{B}$, $(f,g)_h = (f,g)_{\mathcal{H}}$ is the canonical Hilbert functional on $\mathcal{B}$.

Recall that $\mathbb{S}_\infty[\mathcal{B}]$ is the set of compact operators on $\mathcal{B}$. If $A \in \mathbb{S}_\infty[\mathcal{B}]$ then, by the polar representation theorem, A^*A is a non-negative self-adjoint operator and $|A| = [A^*A]^{1/2} \in \mathbb{S}_\infty[\mathcal{B}]$ where A^* is the adjoint of A. Let $s_1(A) \geq s_2(A) \geq \ldots \geq s_k(A)$ $(1 \leq k \leq \infty)$ be the non-zero eigenvalues of $|A|$ with each eigenvalue is repeated as many times as its multiplicity (s-numbers). When $k < \infty$, we assume that $s_j(A) = 0$ for $j = k+1, k+2, \ldots$.

If $A \in \mathbb{S}_p[\mathcal{B}]$, $1 \leq p < \infty$, then by Corollary 2.6, A extends to $\bar{A} \in \mathbb{S}_p[\mathcal{H}]$ with $\|A\|_p^{\mathcal{B}} = \|\bar{A}\|_p^{\mathcal{H}}$. If $A \in \mathbb{S}_1[\mathcal{B}]$, we called it a trace class (or nuclear) operator on $\mathcal{B}$.

Since $\mathbb{S}_p[\mathcal{H}]$ is a two sided *ideal, it follows that the same is true for $\mathbb{S}_p[\mathcal{B}]$. Thus, for $1 \leq p < \infty$, $A \in \mathbb{S}_p[\mathcal{B}]$ and $B \in \mathcal{L}[\mathcal{B}]$ then $AB, BA \in \mathbb{S}_p[\mathcal{B}]$ and

$$\|AB\|_{\mathbb{S}_p[\mathcal{B}]} \leq \|B\|_{\mathcal{L}[\mathcal{B}]} \|A\|_{\mathbb{S}_p[\mathcal{B}]}$$
$$\|BA\|_{\mathbb{S}_p[\mathcal{B}]} \leq \|B\|_{\mathcal{L}[\mathcal{B}]} \|A\|_{\mathbb{S}_p[\mathcal{B}]}$$

We can now return to the main problem of interest. Recall that our problem is now posed on $\mathcal{H}_1 = L^2(\mathcal{B};\ [0,\pi])$ and the inner product on $\mathcal{H}_1$ is defined using our canonical h-representation of $\mathcal{B}$ in $\mathcal{B}^*$ by:

$$(f,g)_{\mathcal{H}_1} = \int_0^\pi (f(t), g(t))_h dt$$

for all $f, g \in \mathcal{H}_1$. Moreover, L_0 and L are differential operators satisfying:

$$L_0(y) = -y''(t) \qquad \text{and} \qquad L(y) = -y''(t) + Q(t)y(t)$$

with the same boundary conditions $y'(0) = y(\pi) = 0$. Where we assume that $Q(t)$ is an operator valued function with the following properties:

(1) $Q(t)$ has a weak second-order derivative in $[0,\pi]$ and for $t \in [0,\pi]$, $Q^{(i)}(t)$ $(i = 0,1,2)$ is a self adjoint trace class operator on $\mathcal{B}$.
(2) $\|Q\|_{\mathcal{H}_1} < 1$.
(3) $\mathcal{H}_1$ has an o.n.b. $\{\varphi_n\}_{n=1}^\infty$ such that $\sum_{n=1}^\infty \|Q\varphi_n\|_{\mathcal{H}_1} < \infty$.
(4) $\|Q^i(t)\|_{S_1[\mathcal{B}]}$ $(i = 0,1,2)$ is a bounded measurable function on $[0,\pi]$.

As in [3], the spectrum of L_0, $\sigma(L_0)$ is the set $\{(m+1/2)^2\}_{m=0}^\infty$ and each $\lambda \in \sigma(L_0)$ is an eigenvalue with infinite multiplicity. The corresponding orthonormal eigenfunctions are of the form

$$\psi_{mn}^\circ(t) = \sqrt{2/\pi}\varphi_n cos(m+1/2)t \quad (m = 0,1,2,\ldots; n = 1,2,\ldots) \tag{3.1}$$

Let $R_0(\lambda)$ and $R(\lambda)$ be resolvents of L_0 and L, respectively.

Lemma 3.1. *If $\lambda \notin \sigma(L_0)$ then $QR_0(\lambda) \in \mathbb{S}_1[\mathcal{H}_1]$*

Proof. Put $\mu_m = (m+1/2)^2$. For the orthonormal basis ψ_{mn}° of $\mathcal{H}_1$ we get:

$$\begin{aligned}
\sum_{m=0}^\infty \sum_{n=1}^\infty \|QR_0(\lambda)\psi_{mn}^\circ\|_{\mathcal{H}_1} &= \sum_{m=0}^\infty \sum_{n=1}^\infty |\mu_m - \lambda|^{-1} \|Q\psi_{mn}^\circ\|_{\mathcal{H}_1} \\
&= \sum_{m=0}^\infty \sum_{n=1}^\infty |\mu_m - \lambda|^{-1} \left[2/\pi \int_0^\pi cos^2(m+1/2)t \|Q(t)\varphi_n\|_h^2 dt\right]^{1/2} \\
&\leq \sum_{m=0}^\infty \sum_{n=1}^\infty |\mu_m - \lambda|^{-1} \left[\int_0^\pi \|Q(t)\varphi_n\|_h^2 dt\right]^{1/2} \\
&= \sum_{m=0}^\infty |\mu_m - \lambda|^{-1} \sum_{n=1}^\infty \|Q\varphi_n\|_{\mathcal{H}_1} < \infty
\end{aligned}$$

Thus, the lemma follows. □

Using this lemma along with conditions (2) and (3) on $Q(t)$, it follows that the spectrum of L, $\sigma(L)$, is a subset of the union of pairwise disjoint intervals $F_m = [\mu_m - \|Q\|_{\mathcal{H}_1}, \mu_m + \|Q\|_{\mathcal{H}_1}]$ $(m = 0,1,2,...)$. Note, each point of $\sigma(L)$ which is different from μ_m is an isolated eigenvalue of finite multiplicity. However, μ_m itself can be an eigenvalue of L with either finite or infinite multiplicity. Moreover,

$$\lim_{n\to\infty} \lambda_{mn} = \mu_m$$

where $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of L in the interval F_m.

Lemma 3.2. *The operator valued function $R(\lambda)-R_0(\lambda)$ is analytic in $\rho(L)$, the resolvent set of L, with respect to the $\mathbb{S}_1[\mathcal{H}_1]$ norm.*

Proof. Clearly, $R(\lambda)-R_0(\lambda) = -R(\lambda)QR_0(\lambda)$ and note that $\rho(L) \subset \rho(L_0)$. By the Hilbert identity $R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu)$ we have:

$$\begin{aligned} D(\lambda, \Delta\lambda) &= \frac{R(\lambda+\Delta\lambda)QR_0(\lambda+\Delta\lambda) - R(\lambda)QR_0(\lambda)}{\Delta\lambda} - R^2(\lambda)QR_0(\lambda) - R(\lambda)QR_0^2(\lambda) \\ &= R(\lambda+\Delta\lambda)QR_0(\lambda)\big(R_0(\lambda+\Delta\lambda) - R_0(\lambda)\big) + \big(R(\lambda+\Delta\lambda) - R(\lambda)\big)QR_0^2(\lambda) \\ &\quad + \big(R(\lambda+\Delta\lambda) - R(\lambda)\big)R(\lambda)QR_0(\lambda) \end{aligned}$$

and thus

$$\begin{aligned} \|D(\lambda,\Delta\lambda)\|_{\mathbb{S}_1[\mathcal{H}_1]} &\le \|R(\lambda+\Delta\lambda)\|_{\mathcal{H}_1} \|QR_0(\lambda)\|_{\mathbb{S}_1[\mathcal{H}_1]} \|R_0(\lambda+\Delta\lambda) - R_0(\lambda)\|_{\mathcal{H}_1} \\ &\quad + \|R(\lambda+\Delta\lambda) - R(\lambda)\|_{\mathcal{H}_1} \|QR_0(\lambda)\|_{\mathbb{S}_1[\mathcal{H}_1]} \big[\, \|R_0(\lambda)\|_{\mathcal{H}_1} + \|R(\lambda)\|_{\mathcal{H}_1} \big] \end{aligned}$$

Therefore, we conclude that

$$\lim_{\Delta\lambda\to 0} \|D(\lambda,\Delta\lambda)\|_{\mathbb{S}_1[\mathcal{H}_1]} = 0$$

and the proof is done. □

Let $\{\psi_{mn}(t)\}_{m,n=1}^{\infty}$ be orthonormal eigenfunctions corresponding to eigenvalues $\{\lambda_{mn}\}_{m,n=1}^{\infty}$ of L. Since the spectra of the operators L_0 and L only consist of their eigenvalues and limit points, from [7], it is well known that

$$R_0(\lambda) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{V_{mn}^{\circ}}{\mu_m - \lambda};\;\; R(\lambda) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{V_{mn}}{\lambda_{mn} - \lambda} \tag{3.2}$$

where

$$V_{mn}^{\circ} = (.,\psi_{mn}^{\circ})_{\mathcal{H}_1}\psi_{mn}^{\circ};\;\; V_{mn} = (.,\psi_{mn})_{\mathcal{H}_1}\psi_{mn}.$$

In view of Lemma 3.2 and by using the equalities (3.2) it can be seen that for each p ($p = 0, 1, 2, ...$), the series $\sum_{n=1}^{\infty}(\lambda_{pn} - \mu_p)$ is absolutely convergent. Since $R(\lambda) - R_0(\lambda) \in \mathbb{S}_1[\mathcal{H}_1]$, for every $\lambda \in \rho(L)$:

$$tr(R(\lambda) - R_0(\lambda)) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{\mu_m - \lambda}\Big).$$

If multiply both sides of this last equality by $\lambda^2/2\pi i$ and integrate over the circle $|\lambda| = b_p = \mu_p + p$ ($p = 1, 2, \dots$) we conclude that for a natural N

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}(\lambda_{mn} - \mu_m) = \sum_{j=1}^{N} M_{pj} + M_p^{(N)} \tag{3.3}$$

where

$$M_{pj} = \frac{(-1)^j}{2\pi i j}\int_{|\lambda|=b_p} \lambda\, tr\big[(QR_0^j(\lambda)\big]d\lambda \tag{3.4}$$

and

$$M_p^{(N)} = \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda \, tr\left[R(\lambda)(QR_0^{N+1}(\lambda))\right] d\lambda. \tag{3.5}$$

Now, note that for $j = 1$ we have:

$$M_{p1} = \frac{-1}{2\pi i} \int_{|\lambda|=b_p} \sum_{m=0}^{p} \sum_{n=1}^{\infty} (QR_0(\lambda)\psi_{mn}^{\circ}, \psi_{mn}^{\circ})_{\mathcal{H}_1} d\lambda \tag{3.6}$$

Lemma 3.3. *The assumptions of the existence of the o.n.b.* $\{\varphi_n\}_{n=1}^{\infty}$ *in* $\mathcal{H}_1$ *and the integrability of the function* $\|Q(t)\|_{S_1[\mathcal{B}]}$ *on* $[0, \pi]$ *imply that*

$$M_{p1} = \frac{p+1}{\pi} \int_0^{\pi} trQ(t)dt + \frac{1}{\pi} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_0^{\pi} (Q(t)\varphi_n, \varphi_n)_h \, cos(2m+1)tdt$$

Proof. Easily, we observe that

$$\begin{aligned} \left|(QR_0(\lambda)\psi_{mn}^{\circ}, \psi_{mn}^{\circ})_{\mathcal{H}_1}\right| &\leq \sqrt{\pi}|\mu_m - \lambda|^{-1} \left[\int_0^{\pi} \|Q(t)\varphi_n\|_h^2 \, dt\right]^{1/2} \\ &= \sqrt{\pi} \, |\mu_m - \lambda|^{-1} \, \|Q\varphi_n\|_{\mathcal{H}_1} \end{aligned}$$

This means that the series

$$\sum_{m=0}^{\infty} \alpha_m(\lambda) \, ; \; \alpha_m(\lambda) = \sum_{n=1}^{\infty} (QR_0(\lambda)\psi_{mn}^{\circ}, \psi_{mn}^{\circ})_{\mathcal{H}_1}$$

is absolutely and uniformly convergent with respect to λ on the circle $|\lambda| = b_p$. So, by (3.6) we get:

$$M_{p1} = \sum_{m=0}^{p} \sum_{n=1}^{\infty} (QR_0(\lambda)\psi_{mn}^{\circ}, \psi_{mn}^{\circ})_{\mathcal{H}_1} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{d\lambda}{\lambda - \mu_m}$$

or

$$\begin{aligned} M_{p1} &= \sum_{m=0}^{p} \sum_{n=1}^{\infty} (QR_0(\lambda)\psi_{mn}^{\circ}, \psi_{mn}^{\circ})_{\mathcal{H}_1} \\ &= \frac{2}{\pi} \sum_{m=0}^{p} \sum_{n=1}^{\infty} \int_0^{\pi} (Q(t)\varphi_n, \varphi_n)_h \, cos^2(m+1/2)tdt \\ &= \frac{1}{\pi} \sum_{m=0}^{p} \sum_{n=1}^{\infty} \int_0^{\pi} (Q(t)\varphi_n, \varphi_n)_h \, (1 + cos(2m+1))tdt \end{aligned}$$

If we consider the equality

$$\sum_{n=1}^{\infty} \int_0^{\pi} (Q(t)\varphi_n, \varphi_n)_h \, dt = \int_0^{\pi} trQ(t)dt$$

then we obtain desired result for M_{p1} □

By considering the equalities

$$QR_0(\lambda)\psi_{mn}^{\circ} = \frac{Q\psi_{mn}^{\circ}}{\mu_m - \lambda}$$

and

$$(QR_0(\lambda))^2\psi_{mn}^{\circ} = (\mu_m - \lambda)^{-1} \left\{ \sum_{r=0}^{\infty} \sum_{q=1}^{\infty} (\mu_r - \lambda)^{-1} (Q\psi_{mn}^{\circ}, \psi_{rq}^{\circ})_{\mathcal{H}_1} Q\psi_{rq}^{\circ} \right\}$$

we can easily get:

$$M_{p2} = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(\mu_m - \mu_r)^{-1}\left|(Q\psi^{\circ}_{mn}, \psi^{\circ}_{rq})_{\mathcal{H}_1}\right|^2 .$$

or

$$|M_{p2}| \leq \sum_{r=p+1}^{\infty}(\mu_r - \mu_p)^{-1}\sum_{q=1}^{\infty}\left\|Q\psi^{\circ}_{rq}\right\|^2_{\mathcal{H}_1}$$

By the fact

$$\sum_{q=1}^{\infty}\left\|Q\psi^{\circ}_{rq}\right\|^2_{\mathcal{H}_1} \leq \sum_{q=1}^{\infty}\int_0^{\pi}\|Q\varphi_q\|_h\,dt = \sum_{q=1}^{\infty}\|Q\varphi_q\|^2_{\mathcal{H}_1} < const.$$

and observation

$$\sum_{r=p+1}^{\infty}(\mu_r - \mu_p)^{-1} < 2p^{-1/2}$$

it follows that

$$\lim_{p\to\infty} M_{p2} = 0. \tag{3.7}$$

Similar calculations show that

$$\lim_{p\to\infty} M_{pj} = 0\ (j = 3, 4, ...) \tag{3.8}$$

and

$$\lim_{p\to\infty} M_p^N = 0\ (N \geq 4). \tag{3.9}$$

All computations above give rise us an explicit formula called the regularized trace formula for operator L on B as follow.

Theorem 3.4. *The regularized trace formula for operator L on B with the conditions on operator function $Q(t)$ is given by*

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn} - \mu_m) - \frac{1}{\pi}\int_0^{\pi} trQ(t)dt\right] = \frac{1}{4}[\sum_{n=1}^{\infty}(Q(0)\varphi_n, \varphi_n)_h - \sum_{n=1}^{\infty}(Q(\pi)\varphi_n, \varphi_n)_h]$$

Proof. By considering the relation (3.3) with Lemma 3.4, and the equations (3.7), (3.8) and (3.9), we conclude:

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn} - \mu_m) - \frac{1}{\pi}\int_0^{\pi} trQ(t)dt\right] = \frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\int_0^{\pi}(Q(t)\varphi_n, \varphi_n)_h cos(2m+1)tdt$$

The right part of this equality easily can be written in terms that are the values at the points 0 and π of the Fourier series of the Hilbert functional $(Q(t)\varphi_n, \varphi_n)_h$, which have continuous derivative of second order, with respect to $\{cosmt\}_{m=0}^{\infty}$ in $[0, \pi]$. And hence we obtain the required formula given in the hypothesis of the theorem. □

4. Conclusions

Now, we have known that the continuous dense embedding of a separable Banach space into a Hilbert space is a powerful tool for studying the structure of operators on Banach spaces. This approach also offers some new insights into the structure of Banach spaces themselves. This embedding shows that the representation of the dual of a Banach space is not unique and also every closed densely defined linear operator A on $\mathcal{B}$ has a unique adjoint A^* defined on $\mathcal{B}$. Moreover, knowing that $\mathcal{L}[\mathcal{B}]$, the bounded linear operators on $\mathcal{B}$, are continuously embedded in $\mathcal{L}[\mathcal{H}]$. This result allowed us to define the Schatten classes for $\mathcal{L}[\mathcal{B}]$ as the restriction of a subset of the ones in $\mathcal{L}[\mathcal{H}]$. In this paper, we have applied these results to the development of a regularized trace formula for a second order differential operator on a separable Banach space, with a bounded operator valued coefficient given on a finite interval, extending the work in [3].

Acknowledgement

The work of the second author was supported by the National Science Foundation under Grant No. DMS-1440140, National Security Agency under Grant No. H98230-20-1-0015, and the Sloan Grant under Grant No. G-2020-12602 while the author participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the summer of 2020.

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Erdal Gül received his B.S., M.S. and Ph.D. degrees all in mathematics from Yıldız Technical University, Istanbul, Turkey, in 1990, 1992 and 1997, respectively. He is now an associate professor in the Department of Mathematics, Faculty of Arts and Sciences in Yıldız Technical University. His current research interest include functional analysis, complex analysis, spectral theory of differential operators, point-set topology, infinite dimensional analysis and applications, Abel convergence, Tauberian theorems and q-calculus.

Tepper L. Gill received his B.S., M.S. and Ph.D. degrees all in mathematics and physics (1966), in applied mathematics (1969), in mathematics (1974), respectively, from Wayne State University, USA. He is a professor in the Department of EECS and Mathematics, Howard University, Washington DC, USA. His current research interest include classical and quantum electrodynamics, stochastic analysis, infinite dimensional analysis and applications, path integrals, Feymann operator calculus and applications.