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EXISTENCE, UNIQUENESS AND STABILITY RESULTS FOR FRACTIONAL NONLINEAR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish some new conditions for the existence and uniqueness of solutions for a class of nonlinear Caputo fractional Volterra-Fredholm integro-differential equations with integral boundary conditions. The desired results are proved by using Banach and Krasnoselskii's fixed point theorems. In addition, the Ulam-Hyers stability and Ulam-Hyers-Rassias stability for solutions of the given problem are also discussed. An example is presented to guarantee the validity of our results.

Keywords: Fractional Volterra-Fredholm integro-differential equation, Caputo sense, Generalized Ulam stability, Fixed point method.

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1. Introduction

Recently, it has been proven that the differential models involving nonlocal derivatives of fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in many fields, for instance, physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth [13, 14, 26]. For instance, fractional differential and fractional integro-differential equations are an excellent tool to describe hereditary properties of viscoelastic materials and, in general, to simulate the dynamics of many processes on anomalous media [4, 10, 11, 12, 15, 23].

Theory of fractional differential equations has been extensively studied by several authors as Baleanu [1], Balachandran and Trujillo [3], Kilbas et al. [13], Lakshmikantham and Rao [14], and also see [7, 8, 9].

The stability theory for functional equations started with a problem related to the stability of group homomorphisms that was considered by Ulam in 1940 [24]. Afterwards, Rassias [16] introduced new ideas e.g., by proposing to consider unbounded right-hand

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sides in the involved inequalities, depending on certain functions, introducing therefore the so-called Hyers-Ulam-Rassias stability. Equation stability is an important subject in the applications. Many authors investigated different types of stability of fractional integro-differential equations, for instance, see [17, 18, 19, 20, 21].

Subsequently several authors have investigated the problem for different types of nonlinear differential equations and integro-differential equations including functional differential equations in Banach spaces. Very recently N'Guer'ekata [22] discussed the existence of solutions of fractional abstract differential equations with nonlocal initial condition. The nonlocal Cauchy problem is discussed by authors in [2] using the fixed-point concepts. Tidke [23] studied the fractional mixed Volterra-Fredholm integro-differential equations with nonlocal conditions using Leray-Schauder Theorem. Naimi et al. [15], studied existence, uniqueness and generalized Ulam stability for fractional integro-differential problem with integral conditions by used the Krasnoselskii and Banach fixed point theorems.

Baleanu et al. [1], by using fixed-point methods, studied the existence and uniqueness of a solution for the nonlinear fractional boundary value problem given by

$$\begin{aligned}&{}^cD^{\nu}u(t) = A(t, u(t)),\ t \in J = [0, T],\ 0 < \nu < 1,\\ &u(0) = u(T), u(0) = \beta_1 u(\eta), u(T) = \beta_2 u(\eta),\ 0 < \eta < T,\ 0 < \beta_1 < \beta_2 < 1.\end{aligned}$$

Devi and Sreedhar [5] used the monotone iterative technique to the Caputo fractional integro-differential equation of the type

$$\begin{aligned}&{}^cD^{\nu}u(t) = A(t, u(t), I^{\nu}u(t)),\ t \in J = [0, T],\ 0 < \nu < 1,\\ &u(0) = u_0.\end{aligned}$$

Wang and Zhou [25] studied the Ulam stability and data dependence for a Caputo fractional differential given by

$$\begin{aligned}&{}^cD^{\nu}u(t) = A(t, u(t)),\ t \in J = [a, +\infty),\ 0 < \nu < 1,\\ &u(a) = \xi.\end{aligned}$$

Dong et al. [6] established the existence and uniqueness of solutions via Banach and Schaude fixed point techniques for the problem given by

$$\begin{aligned}&{}^cD^{\nu}_{0^+}u(t) = A(t, u(t)) + \int_0^t B(t, s, u(s))ds,\ t \in J = [0, T],\ 0 < \nu \leq 1,\\ &u(0) = \xi.\end{aligned}$$

Motivated by the above works, we will study a more general problem of Caputo fractional integro-differential equations wich called Caputo fractional Volterra-Fredholm integro-differential equation with integral boundary condition in Banach Space:

$${}^cD^{\nu+\beta}_{0^+}u(t) = A(t, u(t)) + \int_0^t B(t, s, u(s))ds + \int_0^1 C(t, s, u(s))ds + I^{\nu}_{0^+}F(t, u(t)), \quad (1)$$

$$u(0) = b\int_0^{\eta} u(s)ds,\ b \in \mathbb{R},\ 0 < \eta < 1, \quad (2)$$

where ${}^cD^{\nu+\beta}_{0^+}$ is the Caputo fractional derivative of order $\nu + \beta$, $0 < \nu + \beta \leq 1$, $t \in J := [0, 1]$, $A, F : J \times X \longrightarrow X$, $u : \Omega \longrightarrow \mathbb{R}$, and $\xi : J \longrightarrow L_2(\Omega)$ is a random variable with $E(u^2) < \infty$. $B, C : J \times J \times X \longrightarrow X$ are continuous functions satisfying some conditions which will be stated later. $I^{\nu}_{0^+}$ denotes the left sided Riemann-Liouville fractional integral of order ν.

The paper is organized as follows: Sect. 2, we present some notations, definitions and results which are used throughout this paper. In Sect. 3, we use the fixed point techniques to prove the existence and uniqueness results for the problem (1)-(2). In Sect. 4, we establish the Hyers-Ulam stability of the problem (1)-(2) be also discussed. In Sect. 5, an example is presented to guarantee the validity of our results. Concluding remarks close the paper in Sect. 6.

2. Preliminaries

Here, we present some notations, definitions and results which are used throughout this paper.

Let X, we denote the Banach space equipped with the norm $\|.\|$ and $C(J,X)$, $C^n(J,X)$ denotes respectively the Banach spaces of all continuous bounded functions and n times continuously differentiable functions on J. In addition, we define the norm [14]

$$\|u\|_C = \max_{t\in J} |u(t)|,$$

for any continuous function $u : J \longrightarrow X$.

Definition 2.1. [26] *The left sided Riemann-Liouville fractional integral of order $\nu > 0$ of a function $u : J \longrightarrow X$ is defined as*

$$J^{\nu}_{0^+}u(t) \;=\; \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}u(s)ds, t\in J,$$

where Γ denotes the Gamma function.

Definition 2.2. [13] *The left sided fractional derivative of $u \in C^n(J,X)$ in the Caputo sense is defined by*

$$\begin{aligned} {}^cD^{\nu}_{0^+}u(t) &= J^{m-\nu}_{0^+}D^m u(t) \\ &= \begin{cases} \frac{1}{\Gamma(m-\nu)}\int_0^t (t-s)^{m-\nu-1}\frac{\partial^m u(s)}{\partial s^m}ds, & m-1<\nu<m, \\ \frac{\partial^m u(t)}{\partial t^m}, & \nu = m, \quad m\in N. \end{cases} \end{aligned}$$

Hence, we have

(1) $J^{\nu}_{0^+}J^{v}u = J^{\nu+v}_{0^+}u, \quad \nu, v > 0.$
(2) $J^{\nu}_{0^+}u^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)}u^{\beta+\nu},$
(3) $D^{\nu}_{0^+}u^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\nu+1)}u^{\beta-\nu}, \quad \nu > 0, \quad \beta > -1.$
(4) $J^{\nu}_{0^+}D^{\nu}_{0^+}u(t) = u(t) - u(a), \ 0 < \nu < 1.$
(5) $J^{\nu}_{0^+}D^{\nu}_{0^+}u(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(0^+)\frac{(t-a)^k}{k!}, \quad t > 0.$

Definition 2.3. [26] *The Riemann Liouville fractional derivative of order $\nu > 0$ is normally defined as*

$$D^{\nu}_{0^+}u(t) = D^m_{0^+}J^{m-\nu}_{0^+}u(t), \qquad m-1<\nu\le m, \quad m\in\mathbb{N}.$$

Theorem 2.1. [13] *(Banach fixed point theorem). Let $(S, \|.\|)$ be a complete normed space, and let the mapping $F : S \longrightarrow S$ be a contraction mapping. Then F has exactly one fixed point.*

Theorem 2.2. [13] *(Krasnoselskii fixed point theorem). Let w be bounded, closed and convex subset in a Banach space X. If $T_1, T_2 : w \longrightarrow w$ are two applications satisfying the following conditions:*
1) $T_1x + T_2y \in w$ for every $x, y \in w$;
2) T_2 is a contraction;

3) T_1 is compact and continuous.
Then there exists $a \in w$ such that $T_1 a + T_2 a = a$.

Lemma 2.1. [6] *Let $u(t), A(t), q(t) \in C(J, \mathbb{R}_+)$ and let $n(t) \in C(J, \mathbb{R}_+)$ be nondecreasing for which the inequality*

$$u(t) \leq n(t) + \int_0^t A(s)u(s)ds + \int_0^t A(s) \int_0^s q(r)u(r)drds,$$

holds for any $t \in J$. Then

$$u(t) \leq n(t)\Big[1 + \int_0^t A(s)\Big(\int_0^s (A(r) + q(r))dr\Big)ds\Big].$$

3. Existence and uniqueness results

In this section, we shall give an existence and uniqueness results of Eq.(1), with the condition (2). Before starting and proving the main results, we introduce the following hypotheses:
(A1) For any $t \in J$ and $u, v \in X$, $(t, s) \in G = \{(t, s) : 0 \leq s \leq t \leq 1\}$, there exist positive constants L_1, L_2, L_3 and L_4 such that

$$\begin{aligned}
&\|A(t, u) - A(t, v)\| \leq L_1\|u - v\|,\\
&\|F(t, u) - F(t, v)\| \leq L_2\|u - v\|,\\
&\|B(t, s, u) - B(t, s, v)\| \leq L_3\|u - v\|,\\
&\|C(t, s, u) - C(t, s, v)\| \leq L_4\|u - v\|,
\end{aligned}$$

with $L = \max\{L_1, L_2, L_3, L_4\}$.
(A2) Assume that A, B, C and F satisfy

$$\begin{aligned}
&\|A(t, u)\| \leq \mu_1(t)\|u\|,\\
&\|F(t, u)\| \leq \mu_2(t)\|u\|,\\
&\|B(t, s, u)\| \leq \mu_3(t)\|u\|,\\
&\|C(t, s, u)\| \leq \mu_4(t)\|u\|,
\end{aligned}$$

where $\mu_i \in L^\infty(J, \mathbb{R}_+),\ i = 1, 2, 3, 4,\ t \in J,\ u \in X$ and $(t, s) \in G$.
(A3) Let $t \in J$, and the functions A, F, B and C are continuous on J. Then there exist positive constants N_1, N_2, N_3 and N_4 such that

$$\begin{aligned}
&\|A(t, u(t))\| \leq L_1\|u\| + N_1,\\
&\|F(t, u(t))\| \leq L_2\|u\| + N_2,\\
&\|B(t, s, u)\|_2 \leq L_3\|u\| + N_3,\\
&\|C(t, s, u)\|_2 \leq L_4\|u\| + N_4.
\end{aligned}$$

Lemma 3.1. *Let $0 < \nu+\beta < 1,\ b \neq \frac{1}{\eta}$ and $u \in C(J,X)$ is called a solution of the problem (1)-(2) $\iff$ u satisfies*

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[A(s,u(s)) + \int_0^s B(s,r,u(r))dr + \int_0^1 C(s,r,u(r))dr \\ &\quad + \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)} F(r,u(r))dr\Big]ds \\ &\quad + \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[A(r,u(r)) + \int_0^r B(r,\sigma,u(\sigma))d\sigma + \int_0^1 C(r,\sigma,u(\sigma))d\sigma \\ &\quad + \int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}F(\sigma,u(\sigma))d\sigma\Big]dr. \end{aligned} \tag{3}$$

Proof. Let $u \in C(J,X)$ be a solution of the problem (1)-(2). Firstly, we show that u is solution of integral equation (3). By 2.2, we obtain

$$I_{0^+}^{\nu+\beta}\ {}^cD_{0^+}^{\nu+\beta}u(t) = u(t) - u(0). \tag{4}$$

In addition, from equation in (1) and Definition 2.1, and use the assumption (4) of 2.2, we have

$$\begin{aligned} &I_{0^+}^{\nu+\beta}\ {}^cD_{0^+}^{\nu+\beta}u(t) \\ &= I_{0^+}^{\nu+\beta}\Big(A(t,u(t)) + \int_0^t B(t,s,u(s))ds + \int_0^1 C(t,s,u(s))ds + I_{0^+}^{\nu}F(t,u(t))\Big)ds \\ &= \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[A(s,u(s)) + \int_0^s B(s,r,u(r))dr + \int_0^1 C(s,r,u(r))dr \\ &\quad + \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}F(r,u(r))dr\Big]ds. \end{aligned} \tag{5}$$

By substituting (5) in (4) with nonlocal condition in problem (3), we get the following integral equation

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[A(s,u(s)) + \int_0^s B(s,r,u(r))dr + \int_0^1 C(s,r,u(r))dr \\ &\quad + \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}F(r,u(r))dr\Big]ds + u(0). \end{aligned} \tag{6}$$

From integral boundary condition of our problem with using Fubini's theorem and after some computations, we get:

$$\begin{aligned}
u(0) &= b\int_0^\eta u(s)ds \\
&= b\int_0^\eta \Big[\int_0^s \frac{(s-r)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[A(r,u(r)) + \int_0^r B(r,\sigma,u(\sigma))d\sigma \\
&\quad + \int_0^1 C(r,\sigma,u(\sigma))d\sigma + \int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}F(\sigma,u(\sigma))d\sigma\Big]dr\Big]ds + b\eta u(0) \\
&= b\int_0^\eta\int_0^s \frac{(s-r)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}A(r,u(r))drds \\
&\quad +b\int_0^\eta\int_0^s \frac{(s-r)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}[\int_0^r B(r,\sigma,u(\sigma))d\sigma + \int_0^1 C(r,\sigma,u(\sigma))d\sigma]drds \\
&\quad +b\int_0^\eta\int_0^s \frac{(s-r)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}F(\sigma,u(\sigma))d\sigma drds + b\eta u(0) \\
&= b\int_0^\eta\int_r^\eta \frac{(s-r)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}dsA(r,u(r))dr \\
&\quad +b\int_0^\eta\int_r^\eta \frac{(s-r)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}ds[\int_0^r B(r,\sigma,u(\sigma))d\sigma dr + \int_0^1 C(r,\sigma,u(\sigma))d\sigma]dr \\
&\quad +b\int_0^\eta\int_r^\eta \frac{(s-r)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}ds\int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}F(\sigma,u(\sigma))d\sigma dr + b\eta u(0),
\end{aligned}$$

that is

$$\begin{aligned}
u(0) &= \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[A(r,u(r)) + \int_0^r B(r,\sigma,u(\sigma))d\sigma \\
&\quad + \int_0^1 C(r,\sigma,u(\sigma))d\sigma + \int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}F(\sigma,u(\sigma))d\sigma\Big]dr. \qquad (7)
\end{aligned}$$

Finally, by substituting (7) in (6) we find (3). Conversely, from 2.2 and by applying the operator ${}^cD_{0^+}^{\nu+\beta}$ on both sides of (3), we find

$$\begin{aligned}
{}^cD_{0^+}^{\nu+\beta}u(t) &= {}^cD_{0^+}^{\nu+\beta}I_{0^+}^{\nu+\beta}\Big(A(t,u(t)) + \int_0^t B(t,s,u(s))ds + \int_0^1 C(t,s,u(s))ds \\
&\quad +I_{0^+}^{\nu}F(t,u(t))\Big)ds +{}^c D_{0^+}^{\nu+\beta}u(0) \\
&= A(t,u(t)) + \int_0^t B(t,s,u(s))ds + \int_0^1 C(t,s,u(s))ds + I_{0^+}^{\nu}F(t,u(t)), \qquad (8)
\end{aligned}$$

this means that u satisfies the equation in the problem (1)-(2). Furthermore, by substituting t by 0 in integral equation (3), we have clearly that the integral boundary condition in (2) holds. Therefore, u is solution of problem (1)-(2), which completes the proof. □

Theorem 3.1. *Assume that the assumptions (A1) and (A2) are satisfied and if*

$$\begin{aligned} K \;:=\; & \frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}+\|\mu_4\|_{L^\infty}}{\Gamma(\nu+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}\beta(\nu+1,\nu+\beta)}{\Gamma(\nu+1)\Gamma(\nu+\beta)} \\ & +\frac{|b|\|\mu_1\|_{L^\infty}\eta^{\nu+\beta+1}+|b|\|\mu_3\|_{L^\infty}\eta^{\nu+\beta+1}+|b|\|\mu_4\|_{L^\infty}\eta^{\nu+\beta+1}}{|1-b\eta|\Gamma(\nu+\beta+2)} \\ & +\frac{|b|\|\mu_2\|_{L^\infty}\eta^{2\nu+\beta+1}\beta(\nu+1,\nu+\beta+1)}{|1-b\eta|\Gamma(\nu+\beta1)\Gamma(\nu+\beta+1)}\leq 1, \end{aligned} \tag{9}$$

and

$$LK_1 := L\frac{|b|}{|1-b\eta|}\Big[\frac{3\eta^{\nu+\beta+1}}{\Gamma(\nu+\beta+2)}+\frac{\eta^{2\nu+\beta+1}\beta(\nu+1,\nu+\beta+1)}{\Gamma(\nu+1)\Gamma(\nu+\beta+1)}\Big]\leq 1. \tag{10}$$

Then the problem (1)-(2) has at least one solution $u(t)$ on J.

Proof. Consider the operator $\Upsilon : B_\psi \longrightarrow B_\psi$ by

$$\begin{aligned} \Upsilon u(t) \;=\; & \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[A(s,u(s))+\int_0^s B(s,r,u(r))dr+\int_0^1 C(s,r,u(r))dr \\ & +\int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}F(r,u(r))dr\Big]ds+\frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)} \\ & \times\Big[A(r,u(r))+\int_0^r B(r,\sigma,u(\sigma))d\sigma+\int_0^1 C(r,\sigma,u(\sigma))d\sigma \\ & +\int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}F(\sigma,u(\sigma))d\sigma\Big]dr. \end{aligned} \tag{11}$$

For any function $u \in C(J,X)$ we define the norm

$$\|u\|_1=\max\{e^{-t}\|u(t)\| : t\in[0,1]\},$$

and consider the closed ball

$$B_\psi=\{u\in C(J,X) \text{ such that } \|u\|_1\leq\psi\}.$$

For any $t\in[0,r]$. It is easy to see that the operator Υ is well-defined.

Let us define the operators Υ_1, Υ_2 on B_ψ as follows

$$\begin{aligned} \Upsilon_1 u(t) \;=\; & \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[A(s,u(s))+\int_0^s B(s,r,u(r))dr \\ & +\int_0^1 C(s,r,u(r))dr+\int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}F(r,u(r))dr\Big]ds, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \Upsilon_2 u(t) \;=\; & \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[A(r,u(r))+\int_0^r B(r,\sigma,u(\sigma))d\sigma \\ & +\int_0^1 C(r,\sigma,u(\sigma))d\sigma+\int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}F(\sigma,u(\sigma))d\sigma\Big]dr. \end{aligned} \tag{13}$$

For $u, v \in B_\psi, t \in J$ and by the assumption (A2), we find

$$
\begin{aligned}
&\|\Upsilon_1 u(t) + \Upsilon_2 v(t)\| \\
\leq\ & \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[\|A(s,u(s))\| + \int_0^s \|B(s,r,u(r))\|dr \\
&+ \int_0^1 \|C(s,r,u(r))\|dr + \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}\|F(r,u(r))\|dr\Big]ds \\
&+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[\|A(r,v(r))\| + \int_0^r \|B(r,\sigma,v(\sigma))\|d\sigma \\
&+ \int_0^1 \|C(r,\sigma,v(\sigma))\|d\sigma + \int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}\|F(\sigma,v(\sigma))\|d\sigma\Big]dr \\
\leq\ & \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[\mu_1(s)\|u(s)\| + \int_0^s \mu_3(s)\|u(r)\|dr + \int_0^1 \mu_4(s)\|u(r)\|dr \\
&+ \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}\mu_2(s)\|u(r)\|dr\Big]ds \\
&+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[\mu_1(r)\|v(r)\| + \int_0^r \mu_3(r)\|v(\sigma)\|d\sigma \\
&+ \int_0^1 \mu_4(r)\|v(\sigma)\|d\sigma + \int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}\mu_2(\sigma)\|v(\sigma)\|d\sigma\Big]dr \\
\leq\ & \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[\|\mu_1\|_{L^\infty}\|u\|_1 e^s + \|\mu_3\|_{L^\infty}\|u\|_1(e^s-1) \\
&+\|\mu_4\|_{L^\infty}\|u\|_1(e^s-1) + \|\mu_2\|_{L^\infty}\|u\|_1\int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}e^r dr\Big]ds \\
&+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[\|\mu_1\|_{L^\infty}\|v\|_1 e^r + \|\mu_3\|_{L^\infty}\|v\|_1(e^r-1) \\
&+\|\mu_4\|_{L^\infty}\|v\|_1(e^r-1) + \|\mu_2\|_{L^\infty}\|v\|_1\int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}e^\sigma d\sigma\Big]dr.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\|\Upsilon_1 u+\Upsilon_2 v\|_1\\
\leq\;& \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[\|\mu_1\|_{L^\infty}\|u\|_1\frac{e^s}{e^t}+\|\mu_3\|_{L^\infty}\|u\|_1\frac{(e^s-1)}{e^t}\\
&+\|\mu_4\|_{L^\infty}\|u\|_1\frac{(e^s-1)}{e^t}+\|\mu_2\|_{L^\infty}\|u\|_1\int_0^s\frac{(s-r)^{\nu-1}}{\Gamma(\nu)}\frac{e^r}{e^t}dr\Big]ds\\
&+\frac{|b|}{|1-b\eta|}\int_0^\eta\frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[\|\mu_1\|_{L^\infty}\|v\|_1\frac{e^r}{e^t}+\|\mu_3\|_{L^\infty}\|v\|_1\frac{(e^r-1)}{e^t}\\
&+\|\mu_4\|_{L^\infty}\|v\|_1\frac{(e^r-1)}{e^t}+\|\mu_2\|_{L^\infty}\|v\|_1\int_0^r\frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}\frac{e^\sigma}{e^t}d\sigma\Big]dr\\
\leq\;& \psi\Big[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}+\|\mu_4\|_{L^\infty}}{\Gamma(\nu+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}}{\Gamma(\nu+1)\Gamma(\nu+\beta)}\int_0^1(1-s)^{\nu+\beta+1}s^\nu ds\\
&\frac{|b|\|\mu_1\|_{L^\infty}\eta^{\nu+\beta+1}+|b|\|\mu_3\|_{L^\infty}\eta^{\nu+\beta+1}+|b|\|\mu_4\|_{L^\infty}\eta^{\nu+\beta+1}}{|1-b\eta|\Gamma(\nu+\beta+1)}\\
&+\frac{|b|\|\mu_2\|_{L^\infty}}{|1-b\eta|\Gamma(\nu+1)\Gamma(\nu+\beta+1)}\int_0^\eta(\eta-r)^{\nu+\beta}r^\nu dr\Big]\\
=\;& \psi\Big[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}+\|\mu_4\|_{L^\infty}}{\Gamma(\nu+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}\beta(\nu+1,\nu+\beta)}{\Gamma(\nu+1)\Gamma(\nu+\beta)}\\
&+\frac{|b|\|\mu_1\|_{L^\infty}\eta^{\nu+\beta+1}+|b|\|\mu_3\|_{L^\infty}\eta^{\nu+\beta+1}+|b|\|\mu_4\|_{L^\infty}\eta^{\nu+\beta+1}}{|1-b\eta|\Gamma(\nu+\beta+2)}\\
&+\frac{|b|\|\mu_2\|_{L^\infty}\eta^{2\nu+\beta+1}\beta(\nu+1,\nu+\beta+1)}{|1-b\eta|\Gamma(\nu+1)\Gamma(\nu+\beta+1)}\Big]\\
=\;& \psi K\\
\leq\;& \psi.
\end{aligned}$$

This implies that $(\Upsilon_1 u+\Upsilon_2 v)\in B_\psi$. Here we used the computations

$$\begin{aligned}
&\int_0^1(1-s)^{\nu+\beta+1}s^\nu ds=\beta(\nu+1,\nu+\beta)\\
&\int_0^\eta(\eta-r)^{\nu+\beta}r^\nu dr=\eta^{2\nu+\beta+1}\beta(\nu+1,\nu+\beta+1),
\end{aligned}$$

and the estimations: $\frac{e^s}{e^t}\leq 1, \frac{e^r}{e^t}\leq 1, \frac{e^\sigma}{e^t}\leq 1$. Now, we establish that Υ_2 is a contraction mapping. For $u,v\in X$ and $t\in J$, we have

$$\begin{aligned}
&\|\Upsilon_2 u(t)-\Upsilon_2 u(t)\|\\
\leq\;& \frac{|b|}{|1-b\eta|}\int_0^\eta\frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[\|A(r,u(r))-A(r,v(r))\|\\
&+\int_0^r\|B(r,\sigma,u(\sigma))-B(r,\sigma,v(\sigma))\|d\sigma+\int_0^1\|C(r,\sigma,u(\sigma))-C(r,\sigma,v(\sigma))\|d\sigma\\
&+\int_0^r\frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}\|F(\sigma,u(\sigma))-F(\sigma,v(\sigma))\|d\sigma\Big]dr
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[L_1\|u-v\|_1 e^r + \int_0^r L_3\|u-v\|_1 e^\sigma d\sigma \\
&\quad + \int_0^1 L_4\|u-v\|_1 e^\sigma d\sigma + \int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)} L_2\|u-v\|_1 e^\sigma d\sigma\Big] dr \\
&\leq \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[L\|u-v\|_1 e^r + L\|u-v\|_1 (e^r-1) \\
&\quad + L\|u-v\|_1(e^r-1) + \int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)} L\|u-v\|_1 e^\sigma d\sigma\Big] dr.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|\Upsilon_2 u(t) - \Upsilon_2 u(t)\| \\
&\leq \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[L\|u-v\|_1 \frac{e^r}{e^t} + L\|u-v\|_1 \frac{(e^r-1)}{e^t} \\
&\quad + L\|u-v\|_1\frac{(e^r-1)}{e^t} + \int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)} L\|u-v\|_1 \frac{e^\sigma}{e^t} d\sigma\Big] dr \\
&\leq \frac{|b|L}{|1-b\eta|}\Big[\frac{3\eta^{\nu+\beta+1}}{\Gamma(\nu+\beta+2)} + \frac{\eta^{2\nu+\beta+1}\beta(\nu+1,\nu+\beta+1)}{\Gamma(\nu+1)\Gamma(\nu+\beta+1)}\Big]\|u-v\|_1.
\end{aligned}
$$

Then since $LK_1 \leq 1$, Υ_2 is a contraction mapping. The continuity of the functions A, B, C and F implies that the operator Υ_1 is continuous. Also, $\Upsilon_1 B_\psi \subset B_\psi$, for each $u \in B_\psi$, i.e., Υ_1 is uniformly bounded on B_ψ as

$$
\begin{aligned}
\|\Upsilon_1 u(t)\| &\leq \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[\|A(s,u(s))\| + \int_0^s \|B(s,r,u(r))\| dr \\
&\quad + \int_0^1 \|C(s,r,u(r))\| dr + \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}\|F(r,u(r))\| dr\Big] ds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\Upsilon_1 u\|_1 &\leq \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[\|\mu_1\|_{L^\infty}\|u\|_1\frac{e^s}{e^t} + \|\mu_3\|_{L^\infty}\|u\|_1 \frac{(e^s-1)}{e^t} \\
&\quad + \|\mu_4\|_{L^\infty}\|u\|_1\frac{(e^s-1)}{e^t} + \|\mu_2\|_{L^\infty}\|u\|_1 \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}\frac{e^r}{e^t} dr\Big] ds \\
&\leq \psi\Big[\frac{\|\mu_1\|_{L^\infty} + \|\mu_3\|_{L^\infty} + \|\mu_4\|_{L^\infty}}{\Gamma(\nu+\beta+1)} + \frac{\|\mu_2\|_{L^\infty}\beta(\nu+1,\nu+\beta)}{\Gamma(\nu+1)\Gamma(\nu+\beta)} \\
&\leq \psi K \\
&\leq \psi.
\end{aligned} \tag{14}
$$

Finally, we will show that $(\Upsilon_1 B_\psi)$ is equi-continuous. For this end, we define

$$
\bar{A} = \sup_{(s,u)\in J\times B_\psi} \|A(s,u)\|, \quad \bar{F} = \sup_{(s,u)\in J\times B_\psi}\|F(s,u)\|,
$$

$$
\bar{B} = \sup_{(s,r,u)\in G\times B_\psi}\int_0^s \|B(s,r,u)\| dr, \ \bar{C} = \sup_{(s,r,u)\in G\times B_\psi}\int_0^1 \|C(s,r,u)\| dr.
$$

Let for any $u \in B_\psi$ and for each $t_1, t_2 \in J$ with $t_1 \leq t_2$, we have:

$$\begin{aligned}
&\|(\Upsilon_1 u)(t_1) - (\Upsilon_1 u)(t_2)\| \\
\leq\;& \int_{t_1}^{t_2} \frac{(t_2 - s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)} \Big[\|A(s,u(s))\| + \int_0^s \|B(s,r,u(r))\| dr \\
&+ \int_0^1 \|C(s,r,u(r))\| dr + \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)} \|F(r,u(r))\| dr\Big] ds \\
&+ \int_0^{t_1} \frac{[(t_1 - s)^{\nu+\beta-1} - (t_2 - s)^{\nu+\beta-1}]}{\Gamma(\nu+\beta)} \Big[\|A(s,u(s))\| + \int_0^s \|B(s,r,u(r))\| dr \\
&+ \int_0^1 \|C(s,r,u(r))\| dr + \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)} \|F(r,u(r))\| dr\Big] ds \\
\leq\;& \int_{t_1}^{t_2} \frac{(t_2 - s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)} \Big[\bar{A} + \bar{B} + \bar{C} + \frac{\bar{F}}{\Gamma(\nu)} \int_0^s (s-r)^{\nu-1} dr\Big] ds \\
&+ \int_0^{t_1} \frac{[(t_1 - s)^{\nu+\beta-1} - (t_2 - s)^{\nu+\beta-1}]}{\Gamma(\nu+\beta)} \Big[\bar{A} + \bar{B} + \bar{C} + \frac{\bar{F}}{\Gamma(\nu)} \int_0^s (s-r)^{\nu-1} dr\Big] ds \\
\leq\;& \int_{t_1}^{t_2} \frac{(t_2 - s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)} \Big[\bar{A} + \bar{B} + \bar{C} + \frac{\bar{F}}{\Gamma(\nu+1)}\Big] ds \\
&+ \int_0^{t_1} \frac{[(t_1 - s)^{\nu+\beta-1} - (t_2 - s)^{\nu+\beta-1}]}{\Gamma(\nu+\beta)} \Big[\bar{A} + \bar{B} + \bar{C} + \frac{\bar{F}}{\Gamma(\nu+1)}\Big] ds \\
\leq\;& \frac{1}{\Gamma(\nu+\beta+1)} \Big[\bar{A} + \bar{B} + \bar{C} + \frac{\bar{F}}{\Gamma(\nu+1)}\Big] \Big[2(t_2 - t_1)^{\nu+\beta} + t_1^{\nu+\beta} - t_2^{\nu+\beta}\Big] \\
&\longrightarrow 0 \text{ as } t_1 \longrightarrow t_2,
\end{aligned}$$

this means that $|\Upsilon_1 u(t_2) - \Upsilon_1 u(t_1)| \longrightarrow 0$, which implies that $(\overline{\Upsilon_1 B_\psi})$ is equi-continuous, then Υ_1 is relatively compact on B_ψ. Hence by Arzela-Ascoli theorem, Υ_1 is compact on B_ψ. Now, all hypothesis of Theorem 2.2 hold, therefore the operator Υ has a fixed point on B_ψ. So the problem (1)-(2) has at least one solution on J. This proves the theorem. □

Theorem 3.2. *Assume that the assumptions (A1) and (A3) are satisfied and if $LK < 1$. Then the problem (1)-(2) has a unique solution on J.*

Proof. Let the operator Υ as in Theorem 3.1. Define

$$R_\psi = \{u \in C(J,X) : \|u\| \leq \psi\}.$$

We fix $\psi \geq \frac{NK}{1-LK}$, where $N = \max\{N_1, N_2, N_3, N_4\}$, such that $N_1 = \sup_{t\in J} \|A(t,0)\|$, $N_2 = \sup_{t\in J} \|F(t,0)\|$, $N_3 = \sup_{(t,s)\in G} \|B(t,s,0)\|$, $N_4 = \sup_{(t,s)\in G} \|C(t,s,0)\|$.

Firstly, we will prove that $\Upsilon R_\psi \subset R_\psi$. For any $u \in R_\psi$, we have

$$\begin{aligned}
\|(\Upsilon u)(t)\| \;\leq\;& \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)} \Big[\|A(s,u(s))\| + \int_0^s \|B(s,r,u(r))\| dr \\
&+ \int_0^1 \|C(s,r,u(r))\| dr + \int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)} \|F(r,u(r))\| dr\Big] ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[\|A(r,u(r))\|+\int_0^r \|B(r,\sigma,u(\sigma))\|d\sigma \\
&+\int_0^1 \|C(r,\sigma,u(\sigma))\|d\sigma+\int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}\|F(\sigma,u(\sigma))\|d\sigma\Big]dr \\
\leq\ & (L\psi+N)K \\
\leq\ & \psi.
\end{aligned}
$$

Hence, $\Upsilon R_\psi \subset R_\psi$.

Secondly, We shall show that $\Upsilon : R_\psi \longrightarrow R_\psi$ is a contraction. From the assumption (A1) we have for any $u, v \in R_\psi$ and for each $t \in J$

$$
\begin{aligned}
\|(\Upsilon u)(t)-(\Upsilon v)(t)\| \leq\ & \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[\|A(s,u(s))-A(s,v(s))\| \\
&+\int_0^s \|B(s,r,u(r))-B(s,r,v(r))\|dr \\
&+\int_0^1 \|C(s,r,u(r))-C(s,r,v(r))\|dr \\
&+\int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}\|F(r,u(r))-F(r,v(r))\|dr\Big]ds \\
&+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[\|A(r,u(r))-A(r,v(r))\| \\
&+\int_0^r \|B(r,\sigma,u(\sigma))-B(r,\sigma,v(\sigma))\|d\sigma \\
&+\int_0^1 \|C(r,\sigma,u(\sigma))-C(r,\sigma,v(\sigma))\|d\sigma \\
&+\int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}\|F(\sigma,u(\sigma))-F(\sigma,v(\sigma))\|d\sigma\Big]dr \\
\leq\ & LK\|u-v\|.
\end{aligned}
$$

Since $LK < 1$, it follows that Υ is a contraction, from Theorem 2.1, then there exists $u \in C(J,X)$ such that $\Upsilon u = u$, which is the unique solution of the problem (1)-(2) in $C(J,X)$. This proof is completed. □

4. Ulam-Hyers stability

In this section, we establish the Hyers-Ulam stability of the problem (1)-(2).

We say that the problem (1)- (2) has the Hyers-Ulam stability, if for all $\epsilon > 0$ and all function $v \in C(J,X)$ satisfying

$$
\begin{aligned}
v(t) =\ & \int_0^t \frac{(t-s)^{\nu+\beta-1}}{\Gamma(\nu+\beta)}\Big[A(s,v(s))+\int_0^s B(s,r,v(r))dr \\
&+\int_0^1 C(s,r,v(r))dr+\int_0^s \frac{(s-r)^{\nu-1}}{\Gamma(\nu)}F(r,v(r))dr\Big]ds
\end{aligned}
$$

$$+\frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-r)^{\nu+\beta}}{\Gamma(\nu+\beta+1)}\Big[A(r,v(r))+\int_0^r B(r,\sigma,v(\sigma))d\sigma$$
$$+\int_0^1 C(r,\sigma,v(\sigma))d\sigma+\int_0^r \frac{(r-\sigma)^{\nu-1}}{\Gamma(\nu)}F(\sigma,v(\sigma))d\sigma\Big]dr. \tag{15}$$

We define the nonlinear continuous operator $\Delta : C(J,X) \longrightarrow C(J,X)$, as follows

$$\Delta v(t) =^c D_{0^+}^{\nu+\beta}v(t) - I_{0^+}^{\nu}F(t,v(t)) - A(t,v(t)) - \int_0^t B(t,s,v(s))ds - \int_0^1 C(t,s,v(s))ds.$$

Definition 4.1. [19] *For each $\epsilon > 0$ and for each solution v of the problem (1)- (2), such that*

$$\|\Delta v\| \leq \epsilon, \tag{16}$$

the problem (1), is said to be Ulam-Hyers stable if we can find a positive real number α and a solution $u \in C(J,X)$ of the problem (1), satisfying the inequality

$$\|u-v\| \leq \alpha\epsilon^*, \tag{17}$$

where ϵ^* is a positive real number depending on ϵ.

Definition 4.2. [20] *Let $m \in C(\mathbb{R}^+,\mathbb{R}^+)$ such that for each solution v of the problem (1), we can find a solution $u \in C(J,X)$ of the problem (1) such that*

$$\|u(t)-v(t)\| \leq m\epsilon^*, \ t \in J. \tag{18}$$

Then the problem (1), is said to be generalized Ulam-Hyers stable.

Definition 4.3. [19] *For each $\epsilon > 0$ and for each solution v of the problem (1) is called Ulam-Hyers-Rassias stable with respect to $\Theta \in C(J,\mathbb{R}^+)$ if*

$$\|\Delta v(t)\| \leq \epsilon\Theta(t), \ t \in J, \tag{19}$$

and there exist a real number $\alpha > 0$ and a solution $v \in C(J,X)$ of the problem (1) such that

$$\|u(t)-v(t)\| \leq \alpha\epsilon_*\Theta(t), \ t \in J, \tag{20}$$

where ϵ_* is a positive real number depending on ϵ.

Theorem 4.1. *Assume that (A1) holds, with $LK < 1$. The problem (1) is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $u \in C(J,X)$ be a solution of (1), satisfying (3) in the sense of Theorem 3.2 Let v be any solution satisfying (16). Lemma 3.1 implies the equivalence between the operators Δ and $\Upsilon - \mathrm{Id}$ (where Id is the identity operator) for every solution $v \in C(J,X)$ of (1) satisfying $LK < 1$. Therefore, we deduce by the fixed-point property of the operator Υ that:

$$\begin{aligned}\|v(t)-u(t)\| &= \|v(t)-\Upsilon v(t)+\Upsilon v(t)-u(t)\|\\ &= \|v(t)-\Upsilon v(t)+\Upsilon v(t)-\Upsilon u(t)\|\\ &\leq \|\Upsilon v(t)-\Upsilon u(t)\|+\|\Upsilon v(t)-v(t)\|\\ &\leq LK\|u-v\|+\epsilon,\end{aligned}$$

because $LK < 1$ and $\epsilon > 0$, we find

$$\|u-v\| \leq \frac{\epsilon}{1-LK}.$$

Fixing $\epsilon_* = \frac{\epsilon}{1-LK}$, and $\alpha = 1$, we obtain the Ulam-Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $m\epsilon = \frac{\epsilon}{1-LK}$. □

Theorem 4.2. *Assume that (A1) holds with $L < \frac{1}{K}$, and there exists a function $\Theta \in C(J, \mathbb{R}^+)$ satisfying the condition* (19)*. Then the problem* (1) *is Ulam-Hyers-Rassias stable with respect to* Θ*.*

Proof. We have from the proof of Theorem (4.1),

$$\|u(t) - v(t)\| \leq \epsilon_* \Theta(t),\ t \in J.$$

where $\epsilon_* = \frac{\epsilon}{1-LK}$. This completes the proof. □

5. Example

Example 5.1. Consider the fractional Volterra-Fredholm integro-differential equation (1) with condition

$$u(0) = 3 \int_0^{0.2} u(s)ds,$$

where $\nu = \beta = 0.2,\ b = 3,\ \eta = 0.2$. By the above, we find that $K = 0.2248,\ K_1 = 1.4489$.

To illustrate our results in Theorem 3.1 and Theorem 4.1, we take for $u, v \in X = \mathbb{R}^+$ and $t \in [0,1]$ the following continuous functions: $A(t, u(t)) = \frac{(2-t)u(t)}{60},\ F(t, u(t)) = \frac{(3-t^2)u(t)}{72},\ B(t, s, u(s)) = \frac{e^{-(s+t)}}{64}u(s),\ C(t, s, u(s)) = \frac{\cos(s+t)}{32}u(s)$. Then, we get

$$L = 0.0555,\ K_1 = 1.4489,\ LK_1 = 0.0805 < 1,\ K = 0.2248 < 1,\ LK = 0.0125.$$

All assumptions of Theorem 3.1 are satisfied. Hence, there exists at least one solution on $[0,1]$.

By take the same functions, we result the assumption

$$LK = 0.0125 < 1,$$

then there exists a unique solution on $[0,1]$.

In order to illustrate our stability result, we consider the same above example:

$$LK = 0.0125 < 1.$$

This implies that the problem is Ulam-Hyers stable, then it is generalized Ulam-Hyers stable. It is Ulam-Hyers-Rassias stable if there exists a continuous and positive function.

6. Concluding remarks

In this paper, we studied the existence and uniqueness of solutions for a class of nonlinear Caputo fractional Volterra-Fredholm integro-differential equations with the integral conditions. In addition, the Ulam-Hyers stability and generalized Ulam-Hyers stability for solutions of the given problem are also discussed. The desired results are proved by using via using Banach and Krasnoselskii fixed point theorems.

Remark 1. By taking $C \equiv 0$ and $\int_0^\eta u(s)ds = 1$, in the problem (1)-(2), the results of reference [6] appear as a special case of our results. Also, by taking $B \equiv 0$ in the problem (1)-(2), the results of reference [25] appear as a special case of our results.

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