TWMS J. App. and Eng. Math. V.13, N.2, 2023, pp. 517-530

F-INDEX FOR FUZZY GRAPH WITH APPLICATION

SK R. ISLAM[1*], M. PAL[1], §

Abstract. Different types of topological index of a graph have many applications and many results are available for crisp graphs. But in many practical applications it is seen that many situations cannot be modeled using crisp graphs. In these cases, to handle such a situation, those topological indices are needed to define in a fuzzy graph. In this article, F-index for fuzzy graphs is introduced and some bounds on F-index are provided for several fuzzy graphs such as path, cycle, star, complete fuzzy graph, etc. In this article, the maximal fuzzy graph is calculated with respect to the F-index for a given vertex set. At the end of this article, a group of researchers are considered and a decision making method is provided by using F-index for fuzzy graph to find the most influenced researcher in that group.

Keyword: Fuzzy graph, topological indices, Forgotten Topological Index, Forgotten Topological Index for fuzzy graph.

AMS Subject Classification: 05C40, 05C62.

1. Introduction

1.1. **Research background.** Nowadays, due to various applications of fuzzy graph theory, a huge number of researchers working on topological indices (TIs). The idea of fuzzy set (FS) was first introduced by Zadeh [29], in 1965. Motivated by this, in 1975, Rosenfeld [24] defined the fuzzy graph (FG). In the same time, Yeh et al. [28] also introduced the FG independently and define some connectivity parameter of a FG and their applications. Degree of a vertex in a FG is also discussed in [19] and also discussed strong degree, strong neighbour of a FG. In [14, 15, 16, 19, 20, 25] one can see more details on fuzzy graph theory.

In the field of mathematical chemistry, molecular topology and chemical graph theory, TIs are molecular descriptors which are calculated on the molecular graph of a chemical compound. These TIs are numerical quantity, of a graph which describe its topology. In

[1] Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, 721102, India.
e-mail: islamskrabiul61@gmail.com; ORCID: https://orcid.org//0000-0003-0486-3412.
[*] Corresponding author.
e-mail: mmpalvu@gmail.com; ORCID: https://orcid.org//0000-0002-6709-836X.
§ Manuscript received: January 11, 2021; accepted: April 12, 2021.

The first author is supported by University Grant Commission (UGC) Govt. of India under UGC-Ref. No.: 1144/ (CSIR-UGC NET JUNE 2017) dated 26/12/2018.

1947 [27], Harold Wiener first introduced wiener index (WI) which is used to calculate the boiling point of paraffins. Zagreb index (ZI) is degree-based TI and established by Gutman and Trinajstic in 1972 [8] and used to calculate π-electron energy of a conjugate system. Followed by this topological index, in 2015, Fortula and Gutman defined another degree based topological index called forgotten topological index (F-index) [7]. Some neighbourhood degree-based topological indices are discussed by Mondal et al. [17, 18]. In [5, 6, 9, 23, 26], one can see for different types of TIs. Amin and Nayeem [2] provided the F-index of subdivision graph and line graph.

1.2. **Motivation.** Topological indices has an important role in molecular chemistry, chemical graph theory, spectral graph theory, network theory, etc. Zagreb index (ZI) is one such TIs which is degree-based TI and established by Gutman and Trinajstic in 1972 [8] and this TIs are used to calculate π-electron energy of a conjugate system. Followed by this topological index, in 2015, Fortula and Gutman defined another degree based topological index called forgotten topological index (F-index) [7] and they proved that first ZI and F-index have the almost same entropy, predictive ability and acentric factor and F-index obtain correlation coefficients greater than 0.95. So this TI is very much useful in molecular chemistry and used in spectral graph theory, network theory, and several field of mathematics. Those indices are defined in crisp graph only. Recently, in 2020, Binu et. al. [3, 4] introduced connectivity index and Wiener index in fuzzy graph and given some application in illegal immigration. Followed those paper [1, 10, 11, 12, 21, 22] introduced and studied some topological indices on fuzzy graph also. In this paper, F-index for fuzzy graph is introduced and provided some interesting results on it. In the end of this article, an decision making problem is presented and solved by using F-index for fuzzy graph.

1.3. **Significance and objective of the article.** Different types of topological index of a graph has many applications and many results are available for crisp graphs. But in many practical applications it is seen that many situations cannot be modeled using crisp graphs. In these cases, to handle such a situation, those topological indices are needed to define in a fuzzy graph. In this paper, F-index for fuzzy graph is defined and given some bounds for various fuzzy graphs like path, cycle, star, complete fuzzy graph (CFG), etc. In this article, it is shown that CFG is the maximal FG with respect to F-index for given vertex set. In the end of this article, an decision making problem is presented and solved by using F-index for fuzzy graph.

1.4. **Framework of the article.** Structure of the article is as follows: Section 2 provides some basic definitions which are essential to develop our main results. In section 3, F-index of a FG is studied and provided some bounds of it. An application of F-index in social network has been provided in the Section 4.

2. Preliminaries

In this portion, some basic definitions are provided which are essential to develop our results are given, most of them one can be found in [19, 20].

Let X be a universal set. A FS S on X is a mapping $\Omega : X \to [0, 1]$. Here Ω is called the membership function of the FS S. Generally a FS is denoted by $S = (x, \Omega)$.

Definition 2.1. *Let $X(\neq \phi)$ be a given finite set. The FG is a triplet, $\Gamma = (\Upsilon, \Psi, \Omega)$, where Υ is nonempty finite subset of X with $\Psi : \Upsilon \to [0, 1]$ and $\Omega : \Upsilon \times \Upsilon \to [0, 1]$ satisfying $\Omega(x, y) \leq \Psi(x) \wedge \Psi(y)$, where $\wedge$ represents the minimum.*

TABLE 1. The list of abbreviation.

Abbreviation	Meaning
FS	Fuzzy set
FSS	Fuzzy subset
FG	Fuzzy graph
FSG	Fuzzy subgraph
PFSG	Partial fuzzy subgraph
CFG	Complete fuzzy graph
TI	Topological index
ZI	Zagreb index
F-index	Forgotten Topological Index
MV	Membership value

The set Υ is the set of vertices and $\mathcal{E} := \{(x, y) : \Omega(x, y) > 0\}$ is the set of edge of the FG. $\Psi(x)$ represents the vertex MV of x and $\Omega(x, y)$ represents the edge MV of (x, y) (or simply xy).

Definition 2.2. *Let $\Gamma = (\Upsilon, \Psi, \Omega)$ be a FG. Then $\mathcal{H} = (\Upsilon', \Psi', \Omega')$ is called PFSG of the FG Γ if $\Upsilon' \subset \Upsilon, \Psi'(x) \leq \Psi(x), \Omega'(x, y) \leq \Omega(x, y)$ for all $x, y \in \Upsilon'$.*

If $\Psi'(x) = \Psi(x)$ and $\Omega'(x, y) \leq \Omega(x, y)$ for all $x, y \in \Upsilon'$ then $\mathcal{H}$ is called FSG of the FG Γ. For $x \in \Upsilon$, we denote Γ_x is a FSG of the FG Γ with $\Psi(x) = 0$ and for $xy \in \mathcal{E}$, Γ_{xy} represents the FSG of the FG Γ with $\Omega(xy) = 0$.

Definition 2.3. *Let $x_0, x_1, \cdots, x_n$ be distinct vertices of a FG Γ. Then the sequence of vertices $P(x_0x_1 \cdots x_n)$ is called a path in Γ if $\Omega(x_i, x_{i+1}) \neq 0$ for $i = 0, 1, \cdots, n-1$.*

The path, P is called cycle if $\Omega(x_0, x_n) > 0$.

Definition 2.4. *Let $x_0, x_1, \cdots, x_n$ be the vertices of a FG Γ. Then $\Gamma = (x_0; x_1, x_1 \cdots x_n)$ is called a star if $\Omega(x_0, x_i) \neq 0$ and $\Omega(x_i, x_j) = 0$ for $i, j = 1, 2, \cdots, n$.*

Here x_0 is called the center and $x_1, x_2, \cdots, x_n$ are called the pendent vertices of the star.

Definition 2.5. *For a FG Γ is called a CFG if for all $x, y \in \Upsilon, \Omega(x, y) = \Psi(x) \wedge \Psi(y)$.*

Definition 2.6. *Two FGs $\Gamma_1 = (\Upsilon_1, \Psi_1, \Omega_1)$ and $\Gamma_2 = (\Upsilon_2, \Psi_2, \Omega_2)$ are called isomorphic if there exist a bijective map $h : \Upsilon_1 \to \Upsilon_2$ with any $x, y \in \Upsilon_1, \Psi_1(x) = \Psi_2(h(x))$ and $\Omega_2(h(x), h(y)) = \Omega_1(x, y)$.*

Definition 2.7. *Let $v \in \Upsilon$ then degree of v is denoted by $d_\Gamma(v)$ or simply $d(v)$ and defined as $d(v) = \sum_{x \in \Upsilon} \Omega(xv)$.*

Let $\Delta(\Gamma)$ or Δ be the maximum degree of Γ and defined as $\Delta = \vee_{v \in \Upsilon} d(v)$ and $\delta(\Gamma)$ or δ be the minimum degree of G and defined as $\delta = \wedge_{v \in \Upsilon} d(v)$. The total degree of Γ is denoted by $T(\Gamma)$ or simply T, i.e. $T = \sum_{v \in \Upsilon} d(v) = 2 \sum_{uv \in \mathcal{E}} \Omega(uv) \leq 2 \cdot m$, where $m =$ no. of edges.

3. F-index of fuzzy graphs

Gutman et al. [8] introduced the first and second ZI of a crisp graph in 1972.

Definition 3.1. [8] *Let $\Gamma = (\Upsilon, \mathcal{E})$ be a crisp graph. Then the first ZI of the graph Γ is denoted by $M_1(\Gamma)$ and is defined by:*

$$M_1(\Gamma) = \sum_{v \in \Upsilon} d^2(v).$$

Definition 3.2. [8] *Let $\Gamma = (\Upsilon, \mathcal{E})$ be a crisp graph. Then the second ZI of the Γ is denoted by $M_2(\Gamma)$ and is defined by:*

$$M_2(\Gamma) = \sum_{uv \in \mathcal{E}} d(u)d(v).$$

This two TIs are degree based TIs. Followed by this two topological indices, in 2015, Fortula and Gutman defined another degree based topological indices called forgotten topological [7].

Definition 3.3. [7] *Suppose $\Gamma = (\Upsilon, \mathcal{E})$ be a graph (crisp). F-index of Γ is indicated by $F(\Gamma)$ and is defined by:*

$$F(\Gamma) = \sum_{v \in \Upsilon} d^3(v).$$

In 2019, Kalathian et al. [12] studied first and second ZI for a FG as follows:

Definition 3.4. [12] *Suppose $\Gamma = (\Upsilon, \Psi, \Omega)$ be a FG. Then first ZI of the FG Γ is indicated by $M(\Gamma)$ and is defined by:*

$$M(\Gamma) = \sum_{v \in \Upsilon} \Psi(v)d^2(v).$$

Definition 3.5. [12] *Suppose $\Gamma = (\Upsilon, \Psi, \Omega)$ be a FG. Then the second ZI of the FG Γ is indicated by $ZF_2(\Gamma)$ and is defined by:*

$$ZF_2(\Gamma) = \sum_{uv \in \mathcal{E}} \Psi(u)d(u)\Psi(v)d(v).$$

F-index of a FG is defined as:

Definition 3.6. *Suppose $\Gamma = (\Upsilon, \Psi, \Omega)$ be a FG. Then F-index of the FG Γ is indicated by $FF(\Gamma)$ and is defined by:*

$$FF(\Gamma) = \sum_{v \in \Upsilon} [\Psi(v)d(v)]^3.$$

In the next theorem, an upper bound of F-index is given for any FG.

Theorem 3.1. *Suppose Γ be n-vertex FG with m edges. Then*
(i) $FF(\Gamma) \leq n^3T^3$,
(ii) $FF(\Gamma) \leq 8n^3m^3$.

Proof. (i) Using the fact $\Psi(v) \leq 1$, the following inequality holds:

$$\begin{aligned} FF_1(\Gamma) &= \sum_{v \in \Upsilon} [\Psi(v)d(v)]^3 \\ &\leq [\sum_{v \in \Upsilon} \Psi(v)]^3 [\sum_{v \in \Upsilon} d(v)]^3 \\ &\leq n^3T^3. \end{aligned}$$

(ii) Using (i) and the fact $T(\Gamma) = 2\sum_{i=1}^{m} \Omega_i \leq 2m$, the required inequality follows. □

Now, the first ZI at a vertex of a FG is defined below.

Definition 3.7. *Suppose $\Gamma = (\Upsilon, \Psi, \Omega)$ be a FG and $v \in \Upsilon$. Then the first ZI at the vertex v of the FG Γ is indexed by $ZF_1(v; G)$ or simply $ZF_1(v)$ and is defined by*

$$ZF_1(v) = FF(\Gamma) - FF(\Gamma_v).$$

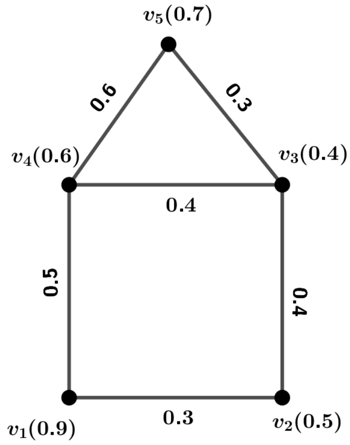

FIGURE 1. A FG with $ZF_1(\Gamma) = 1.480354$.

In the next example, F-index the FG in Fig. 1 is calculated.

Example 3.1. *Suppose Γ be a FG shown in Fig. 1 with vertex set $\Upsilon = \{v_1, v_2, \cdots, v_5\}$ such that, $\Psi(v_1) = 0.9, \Psi(v_2) = 0.5, \Psi(v_3) = 0.4, \Psi(v_4) = 0.6, \Psi(v_5) = 0.7, \Omega(v_1, v_2) = 0.3, \Omega(v_1, v_4) = 0.5, \Omega(v_2, v_3) = 0.4, \Omega(v_3, v_4) = 0.4, \Omega(v_3, v_5) = 0.3, \Omega(v_4, v_5) = 0.6$. Then $d(v_1) = 0.8, d(v_2) = 0.7, d(v_3) = 1.1, d(v_4) = 1.5, d(v_5) = 0.9$. Therefore,*

$$\begin{aligned} FF(\Gamma) = &\sum_{v \in \Upsilon} [\Psi(v)d(v)]^3 \\ =&[0.9 \times 0.8]^3 + [0.5 \times 0.7]^3 + [0.4 \times 1.1]^3 + [0.6 \times 1.5]^3 + [0.7 \times 0.9]^3 \\ =&1.480354. \end{aligned}$$

The next example shows that the F-index of a FSG is less than the original FG.

Example 3.2. *Let $\mathcal{H}$ be a FSG of the FG Γ shown in Fig. 2 obtained by deletion of the edge v_3v_5. Then $d(v_1) = 0.8, d(v_2) = 0.7, d(v_3) = 0.7, d(v_4) = 1.1, d(v_5) = 0.9$. Therefore,*

$$\begin{aligned} FF(\mathcal{H}) = &\sum_{v \in \Upsilon} [\Psi(v)d(v)]^3 \\ =&[0.9 \times 0.8]^3 + [0.5 \times 0.7]^3 + [0.4 \times 0.7]^3 + [0.6 \times 1.1]^3 + [0.7 \times 0.9]^3 \\ =&0.975618 \leq ZF_1(\Gamma). \end{aligned}$$

Note that, from the Examples 3.1 and 3.2, we get $FF(\mathcal{H}) \leq FF(\Gamma)$. In the next proposition we proved the fact in general.

Proposition 3.1. *Let $\mathcal{H} = (\Upsilon', \Psi', \Omega')$ be a PFSG of a FG $\Gamma = (\Upsilon, \Psi, \Omega)$. Then $FF(\mathcal{H}) \leq FF(\Gamma)$.*

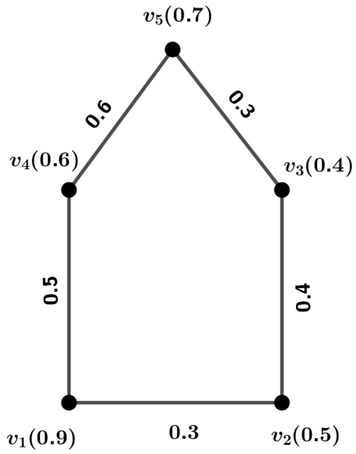

FIGURE 2. A FSG of the FG G in the Fig. 1 with $FF(\mathcal{H}) \leq ZF_1(\Gamma)$.

Proof. As $\mathcal{H}$ be a PFSG of Γ then for any $u, v \in \Upsilon', \Psi'(u) \leq \Psi(u)$ and $\Omega'(u,v) \leq \Omega(u,v)$. so,

$$d_{\mathcal{H}}(u) = \sum_{v\in\Upsilon'} \Omega'(u,v) \leq \sum_{v\in\Upsilon'} \Omega(u,v) \leq \sum_{v\in\Upsilon} \Omega(u,v) = d_\Gamma(u).$$

Therefore,

$$\begin{aligned} FF(\mathcal{H}) &= \sum_{u\in\Upsilon'} [\Psi'(u) d_{\mathcal{H}}(u)]^3 \\ &\leq \sum_{u\in\Upsilon} [\Psi(u) d_\Gamma(u)]^3 \\ &= ZF_1(\Gamma). \end{aligned}$$

Hence, $FF(\mathcal{H}) \leq FF(\Gamma)$. □

Corollary 3.1. *Let $\mathcal{H} = (\Upsilon', \Psi', \Omega')$ be a FSG of a FG $\Gamma = (\Upsilon, \Psi, \Omega)$. Then $FF(\mathcal{H}) \leq FF(\Gamma)$.*

Let $0 \leq p \leq 1$, the FG $\Gamma_p = (\Upsilon', \Psi', \Omega')$ is a FSG of the FG $\Gamma = (\Upsilon, \Psi, \Omega)$ and is defined as $\Upsilon' = \{v \in \Upsilon : \Psi(v) \leq p\}$ and $\Psi'(v) = \Psi(v), \Omega'(uv) = \Omega(uv)$ for $(uv) \in \Omega^*$.

Theorem 3.2. *Suppose Γ be a FG and let $0 \leq p_1 \leq p_2 \leq 1$. Then $FF(\Gamma_{p_2}) \leq FF(\Gamma_{p_1})$.*

Proof. Γ_{p_2} is PFSG of Γ_{p_1}. Then by Proposition 3.1, the result follows.

Corollary 3.2. *Let Γ be a FG and let $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n \leq 1$. Then*

$$FF(\Gamma_{p_n}) \leq FF(\Gamma_{p_{n-1}}) \leq \cdots \leq FF(\Gamma_{p_2}) \leq FF(\Gamma_{p_1}).$$

Theorem 3.3. *Let $P(v_0, v_1, \cdots, v_n)$ be a path. Then $FF(P) \leq 2(4n-3)$.*

Proof. As $P(v_0, v_1, \cdots, v_n)$ be a path, then $d(v_0) = \Omega_1, d(v_n) = \Omega_n$ and $d(v_i) = \Omega_i + \Omega_{i+1}$ for $i = 1, 2, \cdots, n-1$. Therefore,

$$\begin{aligned} FF(P) &= \Psi_0^3\Omega_1^3 + \sum_{i=1}^{n-1} \Psi_i^3(\Omega_i + \Omega_{i+1})^3 + \Psi_n^3\Omega_n^3 \\ &\leq 1 + \sum_{i=1}^{n-1} 2^3 + 1 \\ &= 2(4n-3). \end{aligned}$$

□

Theorem 3.4. *Let $C(v_0, v_1, \cdots, v_n)$ be a cycle. Then $FF(C) \leq 8(n+1)$.*

The proof is similar to the proof of Theorem 3.3. □

Theorem 3.5. *Let $S(v_0; v_1, \cdots, v_n)$ be a star. Then $FF(S) \leq n^3(n^3+1)$.*

Proof. As v_0 is the center of the star S, each v_i is adjacent to v_0, then $d(v_0) = \sum_{i=1}^{n} \Omega(v_0v_i)$ and v_i is pendent vertex of the star S for each $i = 1, 2, \cdots, n$, so only v_0 is adjacent to v_i, so, $d(v_i) = \Omega(v_0v_i)$ for $i = 1, 2, \cdots, n$. Therefore,

$$\begin{aligned} FF(S) &= \sum_{v\in\Upsilon} [\Psi(v)d(v)]^3 \\ &= [\Psi_0 d(v_0)]^3 + \sum_{i=1}^{n} [\Psi_i d(v_i)]^3 \\ &= \Psi_0^3(\sum_{i=1}^{n} \Omega_i)^3 + \sum_{i=1}^{n} \Psi_i^3\Omega_i^3 \\ &\leq (n^3+1)[\sum_{i=1}^{n} \Omega_i]^3 \\ &\leq n^3(n^3+1). \end{aligned}$$

□

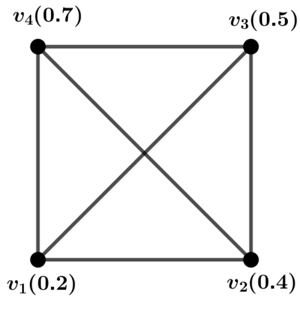

FIGURE 3. A CFG with $ZF_1(\Gamma) = 1.0698$.

In the next example, F-index of the CFG shown in Figure 3 is calculated below:

Example 3.3. *Let Γ be a CFG shown in Figure 3 with vertex set $\Upsilon = \{v_1, v_2, v_3, v_4\}$, and $\Psi(v_1) = 0.2, \Psi(v_2) = 0.4, \Psi(v_3) = 0.5, \Psi(v_4) = 0.7$. As Γ is a CFG then $\Omega(v_1v_2) = 0.2, \Omega(v_1v_3) = 0.2, \Omega(v_1v_4) = 0.2, \Omega(v_2v_3) = 0.4, \Omega(v_2v_4) = 0.4, \Omega(v_3v_4) = 0.5$. Then the* *degree of the vertices are* $d(v_1) = 0.6, d(v_2) = 1.0, d(v_3) = 1.1, d(v_4) = 1.1$. *Therefore, F-index of the fuzzy graph* Γ *is*

$$\begin{aligned} FF(\Gamma) &= \sum_{v\in\Upsilon}[\Psi(v)d(v)]^3 \\ &= [0.2 \times 0.6]^3 + [0.4 \times 1.0]^3 + [0.5 \times 1.1]^3 + [0.7 \times 1.1]^3 \\ &= 0.688636. \end{aligned}$$

F-index of a CFG is studied in the next theorem.

Theorem 3.6. *Suppose* Γ *be a CFG with vertex set* $\Upsilon = \{v_1, v_2, \cdots, v_n\}$. *Then*

$$0 \leq n(n-1)^3\Psi_0^6 \leq FF(\Gamma) \leq n(n-1)^3\Psi_n^6 \leq n(n-1)^3,$$

where $\Psi(v_i) = \Psi_i$ *and* $\Psi_1 \leq \Psi_2 \leq \cdots \leq \Psi_n$.

Proof. As Γ be a CFG. Then degree of the vertex v_i is $d(v_i) = (n-i)\Psi_i + \sum_{j=1}^{i-1}\Psi_j$. Therefore,

$$\begin{aligned} ZF_1(\Gamma) &= \sum_{v\in\Upsilon}[\Psi(v)d(v)]^3 \\ &= \sum_{i=1}^{n}\Psi_i^3[(n-i)\Psi_i + \sum_{j=1}^{i-1}\Psi_j]^3 \\ &\leq \sum_{i=1}^{n}\Psi_n^3[(n-i)\Psi_n + \sum_{j=1}^{i-1}\Psi_n]^3 \\ &= n(n-1)^3\Psi_n^6. \end{aligned}$$

Other inequalities follows similarly. □

Let $\Gamma = (\Upsilon, \Psi, \Omega)$ be a FG. Now we construct the FG $C[\Gamma] = (\Upsilon, \Psi, \Omega^c)$ with $\Omega^c(uv) = \wedge\{\Psi(u), \Psi(v)\}$ and we called the FG is completion fuzzy graph of the FG Γ.

Theorem 3.7. *Suppose* Γ *be a FG. Then* $FF(\Gamma) \leq FF(C[\Gamma])$.

Proof. As for any $(u, v) \in \mathcal{E}, \Omega(u, v) \leq \wedge\{\Psi(u), \Psi(v)\} = \Omega^c(u, v)$. Therefore, for any $v \in \Upsilon$,

$$\begin{aligned} d_\Gamma(v) &= \sum_{u\in\Upsilon}\Omega(u, v) \\ &\leq \sum_{u\in\Upsilon}\Omega^c(u, v) \\ &= d_{C[\Gamma]}(v). \end{aligned}$$

Now,

$$\begin{aligned} FF(\Gamma) &= \sum_{v\in\Upsilon}[\Psi(v)d_\Gamma(v)]^3 \\ &\leq \sum_{v\in\Upsilon}[\Psi(v)d_{C[\Gamma]}(v)]^3 \\ &= FF(C[\Gamma]). \end{aligned}$$

□

Corollary 3.3. *Among all n-vertex FG with given vertex set, CFG has maximum F-index.*

Corollary 3.4. *For any n-vertex FG Γ, $FF(\Gamma) \leq n(n-1)^3$.*

In the next theorem, F-index is discussed for isomorphic FGs.

Theorem 3.8. *Let Γ_1 and Γ_2 be isomorphic. Then $FF(\Gamma_1) = FF(\Gamma_2)$.*

Proof. As Γ_1 and Γ_2 are isomorphic FGs, there exist a isomorphism ϕ between Γ_1 and Γ_2, i.e. $\phi : \Upsilon_1 \to \Upsilon_2$ is a bijection and for all $u, v \in \Upsilon_1, \Psi_1(v) = \Psi_2(\phi(v))$ and $\Omega_1(u, v) = \Omega_2(\phi(u), \phi(v))$. Then

$$\begin{aligned} d_{\Gamma_1}(v) &= \sum_{u \in \Upsilon_1} \Omega_1(u, v) \\ &= \sum_{u \in \Upsilon_1} \Omega_2(\phi(u), \phi(v)) \\ &= \sum_{\phi(u) \in \Upsilon_2} \Omega_2(\phi(u), \phi(v)) \\ &= d_{\Gamma_2}(\phi(v)). \end{aligned}$$

Therefore,

$$\begin{aligned} FF(\Gamma_1) &= \sum_{v \in \Upsilon_1} [\Psi_1(v) d_{\Gamma_1}(v)]^3 \\ &= \sum_{v \in \Upsilon_1} [\Psi_2(\phi(v)) d_{\Gamma_2}(\phi(v))]^3 \\ &= \sum_{\phi(v) \in \Upsilon_2} [\Psi_2(\phi(v)) d_{\Gamma_2}(\phi(v))]^3 \\ &= FF(\Gamma_2). \end{aligned}$$

Hence the result. □

In the next theorem, bounds for F-index is provided.

Theorem 3.9. $\frac{n\delta^6}{m^3} \leq FF(\Gamma) \leq n\Delta^3$.

Table 2. Calculation of vertex membership value

Name of the researcher	Vertex name	Number of research publication	Vertex membership value
Dr. S Samanta	SS	90	0.21
Dr. S Sahoo	SSA	21	0.05
Dr. G Ghorai	GG	52	0.12
Dr. M Pal	MP	439	1.00

Proof. Now for $v \in \Upsilon, \delta \leq d(v) \leq m\Psi(v)$. Therefore, $\Psi(v) \geq \frac{\delta}{m}$. Now,

$$\begin{aligned} FF(\Gamma) &= \sum_{v\in\Upsilon}[\Psi(v)d(v)]^3 \\ &\leq \sum_{v\in\Upsilon}[\Psi(v)\Delta]^3 \\ &\leq n\Delta^3. \\ \text{Again, } FF(\Gamma) &= \sum_{v\in\Upsilon}[\Psi(v)d(v)]^3 \\ &\geq \delta^3\sum_{v\in\Upsilon}[\Psi(v)]^3 \\ &\geq \delta^3\sum_{1}^{n}(\frac{\delta}{m})^3 \\ &= \frac{n\delta^6}{m^3}. \end{aligned}$$

This completes the proof. □

4. Application of F-index in Social Network

There are many real life problems which we can describe by FGs. In this section, a decision making problem is presented and solved by F-index. In [13] Mahapatra et al. predicted the link in a social network by using RSM index. In this article, a group of researchers are chosen and calculated the best influenced researcher in that group. Here every researcher is consider as a vertex and there is a edge between any two researchers if they have co-authored at least one research article. The vertex membership value of a researcher is calculated by the formula:

$$\text{Vertex MV of a researcher} = \frac{\text{Number of research publication}}{\text{Maximum research publication among the researchers}}.$$

All the data are collected from ResearchGate dated 11th January, 2021. The score and membership values of each vertex is given in the Table 2. The score of a edge is the number of collaborated research article of the researchers. The score of each edges is listed in the Table 3. The membership value of a edges is score of the edge divided by maximum research publication among the researchers. MV of each edge is calculated in the Table 4. Then the source graph is shown in Figure 4 and corresponding fuzzy graph is shown in Figure 5. From Table 4 degree of each vertices are calculated in the Table 5 by the

TABLE 3. Score of edges

	SS	SSA	GG	MP
SS		0	11	63
SSA	0		6	21
GG	11	6		36
MP	63	21	36	

TABLE 4. Calculation of edge membership value

	SS	SSA	GG	MP
SS		0	0.03	0.14
SSA	0		0.01	0.05
GG	0.03	0.01		0.08
MP	0.14	0.05	0.08	

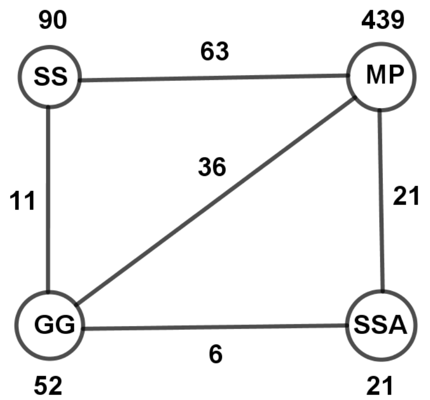

FIGURE 4. Source graph, data taken from ResearchGate (11th January 2021).

TABLE 5. Degree of vertex

Vertex	SS	SSA	GG	MP
Degree	0.17	0.06	0.12	0.27

formula

$$d(v) = \sum_{u \in \Upsilon} \Omega(uv).$$

Therefore F-index of this group is

$$FF(\Gamma) = \sum_{v \in \Upsilon} [\Psi(v) d(v)]^3 = 0.0197315.$$

Similarly one can calculate F-index of Γ_{MP} is $FF(\Gamma_{MP}) = 0.0000003608$. Now score of Dr. M Pal is defined as: $S(MP) = \frac{FF(\Gamma) - FF(\Gamma_{MP})}{FF(\Gamma)} = 0.9999817$. Score of each vertex is given in the Table 6. Note that, score of a researcher is directly related to number of publication for the researcher as well as number of collaboration with other researchers in that group

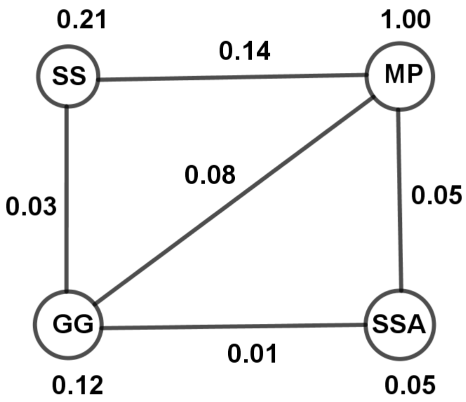

FIGURE 5. Fuzzy graph of the source graph.

TABLE 6. Score of the researcher (correct upto 2 decimal places)

Researcher	Dr. S Samanta	Dr. S Sahoo	Dr. G Ghorai	Dr. M Pal
Score	0.88	0.45	0.65	1.00

and if one of those parameter is increased then score is also increased. So, Higher score implies the more influenced researcher in that research group. From Table 6, it is seen that Dr. M Pal has highest score. Hence, Dr. M Pal is most influenced researcher in that research group.

5. Conclusion

TIs has an important role in spectral graph theory, chemical graph theory, biochemistry, etc. In this article, F-index for FG is defined and provided some upper bound of F-index for path, cycle, star, CFG etc. In this article, it is shown that CFG is the maximal FG with respect to F-index for given vertex set. In the end of this article, an decision making problem is presented and solved by using F-index for fuzzy graph. Also there are so many topological indices in crisp graph which are not studied in fuzzy graph. In future, one can study those topological indices for fuzzy graphs also.

6. Acknowledgments

The authors are highly thankful to the honorable Editor in Chief and anonymous reviewers for their valuable suggestions which significantly improved the quality and representation of the paper.

References

[1] Akram, M., Zafar, F., (2019), A new approach to compute measures of connectivity in rough fuzzy network models, Journal of Intelligent and Fuzzy Systems, 36(1), pp. 449-465.
[2] Amin, R., Nayeem, S. M. A., (2018), On the F-index and F-coindex of the line graphs of the subdivision graphs, Malaya Journal of Atematic, 6(2), pp. 362-368.
[3] Binu, M., Mathew, S., Mordeson, J. N., (2019), Connectivity index of a fuzzy graph and its application to human trafficking, Fuzzy Sets Syst., 360, pp. 117-136.
[4] Binu, M., Mathew, S., Mordeson, J. N., (2020), Wiener index of a fuzzy graph and application to illegal immigration networks, Fuzzy Sets Syst., 384, pp. 132-147.

[5] Dobrynin, A. A., Gutman, I., Klavzar, S., Zigert, P., (2002), Wiener index of hexagonal systems, Acta Appl. Math., 72, pp. 247-294.

[6] Estrada, E., (2000), Characterization of $3D$ molecular structure, Chem. Phys. Lett., 319, pp. 7-13.

[7] Fortual, B., Gutman, I., (2015), A forgotten topological index, J. Math. Chem., 53(4), pp. 1184-1190.

[8] Gutman, I., Trinajstic, N., (1972), Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, Chem. Phys. Lett., 17, pp. 535-538.

[9] Hosoya, H., (1971), Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, Bull. Chem. Sco. Japan, 44(9), pp. 2332-2339.

[10] Islam, S. R., Maity, S., Pal, M., (2020), Comment on "Wiener index of a fuzzy graph and application to illegal immigration networks", Fuzzy Sets Syst., 384, pp. 148-151.

[11] Islam, S. R., Pal, M., (2021), First Zagreb index on a fuzzy graph and its application, Journal of Intelligent and Fuzzy Systems, DOI:10.3233/JIFS-201293.

[12] Kalathian, S., Ramalingam, S., Raman, S., Srinivasan, N., (2020), Some topological indices in fuzzy graphs, Journal of Intelligent and Fuzzy Systems, DOI:10.3233/JIFS-189077.

[13] Mahapatra, R., Samanta, S., Pal, M., Xin, Q., (2019), RSM index: a new way of link prediction in social networks, Journal of Intelligent and Fuzzy Systems, 37(2), pp. 2137–2151.

[14] Mahapatra, T., Ghorai, G., Pal, M., (2020), Fuzzy fractional coloring of fuzzy graph with its application, Journal of Ambient Intelligence and Humanized Computing, 11, pp. 5771-5784.

[15] Mahapatra, T., Sahoo, S., Ghorai, G., Pal, M., (2021), Interval valued m-polar fuzzy planar graph and its application, Artif. Intell. Rev., 54, pp. 1649-1675.

[16] Mahapatra, T., Pal, M., (2021), An investigation on m-polar fuzzy threshold graph and its application on resource power controlling system, Journal of Ambient Intelligence and Humanized Computing, https://doi.org/10.1007/s12652-021-02914-6.

[17] Mondal, S., De, N., Pal, A., (2019), Topological properties of Graphene using some novel neighborhood degree-based topological indices, International Journal of Mathematics for Industry, 11(1), 1950006 (14 Pages).

[18] Mondal, S., De, N., Pal, A., (2019), On some new neighbourhood degree based indices, Acta Chem. Iasi. 27(1), pp. 31–46.

[19] Mordeson, J. N., Mathew, S., (2019), Advanced Topics in Fuzzy Graph Theory, Springer.

[20] Pal, M., Samanta, S., Ghorai, G. (2020), Modern Trends in Fuzzy Graph Theory, Springer, Singapore, https://doi.org/10.1007/978-981-15-8803-7.

[21] Poulik, S., Ghorai, G., (2020), Certain indices of graphs under bipolar fuzzy environment with applications, Soft Computing, 24, 5119-5131.

[22] Poulik, S., Ghorai, G., (2021), Determination of journeys order based on graph's Wiener absolute index with bipolar fuzzy information, Information Sciences, 545, 608–619.

[23] Randic, M. (1993), Novel molecular descriptor for structure-property studies, Chemical Physics Letters, 211(10), pp. 478-483.

[24] Rosenfeld, A., (1975), Fuzzy graphs, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, New York, pp.77-95.

[25] Samanta, S., Pal, M., (2014), A new approach to social networks based on fuzzy graphs, Turkish Journal of fuzzy system, 5(2), pp. 78-99.

[26] Upadhyay, A. K., Maity, S., Islam, S. R., (2018), Corrigendum to "The multiplicative Zagreb co-indices on two graph operators" Bul. Acad. Stiinte Repub. Mold. Mat., Bul. Acad. Stiinte Repub. Mold. Mat., 87(2), pp. 138-140.

[27] Wiener, H., (1947), Structural determination of paraffin boiling points, J. Am. Chem. Soc., 69, pp. 17-20.

[28] Yeh, R. T., Bang, S. Y., (1975), Fuzzy relations, fuzzy graphs and their applications to clustering analysis, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, pp. 125-149.

[29] Zadeh, L. A., (1965), Fuzzy sets, Inf. Control., 8, pp. 338-353.

Sk Rabiul Islam is a research scholar in the department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, India.

Madhumangal Pal for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.9, N.3.