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FUZZY SEMI-ESSENTIAL SUBMODULES AND FUZZY SEMI-CLOSED SUBMODULES

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Abstract. In this paper, we prove some properties of fuzzy semi-essential submodules and fuzzy semi-closed submodules.

Keywords: Fuzzy prime submodule, fuzzy semi-essential submodule, fuzzy semi-closed submodule.

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1. Introduction

In 1965, Zadeh [14] proposed the concept of a fuzzy set. The notion of a fuzzy set was introduced in algebra and various other branches of mathematics. In 1971, Rosenfeld [11] considered fuzzification of algebraic structures and defined a fuzzy subgroupoid. Fuzzy submodules were studied by Mordeson and Malik [7]. Pan [10] studied fuzzy finitely generated modules and fuzzy quotient modules. Acar [2] studied fuzzy prime submodules. Saikia and Kalita [12] defined a fuzzy essential submodule and proved some characteristics of such submodules. Nimbhorkar and Khubchandani [8] studied fuzzy essential submodules with respect to an arbitrary fuzzy submodule. Also, Nimbhorkar and Khubchandani [9] studied fuzzy essential-small submodules and fuzzy small-essential submodules. Ahmed and Abbas [3] introduced the notion of a semi-essential submodule of a module. Mijbass and Abdullah [6] studied semi-essential submodules and semi-uniform modules. Abbas and Al-Aeashi [1] studied fuzzy semi-essential submodules of a fuzzy module.

In this paper, we study the concepts of a fuzzy semi-essential submodule and a fuzzy semi-closed submodule as a generalization of fuzzy essential submodule and fuzzy closed submodule respectively and prove some properties.

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2. Preliminaries

Throughout this paper R denotes a commutative ring with identity, M a unitary R-module with zero element θ. We use the notations "$\subseteq$"and "$\leq$"to denote inclusion and submodule respectively. We recall some definitions and results.

Definition 2.1. [14] *Let S be a nonempty set. A mapping $\omega : S \to [0,1]$ is called a fuzzy subset of S.*

Remark 2.1. [14] *If ω and σ are two fuzzy subsets of R, then*
(i) $\omega \subseteq \sigma$ if and only if $\omega(x) \leq \sigma(x)$;
(ii) $\omega \cup \sigma = \max\{\omega(x), \sigma(x)\}$;
(iii) $\omega \cap \sigma = \min\{\omega(x), \sigma(x)\}$; for all $x \in R$.

Let $N \leq M$, then the characteristic function, χ_N, of N is defined as,

$$\chi_N(x) = \begin{cases} 1, & \text{if } x \in N, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. [7] *Let X and Y be two nonempty sets and $g : X \to Y$ be a mapping. Let $\omega \in [0,1]^X$ and $\sigma \in [0,1]^Y$. Then the image $g(\omega) \in [0,1]^Y$ and the inverse image $g^{-1}(\sigma) \in [0,1]^X$ are defined as follows:*
for all $y \in Y$,

$$g(\omega)(y) = \begin{cases} \vee\{\omega(x) \mid x \in X, g(x) = y\}, & \text{if } g^{-1}(y) \neq \phi, \\ 0, & \text{otherwise.} \end{cases}$$

and $g^{-1}(\sigma)(x) = \sigma(g(x))$, for all $x \in X$.

Definition 2.3. [7] *Let M be an R-module. A fuzzy subset ω of M is said to be a fuzzy submodule, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:*
(i) $\omega(\theta) = 1$;
(ii) $\omega(x - y) \geq min\{\omega(x), \omega(y)\}$;
(iii) $\omega(rx) \geq \omega(x)$.

The set of all fuzzy submodules of M is denoted by $F(M)$.
The support of a fuzzy set ω, denoted by ω^*, is a subset of M defined by
$\omega^* = \{x \in M \mid \omega(x) > 0\}$. We denote by ω_* the set $\omega_* = \{x \in M \mid \omega(x) = 1\}$.

Definition 2.4. [12] *A fuzzy submodule ω of M is called an essential fuzzy submodule of M, denoted by $\omega \trianglelefteq M$, if for every nonzero fuzzy submodule σ of M, $\omega \cap \sigma \neq \chi_\theta$.*

Definition 2.5. [12] *A fuzzy submodule ω of M is said to be a closed submodule of M if ω has no non-constant (proper) essential extension.*

Theorem 2.1. [12] *Let ω be a non zero fuzzy submodule of M. Then $\omega \trianglelefteq M$ if and only if $\omega^* \trianglelefteq M$.*

Definition 2.6. [2, Definition 3.2] *Let ν be an L-fuzzy submodule of μ. Then ν is called an L-fuzzy prime submodule of μ if for $r_t \in F(R)$, $x_s \in F(M)$ ($r \in R, x \in M$and $s, t \in L$), $r_t x_s \in \nu$ implies that either $x_s \in \nu$ or $r_t\mu \subseteq \nu$.*

In particular, taking $\mu = \chi_M$, if for $r_t \in F(R)$, $x_s \in F(M)$ we have $r_t x_s \in \nu$ implies that either $x_s \in \nu$ or $r_t\chi_M \subseteq \nu$, then ν is called an L-fuzzy prime submodule of M.

Corollary 2.1. [2, Corollary 3.5] *Let ν be an L-fuzzy prime submodule of M.*
Then $\nu_* = \{x \in M \mid \nu(x) = \nu(0_M)\}$ is a prime submodule of M.

Theorem 2.2. [2, Theorem 3.6]
(i) Let N be a prime submodule of M and α a prime element in L. If ω is the fuzzy subset of M defined by

$$\omega(x) = \begin{cases} 1, & \text{if } x \in N, \\ \alpha, & \text{otherwise.} \end{cases}$$

for all $x \in M$, then ω is an L-fuzzy prime submodule of M.
(ii) Conversely, any L-fuzzy prime submodule can be obtained as in (i).

Proposition 2.1. [5, Proposition 2.6] *Let $\omega, \nu \in F(M)$.*
Then $(\omega \cap \nu)_* = \omega_* \cap \nu_*$, $(\omega \cup \nu)_* = \omega_* \cup \nu_*$.
Further if, ω and σ have finite images, then $(\omega + \sigma)_* = \omega_* + \sigma_*$.

Definition 2.7. [6] *A nonzero R-submodule N of M is called semi-essential if $N \cap P \neq 0$ for each nonzero prime R-submodule P of M.*

Proposition 2.2. [3, Proposition 1.3] *Let $g : M \to M'$ be an isomorphism. If $N \trianglelefteq_{semi} M$, then $g(N) \trianglelefteq_{semi} M'$.*

Lemma 2.1. [4, Lemma 3.8] *Let $g : M \to N$ be an epimorphism. If $\omega \in F(M)$ and $\sigma \in F(N)$, then*
(i) $g(\omega)_* = g(\omega_*)$;
(ii) $g^{-1}(\sigma)_* = g^{-1}(\sigma_*)$.

Theorem 2.3. [12] *The following conditions are equivalent for a fuzzy submodule δ.*
(i) δ is semisimple;
(ii) δ has no proper essential submodule;
(iii) Every submodule of δ is a direct summand of δ.

Proposition 2.3. [6, Proposition 13] *Let M and L be R-modules. Suppose that $g : M \to L$ is an R-epimorphism such that $ker(g) \subseteq rad(M)$. If N is a semi-essential R-submodule of L, then $g^{-1}(N)$ is a semi-essential R-submodule of M, where $rad(M) = \bigcap_{P \in Spec(M)} P$, and $Spec(M) = \{P : P$ is a prime R-submodule of $M\}$, if no such prime exists then $rad(M) = M$.*

3. Fuzzy Semi-Essential Submodules

The concept of a fuzzy semi-essential submodule is introduced by Abbas and Al-Aeashi [1]. We obtain some properties of such fuzzy submodules.

Definition 3.1. [1] *A fuzzy submodule ω of an R-module M is called a fuzzy semi-essential submodule of M if for any nonzero fuzzy prime submodule η of M, $\omega \cap \eta \neq \chi_\theta$ and then we write $\omega \trianglelefteq_{semi} M$.*

Theorem 3.1. *Let $\omega \in F(M)$. Then $\omega \trianglelefteq_{semi} M$ if and only if $\omega_* \trianglelefteq_{semi} M$.*

Proof. Assume that $\omega \trianglelefteq_{semi} M$. Let η be a fuzzy prime submodule of M. Then $\omega \cap \eta \neq \chi_\theta$ implies that $(\omega \cap \eta)_* \neq \{\theta\}$.
By using Proposition 2.1, we conclude that $\omega_* \cap \eta_* \neq \{\theta\}$. (I)
As η is a fuzzy prime submodule of M, it follows by Corollary 2.1, that η_* is prime submodule of M. Thus by (I), $\omega_* \trianglelefteq_{semi} M$.

Conversely, assume that $\omega_* \trianglelefteq_{semi} M$. Let P be a prime submodule of M. As ω_* is a semi-essential submodule of M, $\omega_* \cap P \neq \{\theta\}$. (II)
Define,

$$\nu(x) = \begin{cases} 1, & \text{if } x \in P. \\ \alpha, & \text{otherwise, where } 0 \leq \alpha < 1. \end{cases}$$

Then by Theorem 2.2, ν is a fuzzy prime submodule of M. Here, $\nu_* = P$. Now, (II) becomes $\omega_* \cap \nu_* \neq \{\theta\}$. Hence $(\omega \cap \nu)_* \neq \{\theta\}$ and thus, $\omega \cap \nu \neq \chi_\theta$.
Hence, $\omega \trianglelefteq_{semi} M$. □

Remark 3.1. *Every fuzzy essential submodule is semi-essential.*

The following example shows that the converse of Remark 3.1 need not be true.

Example 3.1. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{30}$.*
Define $\omega : M \to [0,1]$ as follows:

$$\omega(x) = \begin{cases} 1, & \text{if } x \in (3), \\ 0, & \text{otherwise.} \end{cases}$$

We note that $\omega^* = (3)$ is not an essential submodule of M as $(10) \cap (3) = (0)$.
Hence by Theorem 2.1, ω is not a fuzzy essential submodule of M.
It follows from [13, Theorem 10] *that the prime submodules of M coincide with the prime ideals of M (considering M as a ring). The prime ideals of M are $(2), (3)$ and (5) and the intersection of each of these with (3) is nonzero. Also, here $\omega_* = (3)$. Hence $I = (3)$ is a semi-essential submodule of M and so by Theorem 3.1, $\chi_I = \omega$ is a fuzzy semi-essential submodule of M.*

Definition 3.2. *A fuzzy submodule ω of fuzzy submodule σ is called a semi-essential submodule of σ if for any non-zero fuzzy prime submodule δ of σ, $\omega \cap \delta \neq \chi_\theta$ and then we write $\omega \trianglelefteq_{semi} \sigma$.*

Example 3.2. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{36}$.*
Define $\sigma : M \to [0,1]$ as follows:

$$\sigma(x) = \begin{cases} 1, & \text{if } x = \{0,3,6,9,12,15,18,21,24,27,30,33\} \\ 0.5, & \text{otherwise.} \end{cases}$$

Then $\sigma_* = \{0,3,6,9,12,15,18,21,24,27,30,33\}$ and $B = \{0,9,18,27\}$ is a prime submodule of σ_*.
Define, $\eta : M \to [0,1]$ as follows:

$$\delta(x) = \begin{cases} 1, & \text{if } x \in B, \\ \alpha, & \text{otherwise for all } x \in \sigma_*, \text{ where } 0 \leq \alpha < 1. \end{cases}$$

Then by Theorem 2.2, δ is a fuzzy prime submodule of σ.
Also, define $\omega : M \to [0,1]$ as follows:

$$\omega(x) = \begin{cases} 1, & \text{if } x = \{0,18\} \\ 0.4, & \text{otherwise.} \end{cases}$$

Here, $\omega \subseteq \sigma$ and $\omega_* = \{0,18\}$.
Also, $\omega_* \cap \delta_* \neq \{0\}$. This implies $(\omega \cap \delta)_* \neq \{0\}$ and thus, $\omega \cap \delta \neq \chi_0$.
Hence, $\omega \trianglelefteq_{semi} \sigma$.

Theorem 3.2. *Let $\omega, \sigma \in F(M)$ such that $\omega \subseteq \sigma$. Then $\omega \trianglelefteq_{semi} \sigma$ if and only if $\omega_* \trianglelefteq_{semi} \sigma_*$.*

Proof. Assume that $\omega \trianglelefteq_{semi} \sigma$. Let δ be a non-zero fuzzy prime submodule of σ. Then by definition 3.2, it follows that $\omega \cap \delta \neq \chi_\theta$.
This implies that $(\omega \cap \delta)_* \neq \{\theta\}$. Thus, $\omega_* \cap \delta_* \neq \{\theta\}$. (I)

Also by Corollary 2.1, it follows that δ_* is a prime submodule of σ_*.
Thus, by (I) $\omega_* \trianglelefteq_{semi} \sigma_*$.

Conversely, assume that $\omega_* \trianglelefteq_{semi} \sigma_*$.
Let A be a prime submodule of σ_*. Then $\omega_* \cap A \neq \{\theta\}$. (II)
Define,

$$\gamma(x) = \begin{cases} 1, & \text{if } x \in A. \\ \alpha, & \text{for } x \in \sigma_* - A, \text{ where } 0 \leq \alpha < 1. \end{cases}$$

Then by Theorem 2.2, we conclude that γ is a fuzzy prime submodule of σ and $\gamma_* = A$.
Now (II) becomes, $\omega_* \cap \gamma_* \neq \{\theta\}$. Hence $(\omega \cap \gamma)_* \neq \{\theta\}$ and thus, $\omega \cap \gamma \neq \chi_\theta$.
Hence, $\omega \trianglelefteq_{semi} \sigma$. □

The following result is from [1, Proposition 3.11]

Theorem 3.3. *Let ω_1 and ω_2 be fuzzy submodules of an R-module M. Suppose that ω_1 is a fuzzy submodule of ω_2. If ω_1 is a fuzzy semi-essential submodule of M, then ω_2 is a fuzzy semi-essential submodule of M.*

Remark 3.2. *The converse of Theorem 3.3 may not be true.*

Example 3.3. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{12}$.*
Define fuzzy submodules $\omega, \nu : M \to [0,1]$ as follows:

$$\omega(x) = \begin{cases} 1, & \text{if } x = \{0,4,8\}, \\ 0.7, & \text{otherwise.} \end{cases}$$

Then $\omega_* = \{0,4,8\}$.

$$\nu(x) = \begin{cases} 1, & \text{if } x = \{0,2,4,6,8,10\}, \\ 0.9, & \text{otherwise.} \end{cases}$$

Then $\nu_* = \{0,2,4,6,8,10\}$.
Now we observe that ω_* is a semi-essential submodule of ν_* and ν_* is a semi-essential submodule of M. It follows from Theorem 3.2 and Theorem 3.1 that, $\omega \trianglelefteq_{semi} \nu$ and $\nu \trianglelefteq_{semi} M$ respectively.
Define a fuzzy prime submodule of M as follows:

$$\delta(x) = \begin{cases} 1, & \text{if } x = \{0,3,6,9\}, \\ 0.2, & \text{otherwise.} \end{cases}$$

where $\delta_* = \{0,3,6,9\}$ is a prime submodule of M.
Here we observe that $\omega_* \cap \delta_* = \{0\}$. Hence ω_* is not semi-essential submodule of M.
Thus by Theorem 3.1, ω is not a fuzzy semi-essential submodule of M.

Remark 3.3. *If ω_1 and ω_2 be fuzzy submodules of an R-module M, then $\omega_1 \cap \omega_2$ may not be a fuzzy semi-essential submodule of M.*

Example 3.4. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{36}$.*
Define fuzzy sets ω_1 and ω_2 on M as follows:

$$\omega_1(x) = \begin{cases} 1, & \text{if } x = \{0,12,24\}, \\ 0.7, & \text{otherwise.} \end{cases}$$

Then $\omega_{1*} = \{0,12,24\}$ is semi-essential submodule of $\mathbb{Z}_{36}$.

$$\omega_2(x) = \begin{cases} 1, & \text{if } x = \{0,18\}, \\ 0.5, & \text{otherwise.} \end{cases}$$

Then $\omega_{2*} = \{0, 18\}$ *is a semi-essential submodule of* $\mathbb{Z}_{36}$.
Then by Theorem 3.2, we have $\omega_1 \trianglelefteq_{semi} \mathbb{Z}_{36}$ *and* $\omega_2 \trianglelefteq_{semi} \mathbb{Z}_{36}$.
Now,

$$(\omega_1 \cap \omega_2)(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0.7, & \text{if } x = 18, \\ 0.5, & \text{otherwise.} \end{cases}$$

Then $(\omega_1 \cap \omega_2)_* = \{0\}$ *which is not a semi-essential submodule of* $\mathbb{Z}_{36}$.
Hence by Theorem 3.2, $\omega_1 \cap \omega_2$ *is not a fuzzy semi-essential submodule of* $\mathbb{Z}_{36}$.

Proposition 3.1. *Let* ω_1 *and* ω_2 *be fuzzy submodules of an* R*-module* M. *Suppose that* ω_1 *is fuzzy essential and* ω_2 *is fuzzy semi-essential. Then* $\omega_1 \cap \omega_2$ *is a fuzzy semi-essential submodule of* M.

Proof. Let η be a non-zero fuzzy prime submodule of M. As ω_2 is a fuzzy semi-essential submodule of M, it follows that $\omega_2 \cap \eta \neq \chi_\theta$.
Since ω_1 is fuzzy essential we have $\omega_1 \cap (\omega_2 \cap \eta) \neq \chi_\theta$. This implies that $(\omega_1 \cap \omega_2) \cap \eta \neq \chi_\theta$.
Hence $\omega_1 \cap \omega_2$ is a fuzzy semi-essential submodule of M. □

Corollary 3.1. *Let* ω_1 *and* ω_2 *be two fuzzy submodules of an* R*-module* M. *If* $\omega_1 \cap \omega_2$ *is a fuzzy semi-essential submodule of* M, *then* ω_1 *and* ω_2 *are fuzzy semi-essential.*

Proof. Let η be a non-zero fuzzy prime submodule of M. As $\omega_1 \cap \omega_2$ is fuzzy semi-essential, it follows that $\omega_1 \cap \omega_2 \cap \eta \neq \chi_\theta$. This implies that $\omega_1 \cap \eta \neq \chi_\theta$ and $\omega_2 \cap \eta \neq \chi_\theta$. Thus, ω_1 and ω_2 are semi-essential. □

We give relationships between images and inverse images.

Proposition 3.2. *Let* M *and* M' *be* R*-modules and* g *be an isomorphism from* M *to* M'. *If* ω *is a fuzzy semi-essential submodule of* M, *then* $g(\omega)$ *is a fuzzy semi-essential submodule of* M'.

Proof. Suppose that ω is a fuzzy semi-essential submodule of M then by Theorem 3.1, ω_* is a semi-essential submodule of M. By Proposition 2.2, it follows that $g(\omega_*)$ is a semi-essential submodule of M'.
Using Lemma 2.1, we conclude that $g(\omega)_* = g(\omega_*)$. Thus $g(\omega)_*$ is a semi-essential submodule of M'. Hence by Theorem 3.1, $g(\omega)$ is a fuzzy semi-essential submodule of M'. □

Proposition 3.3. *Suppose that* g *is an* R*-module epimorphism from* M *to* M' *such that* $\chi_{kerg} \subseteq \chi_{radM}$, *where* M *and* M' *are* R*-modules. If* ω *is a fuzzy semi-essential* R*-submodule of* M', *then* $g^{-1}(\omega)$ *is a fuzzy semi-essential of* R*-submodule of* M.

Proof. As ω is a fuzzy semi-essential R-submodule of M', then by Theorem 3.1 we conclude that ω_* is a semi-essential R-submodule of M'.
Since $\chi_{kerg} \subseteq \chi_{radM}$, we have $(\chi_{kerg})_* \subseteq (\chi_{radM})_*$ and so $kerg \subseteq rad(M)$.
As $kerg \subseteq rad(M)$ and ω_* is a semi-essential R-submodule of M', by Proposition 2.3, it follows that $g^{-1}(\omega_*)$ is an semi-essential of R-submodule of M. By Lemma 2.1, we have $g^{-1}(\omega)_* = g^{-1}(\omega_*)$ and so $g^{-1}(\omega)_*$ is a semi-essential of R-submodule of M. It follows from Theorem 3.1, that $g^{-1}(\omega)$ is a fuzzy semi-essential of R-submodule of M. □

Proposition 3.4. *Let* ω_1 *and* ω_2 *be fuzzy submodules of an* R*-module* M. *Suppose that* ω_1 *is a fuzzy semi-essential submodule of* M. *If for any fuzzy prime submodule* ρ *of* M, $\omega_2 \cap \rho$ *is a fuzzy prime submodule of* M, *then* $\omega_1 \cap \omega_2$ *is a fuzzy semi-essential submodule of* M.

Proof. Let η be a fuzzy prime submodule of M. By assumption $\omega_2 \cap \eta$ is a fuzzy prime submodule of M. As ω_1 is fuzzy semi-essential, it follows that $(\omega_1 \cap \omega_2) \cap \eta \neq \chi_\theta$. Thus, $\omega_1 \cap \omega_2$ is a fuzzy semi-essential submodule of M. □

4. Fuzzy Semi-Closed Submodules

In this section, we introduce the concept of a fuzzy semi-closed submodule and prove some results.

Definition 4.1. *A fuzzy submodule ω of an R-module M is called semi-closed if ω has no proper(non-constant) semi-essential extensions in M, i.e. if $\omega \trianglelefteq_{semi} \mu \leq M$, then $\omega = \mu$.*

Remark 4.1. *Every fuzzy semi-closed submodule of an R-module M is a fuzzy closed submodule in M.*

Proof. Let ω be a fuzzy semi-closed submodule of M and η be a fuzzy submodule of M such that $\omega \trianglelefteq \eta \leq M$. We know that if $\omega \trianglelefteq \eta$, then $\omega \trianglelefteq_{semi} \eta$. But ω is semi-closed in M and so $\omega = \eta$. Thus, ω is a fuzzy closed submodule in M. □

Theorem 4.1. *Let ω_1 and ω_2 be fuzzy submodules of an R-module M. If ω_1 is semi-closed in ω_2 and ω_2 is semi-closed in M. Then ω_1 is semi-closed in M, provided that ω_2 is contained in any semi-essential extension of ω_1.*

Proof. Let η be a fuzzy submodule of M such that $\omega_1 \trianglelefteq_{semi} \eta \leq M$.
The following two cases arise:
Case (i): $\eta \leq \omega_2$.
As ω_1 is semi-closed in ω_2, we get $\omega_1 = \eta$. Hence, ω_1 is semi-essential in M.
Case (ii): If $\omega_2 \leq \eta$.
As $\omega_1 \trianglelefteq_{semi} \eta \leq M$ so by Proposition 3.3, $\omega_2 \trianglelefteq_{semi} \eta \leq M$. But ω_2 is semi-closed in M, thus, $\omega_2 = \eta$, that is $\omega_1 \trianglelefteq_{semi} \omega_2$.
Since ω_1 is semi-closed in ω_2, we conclude that $\omega_1 = \omega_2$. Hence, ω_1 is semi-closed in M. □

Proposition 4.1. *Let ω_1 and ω_2 be two fuzzy semi-closed submodules of an R-module M such that $\omega_1 \leq \omega_2 \leq M$. If ω_1 is semi-closed in M, then ω_1 is semi-closed in ω_2.*

Proof. Let η be a fuzzy submodule of ω_2 such that $\omega_1 \trianglelefteq_{semi} \eta \leq \omega_2$. Thus $\omega_1 \leq \eta \leq M$. But ω_1 is semi-closed in M. Therefore, $\omega_1 = \eta$. Hence, ω_1 is semi-closed in ω_2. □

Proposition 4.2. *If ω_1 and ω_2 are fuzzy semi-closed submodules of an R-module M. Then ω_1 and ω_2 are semi-closed in $\omega_1 + \omega_2$.*

Proof. As $\omega_1 \leq \omega_1 + \omega_2 \leq M$ and $\omega_2 \leq \omega_1 + \omega_2 \leq M$, then by Proposition 4.1 the result follows. □

Theorem 4.2. *If every fuzzy submodule of ω is semi-closed, then every fuzzy submodule of ω is a direct summand of ω.*

Proof. Let μ be a fuzzy semi-closed submodule of an R-module M ω. By Remark 3.1, μ is a fuzzy closed submodule of ω, i.e. μ has no proper essential extension in ω.
By Theorem 2.3, μ is a direct summand of ω. □

5. Conclusion

In this paper, we have studied fuzzy semi-essential submodules and fuzzy semi-closed submodules. In future we shall introduce the concepts of a fully fuzzy prime submodule and a fully fuzzy essential submodule.

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