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SUBCLASSES OF UNIVALENT FUNCTIONS RELATED WITH FUNCTIONS OF BOUNDED RADIUS ROTATION

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Abstract. This investigation is in twofold. Firstly, a comprehensive generalization of starlike functions is initiated. This notion gives more insight to the study of functions with bounded radius rotation. In this direction, we examine the geometric characterization of this class, which includes the inclusion, radius results and integral preserving properties. On the other hand, the class of functions that extend the idea of close-to-convex functions is introduced. Also, a necessary condition, radius results, coefficient results and closure property under convex convolution for this novel class are investigated. Overall, some alluring consequences of our results are also presented.

Keywords:Parabolic domain, Analytic functions, Subordination, Univalent functions.

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1. Introduction

Let $p(z)$ be analytic in $E = \{z \in \mathbb{C}\colon |z| < 1\}$ with $p(0) = 1$. Then $p(z)$ belongs to the class P of Caratheodory functions if and only if $\operatorname{Re} p(z) > 0$ for $z \in E$.

For the functions $f(z)$ and $g(z)$ analytic in E, we say $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ or $f \prec g$, if there exists an analytic function $w(z)$ in E with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in E$ such that $f(z) = g(w(z))$. If g is univalent in E, then $f(0) = g(0)$, $f(z) = g(w(z)) \iff f(E) \subset g(E)$.

If $f(z)$, $g(z)$ analytic in E with $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$, $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$, then the convolution (Hadamard product) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n\, b_n z^n = (g * f)(z).$$

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Definition 1.1. *Let $p(z)$ be analytic in E with $p(0) = 1$. Then $p(z)$ is said to belong to the class $P(\phi)$ if and only if $p(z) \prec \phi(z)$, where $\phi(z)$ is convex univalent in E with $\phi(0) = 1$.*

From this definition, we have the following remarks.

Remark 1.1. *Let $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha}, -1 \leq B < A \leq 1, \alpha \in (0,1]$. It can be shown easily that $\left(\frac{1+Az}{1+Bz}\right)^{\alpha}$ is convex univalent in E. Then, in Definition 1.1, $p \in P[A,B;\alpha]$ if $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha}$.*

For $A = 1, B = -1, \alpha = 1, p \in P$ and $p \prec \frac{1+z}{1-z}$ in E. That is, $P[1,-1]$ consists of Caratheodory functions of positive real part.

Also, $P[1-2\beta,-1,1] = P(\beta)$ and $p \in P(\beta)$ implies $\operatorname{Re} p(z) > \beta, z \in E$. It is known [2] *that $P[A,B,\alpha] \subset P(\gamma_\alpha) \subset P$, where*

$$\gamma_\alpha = \left(\frac{1-A}{1-B}\right)^{\alpha}. \tag{1}$$

The class $P[A,B,1] = P[A,B]$ is called Janowski class, see [10]*, and the geometrically interpretation of $p \in P[A,B]$ is that it maps E onto the domain $\Omega[A,B]$ given by*

$$\Omega[A,B] = \left\{w\colon \left|w - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\right\}, \text{ see [6]}. \tag{2}$$

This domain represent an open circular disc with diameter end points $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$ and center on the real axis.

Remark 1.2. *The conic domain $\Omega_k, k \geq 0$ is defined in* [8] *as*

$$\Omega_k = \left\{w = u + iv\colon u > k\sqrt{(u-1)^2 + v^2}\,; k \geq 0\right\}. \tag{3}$$

The domain Ω_0 represents right half plane, $\Omega_k\,(0 < k < 1)$ gives a hyperbola, Ω_1 depicts parabola and $\Omega_k\,(k > 1)$ is an ellipse. For $k = 1$, $\Omega_1 = \Omega_{PAR}$ and the function

$$\phi(z) = p^*_{PAR}(z) = p^*(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \tag{4}$$

plays the role of the extremal function which maps E conformally onto the parabolic domain Ω_{PAR}.

Using this concept and Definition 1, we say that $p(z)$, analytic in E with $p(0) = 1$ belongs to the class $P(p^*)$ if and only if $p(z) \prec p^*$ in E. It is known [8] *that for $p \in P(p^*)$ with $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$, we have*

(i) $|c_n| \leq \frac{8}{\pi^2}$, $n \geq 1$,
(ii) $\operatorname{Re} p(z) > \frac{1}{2}$, $z \in E$,
(iii) $|\arg p(z)| < \frac{\pi}{4}$, $z \in E$.

Geometrically, $p \in P(p^*)$ takes its values in the interior of the parabola in the right-half plane symmetric about the real axis with vertex at $(1/2, 0)$.

Using Definition 1, Remark 1.1 and Remark 1.2, we have

Definition 1.2. *Let $p(z)$ be analytic in E with $p(0) = 1$. Then $p \in P_m(\phi)$ if and only if there exists $p_1, p_2 \in P(\phi)$ such that*

$$p(z) = \left(\frac{m+2}{4}\right)p_1(z) - \left(\frac{m-2}{4}\right)p_2(z),\ m \geq 2,\ z \in E. \tag{5}$$

When $\phi(z)=\left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, we have $P_m\left(\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right)=P_m[A,B;\alpha]$ and with $\phi(z)=p^*(z)$, we have the class $P_m(p^*)$, $m\geq 2$.

Definition 1.3. *Let f be analytic in E and be given by*

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{6}$$

Then $f\in S_m^*(\phi)$ if and only if $\frac{zf'}{f}\in P_m(\phi)$ for $z\in E$.

We have two cases:

(1) when $\phi(z)=\left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, $\frac{zf'}{f}\in P_m[A,B;\alpha]$ and $f\in S_m^*[A,B;\alpha]$. $S_m^*[A,B;\alpha]$ is called the class of generalized starlike functions of Janowski type. The following are special cases:
   (i) $S_2^*[1,-1;\alpha]=\tilde{S}^*(\alpha)$ is the class of strongly starlike functions of order α, defined with the property $\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\alpha\pi}{2}$ [3].
   (ii) $S_m^*[1,-1;1]=R_m$ is the class of functions with bounded radius rotation and $R_2=S^*$ consists of starlike univalent functions [19].
   (iii) $f\in C_m[A,B;\alpha]$ if and only if $zf'\in S_m^*[A,B;\alpha]$ (see [15, 17]). We note that $C_2[1,-1,1]=C$ is the well known class of convex univalent functions

(2) When $\frac{zf'}{f}\in P_m(p^*)$, we have $S_m^*(p^*)$. For $m=2, S_2^*(p^*)=UST$ is the class of uniformly starlike functions [5]

We now define a class of generalized close-to-convex functions as:

Definition 1.4. *Let $f(z)$ be analytic in E and be given by 6. Then $f\in T_m(\phi)$ if and only if there exists $g\in S_m^*(\phi)$ such that $\frac{zf'}{g}\in P(\phi)$ for $z\in E$.*

If $g\in S_m^*[A,B;\alpha]$, we have $T_m[A,B;\alpha]$, and with $g\in S_m^*(p^*), f\in T_m(p^*)$. As a special cases, we have

(i) $T_2[A,B;\alpha]=K[A,B;\alpha]\subset K[1,-1,1]=K$ is the well-known class of close-to-convex univalent functions introduced by Kaplan [9].
(ii) $T_m[1,-1,1]=T_m$ has been studied by Noor (see [14])
(iii) $T_m(p_k(A,B;z))=k-T_m[A,B]$ with $p_k(A,B;z)$ defined in [18] is studied by Saliu and Noor [22, 23]

Next, we present those results that enable us to obtain our assertions in this manuscript.

Lemma 1.1. [11] *Let $\psi(z)$ be convex univalent in E with $\psi(0)=1$. Suppose also that $\lambda(z)$ is analytic in E with $Re\,\lambda(z)\geq 0\,(z\in E)$. If $p(z)$ is analytic in E with p(0)=1, then*

$$p(z)+\lambda(z)zp'(z)\prec\psi(z)\quad\text{in}\quad E \tag{7}$$

implies that

$$p(z)\prec\psi(z)\quad\text{in}\quad E. \tag{8}$$

Lemma 1.2. [11] *Let $\psi(z)$ be convex in E with $Re\,(\beta\psi(z)+\gamma)>0$. If $p(z)$ is analytic in U with $p(0)=\psi(0)$, then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\psi(z)\quad\Rightarrow\quad p(z)\prec\psi(z)\quad\text{in}\quad E. \tag{9}$$

Lemma 1.3. *Let $p\in P_m(p^*)$. Then $p\in P$ for $|z|<\frac{2}{m}$.*

Proof. We can write

$$p(z) = \left(\frac{m+2}{4}\right) p_1(z) - \left(\frac{m-2}{4}\right) p_2(z),\ m \geq 2,\ z \in E,\ p_i \prec p^*,\ i = 1, 2. \tag{10}$$

Since $\operatorname{Re} p_i(z) > \frac{1}{2}$ in E, we have

$$\frac{1}{1+r} \leq \operatorname{Re} p_i(z) \leq \frac{1}{1-r}, \quad i = 1, 2.$$

Thus, from (10), we have

$$\begin{aligned} \operatorname{Re} p(z) \geq & \left(\frac{m+2}{4}\right) \frac{1}{1+r} - \left(\frac{m-2}{4}\right) \frac{1}{1-r} \\ = & \frac{1 - \frac{mr}{2}}{1 - r^2}. \end{aligned} \tag{11}$$

The right hand side of (11) is positive for $r < r_m = \frac{2}{m}$ and this completes the proof. □

Lemma 1.4. [21] *Let $\psi \in C, g \in S^*$ and F be analytic in E with $F(0) = 1$. Then*

$$\frac{\psi * Fg}{\psi * g}(E) \subset \overline{C}_0 F(E),$$

where $\overline{C}_0$ is the closed convex hull

Lemma 1.5. [12] *Let $p \in P$. Then for $z = e^{i\theta}, 0 \leq \theta_1 < \theta_2 \leq 2\pi$,*

$$\max_{p \in P} \left| \int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{zp'(z)}{p(z)} d\theta \right| \leq \pi - 2\cos^{-1} r.$$

Lemma 1.6. [1] *Let $p \in P, \operatorname{Re} \mu \geq 0$. Then*

$$\left| \frac{zp'(z)}{p(z) + \mu} \right| \leq \frac{2r}{(1-r)[1 + r + \operatorname{Re} \mu (1-r)]}$$

Lemma 1.7. [11] *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Phi(u, v)$ be a complex-valued function satisfying the conditions:*

(i) *$\Psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,*
(ii) *$(1, 0) \in D$ and $\Psi(1, 0) > 0$,*
(iii) *$\operatorname{Re} \Psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.*

If $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is a function, analytic in E such that $(h(z), zh'(z)) \in D$ and Re *$(\Psi(h(z), zh'(z))) > 0$ for $z \in E$, then $\operatorname{Re} h(z) > 0$ in E.*

2. Main Results

At this stage, the main findings in this present article are presented.

2.1. **Inclusion Results.** In this section, certain inclusion relation associated with the class $P_m(\phi)$ are proved.

Theorem 2.1. *Let $\beta \geq 0$ and $N(z), D(z)$ be analytic in E with $N(0) = 0$, $D \in S^*$. Then*

$$\left\{(1-\beta)\frac{N}{D} + \beta \frac{N'}{D'}\right\} \in P_m(\phi) \Longrightarrow \frac{N}{D} \in P_m(\phi) \quad \text{in} \quad E.$$

Proof. Let $\frac{N(z)}{D(z)} = p(z)$, where $p(z)$ is given by (5). Then

$$\frac{N'(z)}{D'(z)} = p(z) + \frac{zp'(z)}{h_0(z)}, \quad h_0(z) = \frac{zD'}{D} \in P.$$

With some simple calculations, we get

$$(1-\beta)\frac{N(z)}{D(z)} + \beta\frac{N'(z)}{D'(z)} = p(z) + \beta p_0(z)zp'(z), \quad p_0(z) = \frac{1}{h_0(z)} \in P.$$

Then, from the given hypothesis and (5), we have, for $i = 1, 2$ and $p_0 \in P$,

$$p_i + \beta p_0(z)(zp_i') \in P(\phi).$$

That is

$$p_i(z) + \beta p_0(z)(zp_i'(z)) \prec \phi(z). \tag{12}$$

We now use Lemma 1.1 and this gives us $p_i(z) \prec \phi(z)$, $i = 1, 2$. Consequently, it follows that $p \in P_m(\phi)$ in E. □

By choosing $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, and $\phi(z) = p^*(z)$, $N(z) = zf'(z)$, we obtain several known and new results.

For the application of the above theorem, we have the following.

Theorem 2.2. *Let $\lambda > 0$ and $f(z)$ be analytic in E with $f(0) = 0, f'(0) = 1$. Then, for $0 \le \beta < 1$,*

$$\left\{(1-\beta)\left(\frac{f(z)}{z}\right)^{\lambda-1} f'(z) + \beta\left(\frac{f(z)}{z}\right)^{\lambda}\right\} \in P_m(\phi) \Longrightarrow \left(\frac{f(z)}{z}\right)^{\lambda} \in P_m(\phi), \quad z \in E.$$

Proof. By taking

$$\left(\frac{f(z)}{z}\right)^{\lambda} = h(z) = \frac{m+2}{4}h_1(z) - \frac{m-2}{4}h_2(z),$$

we proceed similarly as in Theorem 2.1 and have

$$\left(h + \frac{(1-\beta)}{\lambda}zh'\right) \in P_m(\phi).$$

This implies

$$\left(h + \frac{zh'}{\gamma}\right) \in P(\phi), \quad \gamma = \frac{\lambda}{1-\beta} \neq 0.$$

Applying a well known result (see [11]), it follows that $h_i(z) \prec \phi(z)$, that is , $h_i \in P(\phi)$, $i = 1, 2$. Consequently, $h \in P_m(\phi)$, $z \in E$. □

As a special case, we have

Corollary 2.1. *Let f be analytic in E and given by (6), satisfies $\left(\frac{f}{z}\right)^{\lambda-1} f' \in P_m(\phi)$, $\lambda > 0$. Then it follows from Lemma 2.2 that $\left(\frac{f}{z}\right)^{\lambda-1} \in P_m(\phi)$ in E.*

If $\phi(z) = \frac{1+Az}{1+Bz}$, $\lambda = \mathbb{N} = \{1, 2, 3, \dots\}$, $m = 2$, we obtain the corresponding result due to Noor [13].

Next, we study certain integral operators for our classes.

Theorem 2.3. *Let $f \in S^*(\phi)$ and define $I_\lambda(f) = F$ as*

$$F(z) = \left[\frac{1}{\lambda}\int_0^z (f(t))^{\frac{1}{\lambda}}\, t^{-1}\, dt\right]^\lambda, \quad \lambda > 0. \tag{13}$$

Then $F \in S^*(\phi)$.

Proof. Differentiation of (13) together with simple computations, we obtain

$$\frac{zf'(z)}{f(z)} = \left\{(1-\lambda)\frac{zF'(z)}{F(z)} + \lambda\left(1 + \frac{zF''(z)}{F'(z)}\right)\right\} \in P(\phi). \tag{14}$$

Let $\frac{zF'(z)}{F(z)} = p(z)$. Then $p(z)$ is analytic in E and $p(0) = 1$. Using this in (14), we obtain

$$p(z) + \lambda\frac{zp'(z)}{p(z)} \prec \phi(z). \tag{15}$$

Applying Lemma 1.2 with $\gamma = 0$, $\beta = \frac{1}{\lambda}$, we obtain the desired result that $\frac{zF'(z)}{F(z)} = p(z) \prec \phi(z)$ in E. This proves that $F \in S^*(\phi)$ in E. □

Corollary 2.2. *Let $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^\alpha$. Then $f \in S^*[A, B; \alpha]$. It can easily be seen that the class $S^*[A, B; \alpha]$ is preserved under under the integral operator defined by (13). Similar assertion holds for the class $S^*(p^*)$.*

Corollary 2.3. *Let $I_\lambda(f) = F$ be defined by (13) and $f \in S^*[A, B; \alpha]$. Then, for $\lambda \geq 1$, $F \in C[A, B; \alpha]$ in E.*

Proof. From Theorem 2.3 and Corollary 2.2,

$$(1-\lambda)\frac{zF'(z)}{F(z)} + \lambda\left(1 + \frac{zF''(z)}{F'(z)}\right) = p_1 \in P[A, B; \alpha],$$

and $\frac{zF'(z)}{F(z)} = p_2(z) \in P[A, B; \alpha]$ in E. Then, for $\lambda \geq 1$

$$\left(1 + \frac{zF''(z)}{F'(z)}\right) = \frac{1}{\lambda}p_1(z) + (1 - \frac{1}{\lambda})p_2(z). \tag{16}$$

Since $P[A, B; \alpha]$ is a convex set, it follows from (16) that $\left(1 + \frac{zF''(z)}{F'(z)}\right) \in P[A, B; \alpha]$. This proves $F \in C[A, B; \alpha]$ in E. □

Corollary 2.4. *Let $f \in S^*[A, B; \alpha]$ and $S^*[A, B; \alpha] \subset S^*(\beta)$ with $\beta = \left(\frac{1-A}{1-B}\right)^\alpha$. Then $I_\lambda(f) = F$, defined by (13) belongs to the class $S^*[1-2\alpha, -1, 1] = S^*(\gamma)$, where*

$$\gamma = \frac{2\lambda}{(2\beta - \lambda) + \sqrt{(2\beta-\lambda)^2 + 8\lambda}}. \tag{17}$$

Proof. It is given that $\frac{zf'}{f} \in P(\beta)$ in E. Now, we set

$$\frac{zF'(z)}{F(z)} = (1-\gamma)p(z) + \gamma\,, \quad F = I_\lambda(f)$$

From Corollary 2.2 together with some computations, we have

$$\mathrm{Re}\left\{(1-\gamma)p(z) + \frac{\lambda(1-\lambda)zp'(z)}{(1-\lambda)p(z) + \gamma} + (\gamma - \beta)\right\} > 0.$$

By taking $p(z) = u = u_1 + iu_2$, $zp'(z) = v = v_1 + iv_2$, we construct the functional $\Psi(u, v)$ as given in Lemma 1.7 and observe that the first two conditions are easily verified. For condition (iii), with $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we have

$$\begin{aligned}\operatorname{Re}\{\Psi(iu_2, v_1)\} =&(\gamma - \lambda) - \frac{\lambda\gamma(1-\gamma)(1+u_2^2)}{2[(1-\gamma)^2u_2^2 + \gamma^2]} \\ \leq&0, \quad \text{for } \gamma \text{ given by (17).}\end{aligned}$$

□

Thus applying Lemma 1.7, $F \in S^*(\gamma)$ in E.

2.2. **Integral Preserving Properties.** The closure properties of some important integral operators are discussed in this section.

Theorem 2.4. *Let $\lambda > 0$, $\sigma > 0$ and $f(z)$ be given by (6). Define $I_{\lambda,\sigma} = F$ as*

$$F(z) = \left[\frac{\lambda+\sigma}{z^\sigma}\int_0^z (f(t))^\lambda\, t^{\sigma-1}\, dt\right]^{\frac{1}{\lambda}}. \tag{18}$$

Then

$$\left(\frac{f(z)}{z}\right)^\lambda \in P_m(\phi) \Longrightarrow \left(\frac{F(z)}{z}\right)^\lambda \in P_m(\phi).$$

Proof. Differentiation of (18) with some simple simplifications gives

$$\begin{aligned}\lambda F^{\lambda-1}(z)F'(z) =& -\frac{\sigma(\lambda+\sigma)}{z^{\sigma+1}}\int_0^z (f(t))^\lambda\, t^{\sigma-1}\, dt\ + \frac{\lambda+\sigma}{z^\sigma}f^\lambda(z)z^{\sigma-1} \\ =& -\frac{\sigma}{z}F^\lambda(z) + \left(\frac{\lambda+\sigma}{z}f^\lambda(z)\right),\end{aligned}$$

or

$$\frac{\lambda}{\lambda+\sigma}\left(\frac{F(z)}{z}\right)^{\lambda-1}F'(z) + \frac{\sigma}{\lambda+\sigma}\left(\frac{F(z)}{z}\right)^\lambda = \left(\frac{f(z)}{z}\right)^\lambda \in P_m(\phi) \quad \text{in E.}$$

Thus, the required result follows from Theorem 2.2. □

As a special cases, this result holds true for the classes $P_m[A, B; \alpha]$ and $P_m(p^*)$.

Theorem 2.5. *Let $f, g \in S^*(\alpha)$. For δ, ν positively real and $\delta + \nu = \lambda$, let*

$$F(z) = \left[\frac{2}{z}\int_0^z (f(t))^\delta\, g(t)^\nu\, dt\right]^{\frac{1}{\lambda}}. \tag{19}$$

Then $F \in S^*(\phi)$ in E.

Proof. Differentiating (19), we get

$$\frac{\left(zF^\lambda(z)\right)'}{2} = (f(z))^\delta(g(z))^\nu. \tag{20}$$

Let $\frac{zF'(z)}{F(z)} = p(z)$. Then, from (20), it follows that

$$\frac{1}{2}F^\lambda(z)[1 + \lambda p(z)] = (f(z))^\delta(g(z))^\nu,$$

and with logarithmic differentiation and some calculations, we obtain

$$\begin{aligned} p(z) + \frac{zp'(z)}{1+\lambda p(z)} =& \frac{\delta}{\lambda}\frac{zf'(z)}{f(z)} + \frac{\nu}{\lambda}\frac{zg'(z)}{g(z)} \\ =& \frac{\delta}{\delta+\nu}p_1(z) + \frac{\nu}{\delta+\nu}p_2(z), \end{aligned}$$

where $p_1, p_2 \in P(\phi)$ in E. Using Lemma 1.2, we have the required result. □

2.3. **Some Radii Results and Properties Associated with $T_m(p^*)$.** We next establish some geometric properties of the class $T_m(p^*)$, which include radii result, closure under convex convolution, coefficient bounds and the rate of growth of coefficients.

Theorem 2.6. *The class $T_m(p^*)$ is closed under convex convolution for $|z| < r_m = \frac{2}{m}$.*

Proof. Let $f \in T_m(p^*)$. That is $\frac{zf'}{g} \in P(p^*)$ for some some $g \in S^*$. For $\psi(z) \in C$, $\psi * g \in S^*$ [21] for $|z| < r_m = \frac{2}{m}$. Then

$$\frac{z(f*\psi)(z)}{(g*\psi)(z)} = \frac{\psi(z)*zf'(z)}{(\psi*g)(z)} = \frac{\psi(z)*\frac{zf'(z)}{g(z)}g(z)}{(\psi*g)}. \tag{21}$$

In view of Lemma 1.4, the proof is completed. □

Application of Theorem 2.6

The class $f \in T_m(p^*)$ is invariant under the following integral operators.

$$\begin{aligned} &f_1(z) = \int_0^z \frac{f(t)}{t}\,dt, \quad f_2(z) = \frac{2}{z}\int_0^z f(t)\,dt, \\ &f_3(z) = \int_0^z \frac{f(t)-f(xt)}{t-xt}, \quad |x|\le 1,\ x \ne 1, \quad f_4(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)\,dt, \quad \operatorname{Re} c > 0. \end{aligned}$$

The proof follows immediately from Theorem 3.2 as we can write $f_i(z) = f(z)*\phi_i(z)$, $i = 1,2,3,4$ with

$$\begin{aligned} &\phi_1(z) = -\log(1-z), \quad \psi_2(z) = \frac{-2[z+\log(1+z)]}{z}, \\ &\psi_3(z) = \frac{1}{1-x}\log\left(\frac{1-xz}{1-z}\right), \quad \psi_4(z) = \sum_{n=1}^{\infty}\frac{1+c}{n+c}z^n, \quad \operatorname{Re} c > 0 \end{aligned}$$

and each $\psi_i(z)$ is convex in E for each $i's$.

Theorem 2.7. *Let $f \in T_m(p^*)$. Then $f(z)$ is close-to convex function (univalent) in the disc $|z| < \frac{2}{m}$.*

Proof. The proof is immediate from Lemma 1.3. □

Theorem 2.8. *Let $f \in T_m(p^*)$ be of the form (6). Then $|a_n| \le \frac{2(\frac{m}{2})_n}{m(n)!}$, $m \ge 2$, $n \ge 2$.*

Proof. Let $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in S_m^*(p^*)$ and $p(z) = 1 + \sum\limits_{n=1}^{\infty} d_n z^n \in P(p^*)$. Then, from the definition of $T_m(p^*)$, we have

$$\begin{aligned} z + \sum_{n=2}^{\infty} a_n z^n &= \left(z + \sum_{n=2}^{\infty} b_n z^n \right) \left(1 + \sum_{n=1}^{\infty} d_n z^n \right) \\ &= z + \sum_{n=2}^{\infty} \left(c_{n-1} + \sum_{j=2}^{n-1} d_j b_{n-j} + b_n \right) z^n. \end{aligned}$$

Comparing the coefficient of z^n, and applying absolute value with the well-known bound for $g \in S_m^*$, we obtain

$$(n-1)|a_n| \leq |c_{n-1}| + \sum_{j=2}^{n-1} |c_{n-j}|\,|d_j| + |b_n|. \tag{22}$$

Since $p \in P(p^*) \Longleftrightarrow \operatorname{Re} p(z) > \frac{1}{2}$ in E, it is easy to see that,

$$|c_n| \leq 1, \quad n \geq 1.$$

Applying this result in (22), and the bound for $g \in S_m^*(p^*)$ (see the procedure for the bound of S_m^* in [20]), we have

$$\begin{aligned} n|a_n| &\leq \frac{(\frac{m}{2})_{n-1}}{(n-1)!} + \frac{1}{\frac{m}{2}} \left(\frac{m}{2} \sum_{j=2}^{n-1} \frac{(\frac{m}{2})_{n-1}}{(n-1)!} \right) \\ &= \frac{(\frac{m}{2})_{n-1}}{(n-1)!} + \frac{1}{\frac{m}{2}} \frac{(\frac{m}{2})_{n-1}}{(n-2)!} \\ &= \frac{(\frac{m}{2})_{n-1}}{(n-1)!} \left(\frac{\frac{m}{2} + n - 1}{\frac{m}{2}(n-1)} \right) \\ &= \frac{(\frac{m}{2})_n}{\frac{m}{2}(n-1)!}. \end{aligned}$$

Hence, we have the result. □

Theorem 2.9. *Let $f \in T_m(p^*)$. Then f maps $|z| < r_c$ onto convex domain, where*

$$r_c = \frac{2}{m+4}.$$

Proof. From the definition of $T_m(p^*)$, we have

$$zf'(z) = g(z) p^{\frac{1}{2}}(z), \quad p \in P,\, g \in S^*.$$

Differentiating logarithmically, and using Lemma 1.6 together with Lemma 1.3, we have

$$\begin{aligned} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} &= \operatorname{Re} \frac{zg'(z)}{g(z)} + \operatorname{Re} \frac{zp'(z)}{2p(z)} \\ &\geq \frac{1 - \frac{m}{2} r}{1 - r^2} - \frac{r}{1 - r^2} \\ &= \frac{1 - (\frac{m}{2} + 1) r}{1 - r^2}. \end{aligned} \tag{23}$$

The right side of (23) is greater than 0 if $r < r_c$ and r_c is given in Theorem 2.9. This completes the proof. □

Corollary 2.5. *Let $f \in T_2(p^*)$. Then f maps $|z| < r_c$ onto convex domain, where $r_c = \frac{1}{4}$.*

Theorem 2.10. *Let $f \in T_2(p^*)$. Then for $m > 4$,*

$$a_n = \mathrm{O}(1)n^{\frac{m-2}{8}-1}, \quad (n \to \infty),$$

where $\mathrm{O}(1)$ is a constant depending on m.

Proof. By Cauchy Theorem,

$$\begin{aligned} n|a_n| \leq & \frac{1}{2\pi r^n}\int_0^{2\pi} |g(z)|\,|p(z)|^{\frac{1}{2}}\, d\theta\,, \quad p \in P,\ g \in S^*(p^*),\ z = re^{i\theta} \\ \leq & \frac{1}{r^n}\left(\frac{1}{2\pi}\int_0^{2\pi} |g(z)|^{\frac{4}{3}}\, d\theta\right)^{\frac{3}{4}} \left(\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^2\, d\theta\right)^{\frac{1}{4}} \\ \leq & \frac{C_1(m)}{r^{n-1}}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^{\frac{m+2}{6}}}\right)^{\frac{3}{4}} \left(\frac{1+3r^2}{1-r^2}\right)^{\frac{1}{4}}, \end{aligned}$$

where $C_1(m)$ is a constant depending on m, and we have used Holder's inequality, subordination property for starlike function $g(z)$ of order $\frac{1}{2}$ and the well-known result for $p \in P$. In view of Hayman's result [7] for $m > 4$ and $r = 1 - \frac{1}{n}$ $(n \to \infty)$, we have

$$n|a_n| \leq C(m)\, n^{\frac{m-2}{8}}, \quad \text{where } C(m) \text{ is a constant depending on } m,$$

which gives the required result. ☐

Theorem 2.11. *Let $f \in T_m(p^*)$. Then*

$$\mathrm{Re}\,\frac{(zf'(z))'}{g'(z)} > \frac{1}{2} \quad \text{in the disc } |z| < r_{c^*} = \frac{2}{m+2}, \quad m \geq 2.$$

Proof. From definition, there exists $g \in S^*_m(p^*)$ such that $\frac{zf'}{g} \prec p^*$. This implies that $\mathrm{Re}\,\frac{zf'(z)}{g(z)} > \frac{1}{2}$ (since $p \prec p^* \Longrightarrow \mathrm{Re}\,p(z) > \frac{1}{2}$ in E). Let $\frac{zf'(z)}{g(z)} = p(z)$ with $p \in P(\frac{1}{2})$ and $g \in S^*_m(p^*)$. Then by logarithmic differentiation and simple simplification, we have

$$\frac{(zf'(z))'}{g'(z)} = p(z) + \frac{g(z)}{g'(z)}p'(z).$$

It follows that

$$\begin{aligned} \mathrm{Re}\left(\frac{(zf'(z))'}{g'(z)} - \frac{1}{2}\right) \geq & \mathrm{Re}\left(p(z) - \frac{1}{2}\right) - \left|\frac{g(z)}{g'(z)}\right| |p'(z)| \\ \geq & \mathrm{Re}\left(p(z) - \frac{1}{2}\right)\left(1 - \frac{2}{1-r^2}\cdot\frac{r(1-r^2)}{1-\frac{m}{2}r}\right) \\ = & \mathrm{Re}\left(p(z) - \frac{1}{2}\right)\frac{1-\left(\frac{m}{2}+2\right)r}{1-\frac{m}{2}r}. \end{aligned} \tag{24}$$

where we have used Lemma 1.3 for $g \in S^*_m(p^*)$ and the well known bound for $p \in P\left(\frac{1}{2}\right)$. Thus the right side of (24) is positive if $r < r_{c^*} = \frac{2}{m+2}$. This completes the proof. ☐

Theorem 2.12. *Let $f \in T_m(p^*)$. Then*

$$\left(\frac{(1-r)^{\frac{m-2}{4}}}{(1+r)^{\frac{m+6}{4}}}\right) \leq |f'(z)| \leq \left(\frac{(1+r)^{\frac{m-2}{4}}}{(1-r)^{\frac{m+6}{4}}}\right), \quad 0 < r < 1, \tag{25}$$

and

$$\frac{2}{m+2}\left[1-\left(\frac{1-r}{1+r}\right)^{\frac{m+2}{4}}\right] \leq |f(z)| \leq \frac{2}{m+2}\left[\left(\frac{1+r}{1-r}\right)^{\frac{m+2}{4}}-1\right], \quad 0<r<1. \tag{26}$$

Proof. The first part of the result is straightforward from the distortion results for stalike function $g \in S_m^*(p^*)$ (see [16]) and $p \in P(p^*)$. For the second part, we let R_r denote the radius of the largest schlicht disc centered at the origin and is contained in the image of $|z|<r$ under $f(z)$. Then there is a point z_0 with $|z_0|=r$ such that $|f(z_0)|=R_r$. The ray from 0 to $f(z_0)$ lies entirely in the image of E and the inverse image of this ray is a curve Γ in $|z|<r$. Thus, using (25), we have

$$\begin{aligned} R_r = |f(z_0)| &= \int_\Gamma |f'(z)||dz| \\ &\geq \int_0^r \left(\frac{1-t}{1+t}\right)^{\frac{m-2}{4}} \frac{1}{(1+t)^2}. \end{aligned} \tag{27}$$

Let $\frac{1-t}{1+t}=u$. Then $dt=-\frac{2}{(1+u)^2}du$ and from (27), we have

$$|f(z_0)| \geq -\frac{1}{2}\int_1^{\frac{1-r}{1+r}} u^{\frac{m-2}{4}}\, du = \frac{2}{m+2}\left[1-\left(\frac{1-r}{1+r}\right)^{\frac{m+2}{4}}\right].$$

As for the upper bound of (26), we have

$$|f(z)| \leq \int_0^r |f'(z)|\, d\rho, \quad z=\rho e^{i\theta}.$$

Applying (25) and integrating, we obtain our result. □

As $r \to 0$ in the lower bound of (26), we have the following covering result.

Corollary 2.6. *The image of E under $f \in T_m(p^*)$ contains the schilicht disc $|z|<\frac{2}{m+2}$.*

As $r \to 0$ in the lower bound of (26) and $m=2$, we have the following covering result.

Corollary 2.7. *The image of E under $f \in T_m(p^*)$ contains the schilicht disc $|z|<\frac{1}{2}$.*

Theorem 2.13. *Let $f \in T_m(p^*)$. Then for $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z=re^{i\theta}$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\frac{(zf'(z))'}{f'(z)}\, d\theta > -\frac{\pi(m+2)}{4}. \tag{28}$$

Proof. From the definition, we have

$$\frac{(zf'(z))'}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{1}{2}\frac{zp'(z)}{p(z)},$$

for some $g \in S_m^*(p^*)$, $p \in P$. Integrating this and using Lemma 1.5 and a result due to Noor in [16] for $g \in S_m^*(p^*)$, we complete the proof. □

Remark 2.1. *Following the definition of $K(\beta)$, $\beta \geq 0$ given by Goodman in* [4]*, it is easy to see that $T_2(p^*) \subset K\left(\frac{m+2}{4}\right)$. Also, the functions $f \in T_m(p^*)$ are not necessary univalent for $m \geq 2$.*

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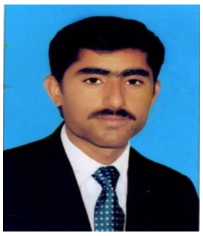

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