TWMS J. App. and Eng. Math. V.13, N.3, 2023, pp. 975-989

CONTROLLABILITY OF A SEMILINEAR NEUTRAL DYNAMIC EQUATION ON TIME SCALES WITH IMPULSES AND NONLOCAL CONDITIONS

C. DUQUE[1], H. LEIVA[2*], §

Abstract. In this paper we consider a control system governed by a neutral differential equation on time scales with impulses and nonlocal conditions. We obtain conditions under which the system is approximately controllable, on one hand, and on the other hand, the exactly controllable is also proved. Concretely, first of all, we prove the existence of solutions. After that, we prove approximate controllability assuming that the associated linear system on time scales is exactly controllable, and applying a technique developed by Bashirov et al. [8, 9, 10] where we can avoid fixed point theorems. Next, assuming certain conditions on the nonlinear term, we can apply Banach Fixed Point Theorem to prove exact controllability. Finally, we propose an example to illustrate the applicability of our results.

Keywords: Controllability, semilinear neutral dynamic equation, impulses, nonlocal conditions, time scales

AMS Subject Classification: 34K42, 93C10, 34K45, 93C23

1. Introduction and Preliminaries

Before formulating the problem to be investigated, we will make a brief introduction to the theory of differential equations on time scales, especially to clarify the notations and definitions, which will help for a better understanding of the reader. For more details about time scales theory, we recommend see the excellent monograph [16].
A time scale, denoted by $\mathbb{T}$, is any closed nonempty subset of $\mathbb{R}$ and was introduced by Stefan Hilger on his doctoral thesis ([33, 34]) with the goal to unify differential and difference calculus, particularly, the theory of dynamic equations on time scales allows to unify the study of evolution equations depending on the chosen time scale, for instance, if $\mathbb{T} = \mathbb{Z}$, we have as result difference equations, and if $\mathbb{T} = \mathbb{R}$, then we have differential equations.

[1] Universidad de Los Andes, Facultad de Ciencias, Departamento de Matemáticas. Mérida-Venezuela.
e-mail: cosduq@gmail.com; ORCID: https://orcid.org/0000-0003-0180-3289.
[2] University YachayTech, School of Mathematical and Computational Sciences. San Miguel de Urcuqui, Imbabura, Ecuador.
e-mail: hleiva@yachaytech.edu.ec; ORCID: https://orcid.org/ 0000-0002-3521-6253.
[*] Corresponding author.
§ Manuscript received: April 08, 2021; accepted: July 21, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.3

In the recent decades the theory on time scales has been attracting the attention of many mathematicians (see for instance [1, 16, 17, 30, 31, 32, 40, 43] and references therein), since this theory represents a powerful tool for applications in different areas such as economics, biomathematics, engineer, quantum physics, among others (see [4, 13, 14, 15, 16, 17, 20, 32, 36, 41]). Particularly in the last two decades the study of controllability of dynamic equations on time scales has been attracting of interest of several researches, we can mention to Bartosiewicz [7] where explored linear positive control systems on time scales, Bartosiewicz and Pawluszewicz [5, 6] reviewed linear systems on time scale, Janglajew and Pawluszewicz [35] analyzed the constrained local controllability of linear dynamic systems on times scales, M. Bohner and N. Wintz [18] studied the controllability and observability of linear systems on time scales. The aproximate and exact controllability of semilinear systems on time scales was studied by Duque, Leiva and Uzcátegui in [27, 28], Malik and Kumar in [39] established the exact controllability for time-varying neutral differential with impulses on time scales. More works can be seen in references [19, 38, 44].
As mentioned at the beginning, a time scale $\mathbb{T}$ is any arbitrary closed nonempty subset of $\mathbb{R}$. For every $t \in \mathbb{T}$, the forward and backward jump operators $\sigma, \rho : \mathbb{T} \longrightarrow \mathbb{T}$ are defined, respectively, as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. We put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$. The function $\mu : \mathbb{T} \longrightarrow [0, \infty)$ defined by $\mu(t) := \sigma(t) - t$ is known as graininess function.

We will assume that $\mathbb{T}$ has the topology inherited from standard topology on the real numbers. The time scale interval $[a, b]_{\mathbb{T}}$ is defined by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$, with $a, b \in \mathbb{T}$ and similarly is defined open intervals and open neighborhoods.

Definition 1.1 ([16])**.** *A function $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ is said to be right dense continuous or just rd-continuous, if f is continuous at every right-dense point $t \in \mathbb{T}$ and $\lim\limits_{s \to t^-} f(s)$ exists (finite) for every left-dense point $t \in \mathbb{T}$.*

The class of all rd-continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$. If $f : \mathbb{T} \to \mathbb{R}^n$ is a function, then we define the function $f \circ \sigma : \mathbb{T} \to \mathbb{R}^n$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e., $f^\sigma = f \circ \sigma$. We define el set $\mathbb{T}^\kappa$ by

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty \end{cases}$$

Definition 1.2 ([16])**.** *A function $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ is called delta differentiable (or simply Δ-differentiable) at $t \in \mathbb{T}^\kappa$ provided there exists a number $f^\Delta(t)$ with the property that given $\varepsilon > 0$, there is a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ for some $\delta > 0$ such that*

$$\left\| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right\| \leq |\sigma(t) - s)| \text{ for all } s \in U.$$

In this case the number $f^\Delta(t)$ will be call the Δ-derivative of f in t.

If f is Δ-differentiable at $t \in \mathbb{T}^\kappa$, then is easy to show that (see [16] Thm. 1.16)

$$f^\Delta(t) = \begin{cases} \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} & \text{if} \quad \sigma(t) > t \\ \\ \lim\limits_{s \to t} \frac{f(t) - f(s)}{t - s} & \text{if} \quad \sigma(t) = t. \end{cases}$$

Definition 1.3 ([16]). *A function $F : \mathbb{T} \longrightarrow \mathbb{R}^n$ is called an antiderivative of $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ if $F^{\Delta}(t) = f(t)$ for $t \in \mathbb{T}^{\kappa}$. The Cauchy integral is defined by*

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s), \quad t, s \in \mathbb{T}.$$

Where F is an antiderivative of f.

It is known that every rd-continuous function has an antiderivative (see [16] Thm. 1.74). A function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0 \quad t \in \mathbb{T}$ and positively regressive if $1 + \mu(t)p(t) > 0 \quad t \in \mathbb{T}$. We will denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions and $\mathcal{R}^+$ the set of all positive regressive and rd-continuous functions.

Definition 1.4 ([16]). *If $p \in \mathcal{R}$, then the generalized exponential function is defined by*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),$$

where

$$\xi_\mu(z) := \begin{cases} \frac{1}{\mu}\text{Log}(1 + \mu z) & \text{if} \quad \mu > 0 \\ z, & \text{if} \quad \mu = 0. \end{cases}$$

Here $z \in \mathbb{C}_\mu := \{z \in \mathbb{C} : z \neq 1/\mu\}$ and $\text{Log} z = \log |z| + i \arg z, \ -\pi < \arg z \leq \pi$. Particularly $e_p(t, 0)$ will be denoted by $e_p(t)$. The function $e_p(t, s)$ satisfy the following properties

Theorem 1.1 ([16] Thm. 2.36). *If $p, q \in \mathbb{R}$, then*
a) $e_0(t, s) \equiv 1, \ e_p(t, t) \equiv 1,$
b) $e_p(t, r)e_p(r, s) = e_p(t, s),$
c) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s),$
d) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t),$
e) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s),$
f) $e_p^{\Delta}(t, s) = p(t)e_p(t, s),$

where $(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t)$, $(\ominus p)(t) = \frac{-p(t)}{1+\mu(t)p(t)}$ and $(p \ominus q)(t) := (p \oplus (\ominus q))(t)$.

Let A be a $n \times m$ matrix valued function on $\mathbb{T}$.

Definition 1.5 ([16]). *We say that A is rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$, and the class of all such rd-continuous $n \times m$ matrix valued functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^{n\times m})$.*

Definition 1.6 ([16]). *We say that A is differentiable on $\mathbb{T}$ provided each entry of A is differentiable on $\mathbb{T}$. In this case, the Δ-derivative of $A(t)$ is defined as*

$$A^{\Delta}(t) = (a_{ij}^{\Delta}(t)), \quad (i = 1, 2, \ldots, n)(j = 1, 2, \ldots m),$$

where $A(t) = (a_{ij}(t))$.

Definition 1.7 ([16]). *A is called regressive (with respect to $\mathbb{T}$) provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^{\kappa}$, and the class of all such regressive and rd-continuous functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.*

Let $t_0 \in \mathbb{T}$ and A be a $n \times n$ regressive matrix valued function defined on $\mathbb{T}$. Then, the unique solution of the initial value problem

$$X^{\Delta} = A(t)X, \quad X(t_0) = I,$$

is called the matrix exponential function and it is denoted by $e_A(t,t_0)$. If $t_0 = 0$, then we denote it by $e_A(t)$. The matrix exponential function has the following properties

Theorem 1.2 ([16] Thm 5.21). *If $A, B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, then*
a) $e_0(t,s) \equiv I$ *and* $e_A(t,t) \equiv I$,
b) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t,s)$,
c) $e_A(t,s) = e_A(s,t)^{-1}$,
d) $e_A(t,s)e_A(s,r) = e_A(t,r)$.

Let $L^2_\Delta(E,\mathbb{R})$ denote the space of the functions $f : E \longrightarrow \mathbb{R}$ Lebesgue Δ-measurable and absolutely continuous such that $\int_E |f(s)|^2 \Delta s < \infty$ (see [2]), where E is an arbitrary closed interval on time scale $\mathbb{T}$. The set $L^2_\Delta(E,\mathbb{R})$ is a Hilbert space endowed with the inner product given by

$$\langle f, g\rangle_{L^2_\Delta(E,\mathbb{R})} = \int_E f(s)g(s)\Delta s.$$

Denote by $L^2_\Delta(E,\mathbb{R}^m)$ the space of the function $f : E \longrightarrow \mathbb{R}^m$, $f(t) = (f_1(t), \ldots, f_m(t))$, such that $f_i \in L^2_\Delta(E,\mathbb{R})$. The space $L^2_\Delta(E,\mathbb{R}^m)$ is a Hilbert space endowed with the inner product given by

$$\langle f, g\rangle_{L^2_\Delta(E,\mathbb{R}^m)} = \int_E \langle f(s), g(s)\rangle_{\mathbb{R}^m} \Delta s.$$

2. Setting the Problem

Once having clear the notations and definitions corresponding to the theory of differential equations on time scales, we will formulate the problem to be investigated in this work. Without further ado, we will study the existence of solutions and the controllability of the following semilinear dynamic equations of neutral type on time scales with impulses and nonlocal conditions

$$\begin{cases} [z(t) - f(t, z_\tau(t))]^\Delta = A(t)z(t) + B(t)u(t) + h(t, z_\tau(t)), \quad t \in [t_0, b]_\mathbb{T} \setminus \{t_1, t_2, \ldots, t_p\} \\ z(s) = g(z)(s) + \phi(s), \quad s \in [\tau(t_0), t_0]_\mathbb{T} \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k^-)), \quad k = 1, 2, \ldots, p, \end{cases} \tag{1}$$

where the nonlocal condition $z(s) = g(z)(s) + \phi(s), \quad s \in [\tau(t_0), t_0]_\mathbb{T}$ means

$$z(s) = g\left(z\Big|_{[\tau(t_0),t_0]_\mathbb{T}}\right)(s) + \phi(s), \quad s \in [\tau(t_0), t_0]_\mathbb{T},$$

$z(t) \in \mathbb{R}^n$ is the state function, $z_\tau(t) = z(\tau(t))$ where $\tau : \mathbb{T} \longrightarrow \mathbb{T}$ is the delay function and is an increasing and unbounded function on $\mathbb{T}$ such that $\tau(t) \leq t$ for $t \in \mathbb{T}$ (see [31]). $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, $B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times m})$, and the control $u \in L^2_\Delta([t_0, b]_\mathbb{T}, \mathbb{R}^m)$. The functions $f, h : [t_0, b]_\mathbb{T} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are suitably defined functions satisfying certain conditions that will be specified later, and $J_k : [0, \infty)_\mathbb{T} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $k = 1, 2, \ldots, p$, are continuous and represents the impulsive effect in the system (1), in this case, we are considering that the system can undergo drastic changes of their state at a given time. These alterations in state might be due to certain external factors, which cannot be well described by pure time scales models, (see, for instance, [3, 12, 26, 29, 37, 38, 43] and reference therein). The points $\{t_1, t_2, \ldots, t_p\} \subset \mathbb{T}$ satisfies $t_0 < t_1 < t_2 < \cdots < t_p < b$, $z(t_k^+)$ and $z(t_k^-)$ represents right and left limits with respect to the time scale, and in addition, if t_k is right-scattered then $z(t_k^+) = z(t_k)$, whereas if t_k is left-scattered, then $z(t_k^-) = z(t_k)$. Moreover, it is usually assumed that the solution z should be left-continuous (see [12, 29, 37]), in this case $z(t_k^+) = z(t_k) + J(t_k, z(t_k))$, $k = 1, 2, \ldots, p$. On the other hand, if t_k is right scattered, then $J(t_k, z(t_k)) = 0$, in other words, it make sense to consider impulses at right-dense points only. $\phi \in \mathcal{PC}_{rd}$, where $\mathcal{PC}_{rd}$ is the Banach space

$$\mathcal{PC}_{rd} = \big\{\phi : [\tau(t_0), t_0]_{\mathbb{T}} \longrightarrow \mathbb{R}^n : \phi \text{ is rd-continuous except in a finite number of points } \theta_k, \quad k = 1, \dots, p, \text{ where } \phi(\theta_k^+), \phi(\theta_k^-) \text{ exist and } \phi(\theta_k^-) = \phi(\theta_k)\big\}$$

endowed with the norm

$$\|\phi\|_{\mathcal{PC}_{rd}} = \sup\big\{\, |\phi(t)| : t \in [\tau(t_0), t_0]_{\mathbb{T}}\big\}.$$

The continuous function $g : \mathcal{PC}_{rd} \longrightarrow \mathcal{PC}_{rd}$ represent the nonlocal conditions, this function acts as a feedback operator which adjusts a part of the past when the initial function is present, or even, the whole past when the function ϕ is absent according to some precise future requirements (see [21]). The advantage of using nonlocal conditions is that measurements at more places can be incorporated to get better models. For more details and physical interpretations about non local condition see [21, 22, 23, 24, 25, 42] and references therein.

The main goal of this paper is to study the controllability of system (1). Specifically, we shall show that under certain conditions the controllability of the associated linear system implies the controllability of the semilinear neutral differential equations with impulses and nonlocal conditions on time scales. In order to prove this asseveration, we impose some conditions on the nonlinear terms presented in the system, and applying a direct approach developed by A.E. Bashirov et al. ([8, 9, 10]) to avoid fixed point theorems, we prove the approximate controllability of the system; then assuming different conditions on the nonlinear terms presented in the system, we can apply Banach Fixed Point Theorem to achieve exact controllability. Finally, we consider an example where our results can be applied.

3. Existence and Uniqueness of Solutions

In this section we will show that system (1) is well posed, i.e., we will prove that system (1) admits a unique solution defined on $[\tau(t_0), b]_{\mathbb{T}}$. We shall also fix the basis to study the controllability of system (1). In this regards, we define the space

$$\mathcal{PC}_p = \big\{\phi : [\tau(t_0), b]_{\mathbb{T}} \longrightarrow \mathbb{R}^n : \phi|_{[\tau(t_0), t_0]_{\mathbb{T}}} \in \mathcal{PC}_{rd} \text{ and } \phi|_{[t_0, b]_{\mathbb{T}}} \text{ is rd-continuous except in a finite number of points } t_k, k = 1, 2, \dots, p, \text{ where } \phi(t_k^-), \phi(t_k^+) \text{ exist and } \phi(t_k) = \phi(t_k^-)\big\}$$

A straightforward computation shows the following theorem

Theorem 3.1. *Consider a control $u \in L^2_\Delta([t_0, b]_{\mathbb{T}}; \mathbb{R}^m)$. z is solution of system* (1) *if and only if z satisfy the integral equation*

$$z(t) = \begin{cases} g(z)(t) + \phi(t), \quad t \in [\tau(t_0), t_0]_{\mathbb{T}}, \\ f(t, z_\tau(t)) + e_A(t, t_0)\big[g(z)(t_0) + \phi(t_0) - f(t_0, g_\tau(z)(t_0) + \phi_\tau(t_0)\big] \\ + \displaystyle\int_{t_0}^t e_A(t, \sigma(s))A(s)f(s, z_\tau(s))\Delta s + \int_{t_0}^t e_A(t, \sigma(s))B(s)u(s)\Delta s \\ + \displaystyle\int_{t_0}^t e_A(t, \sigma(s))h(s, z_\tau(s))\Delta s + \sum_{0<t_k<t} e_A(t, t_k)J_k(t_k, z(t_k)), \quad t \in [t_0, b]_{\mathbb{T}}, \end{cases} \tag{2}$$

where $g_\tau(z)(t) = g(z)(\tau(t))$.

Henceforth, we will assume the following hypotheses

H1) There exist positive constants L_f and L_h such that for all $t \in [t_0, b]_{\mathbb{T}}$, $z, \tilde{z} \in \mathbb{R}^n$

$$|f(t,z) - f(t,\tilde{z})| \leq L_f |z - \tilde{z}|,$$
$$|h(t,z) - h(t,\tilde{z})| \leq L_h |z - \tilde{z}|,$$

H2) there exist nonnegative constants d_k, $k = 1, 2, \ldots, p$ such that for all $t \in [0, \infty)_{\mathbb{T}}$, $z, \tilde{z} \in \mathbb{R}^n$

$$|J_k(t,z) - J_k(t,\tilde{z})| \leq d_k |z - \tilde{z}|,$$

H3) there exists a nonnegative constant L_g such that for all $\phi, \psi \in \mathcal{PC}_{rd}$

$$\|g(\phi) - g(\psi)\|_{\mathcal{PC}_{rd}} \leq L_g \|\phi - \psi\|_{\mathcal{PC}_{rd}}.$$

H4) $L_f + M\left[L_g + L_f L_g + \|A\| L_f b + L_h b + \sum_{k=1}^{p} d_k\right] < 1$, where $M = \sup\{\|e_A(t, \sigma(s))\| : t, s \in [t_0, b]_{\mathbb{T}}\}$ and $\|A\| = \max\{\|A(t)\| : t \in [t_0, b]_{\mathbb{T}}\}$.

Theorem 3.2. *Suppose that H1)- H4) hold. Then for $\phi \in \mathcal{PC}_{rd}$ the system* (1) *has a unique solution defined on $[\tau(t_0), b]_{\mathbb{T}}$.*

Proof. Let $u \in L^2_\Delta([t_0, b]_{\mathbb{T}}; \mathbb{R}^m)$ and we define the operator $\mathcal{T} : \mathcal{PC}_p \longrightarrow \mathcal{PC}_p$ by

$$\mathcal{T}(t) = \begin{cases} g(z)(t) + \phi(t), \quad t \in [\tau(t_0), t_0]_{\mathbb{T}}, \\ f(t, z_\tau(t)) + e_A(t, t_0)\big[g(z)(t_0) + \phi(t_0) - f(t_0, g_\tau(z)(t_0) + \phi_\tau(t_0)\big] \\ + \int_{t_0}^{t} e_A(t, \sigma(s)) A(s) f(s, z_\tau(s)) \Delta s + \int_{t_0}^{t} e_A(t, \sigma(s)) B(s) u(s) \Delta s \\ + \int_{t_0}^{t} e_A(t, \sigma(s)) h(s, z_\tau(s)) \Delta s + \sum_{0 < t_k < t} e_A(t, t_k) J_k(t_k, z(t_k)), \quad t \in [t_0, b]_{\mathbb{T}}. \end{cases} \tag{3}$$

If $t \in [\tau(t_0), t_0]_{\mathbb{T}}$ then

$$\begin{aligned} |(\mathcal{T}z)(t) - (\mathcal{T}\tilde{z})(t)| &= |g(z)(t) - g(\tilde{z})(t)| \leq \left\| (g(z) - g(\tilde{z}))\Big|_{[\tau(t_0), t_0]_{\mathbb{T}}} \right\|_{\mathcal{PC}_{rd}} \\ &\leq L_g \left\| (z - \tilde{z})\Big|_{[\tau(t_0), t_0]_{\mathbb{T}}} \right\|_{\mathcal{PC}_{rd}} \leq L_g \|z - \tilde{z}\|_{\mathcal{PC}_p}. \end{aligned}$$

If $t \in [t_0, b]_{\mathbb{T}}$, then

$$
\begin{aligned}
|(\mathcal{T}z)(t)-(\mathcal{T}\tilde{z})(t)| &\leq |f(t,z_\tau(t))-f(t,\tilde{z}_\tau(t))|+\|e_A(t,t_0)\|\left[\,|g(z)(t_0)-g(\tilde{z})(t_0)|\right.\\
&+ \left.|f(t_0,g_\tau(z)(t_0)+\phi_\tau(t_0))-f(t_0,g_\tau(\tilde{z})(t_0)+\phi_\tau(t_0))|\,\right]\\
&+ \int_{t_0}^{t}\|e_A(t,\sigma(s))\|\,\|A(s)\|\,|f(s,z_\tau(s))-f(s,\tilde{z}_\tau(s))|\,\Delta s\\
&+ \int_{t_0}^{t}\|e_A(t,\sigma(s))\|\,|h(s,z_\tau(s))-h(s,\tilde{z}_\tau(s))|\,\Delta s\\
&+ \sum_{0<t_k<t}\|e_A(t,t_k)\|\,|J_k(t_k,z(t_k))-J_k(t_k,\tilde{z}(t_k))|\\
&\leq L_f\,|z_\tau(t)-\tilde{z}_\tau(t)|+M\big[L_g\,\|z-\tilde{z}\|_{\mathcal{PC}_p}+L_f\,|g_\tau(z)(t_0)-g_\tau(\tilde{z}(t_0))|\,\big]\\
&+ M\,\|A\|\,L_f\int_{t_0}^{t}|z_\tau(s)-\tilde{z}_\tau(s)|\,\Delta s+ML_h\int_{t_0}^{t}|z_\tau(s)-\tilde{z}_\tau(s)|\,\Delta s\\
&+ M\sum_{0<t_k<t}d_k\,|z(t_k)-\tilde{z}(t_k)|\\
&\leq \left(L_f+M\left[L_g+L_fL_g+\|A\|\,L_fb+L_hb+\sum_{k=1}^{p}d_k\right]\right)\|z-\tilde{z}\|_{\mathcal{PC}_p}.
\end{aligned}
$$

Thus,

$$
\|\mathcal{T}z-\mathcal{T}\tilde{z}\|_{\mathcal{PC}_p}\leq\left(L_f+M\left[L_g+L_fL_g+\|A\|\,L_fb+L_hb+\sum_{k=1}^{p}d_k\right]\right)\|z-\tilde{z}\|_{\mathcal{PC}_p},
$$

so, the operator $\mathcal{T}$ satisfies all the assumptions of the Banach contraction theorem, and therefore $\mathcal{T}$ has only one fixed point in the space $\mathcal{PC}_p$ which is the solution of problem (1). □

4. Controllability of System (1)

As we have mentioned from the beginning, our fundamental objective is to study the controllability of the system (1), both the approximate controllability and the exact controllability; in this regard, we will begin by giving the corresponding definitions of controllability.

Definition 4.1. *System* (1) *is said to be approximately controllable on* $[t_0,b]_{\mathbb{T}}$ *if for every* $\phi\in\mathcal{PC}_{rd}$, $z^1\in\mathbb{R}^n$ *and* $\varepsilon>0$ *there exists* $u\in L^2_\Delta([t_0,b]_{\mathbb{T}};\mathbb{R}^m)$ *such that the solution of* (1) *corresponding to u verifies:*

$$
z(t_0)=g(z)(t_0)+\phi(t_0)\quad\textit{and}\quad\left|z(b)-z^1\right|<\varepsilon.
$$

Definition 4.2. *System* (1) *is said to be exactly controllable on* $[t_0,b]_{\mathbb{T}}$ *if for every* $\phi\in\mathcal{PC}_{rd}$, $z^1\in\mathbb{R}^n$, *there exists* $u\in L^2_\Delta([t_0,b]_{\mathbb{T}};\mathbb{R}^m)$ *such that the solution* $z(t)$ *of* (1) *corresponding to u verifies:*

$$
z(t_0)=g(z)(t_0)+\phi(t_0)\quad\textit{and}\quad z(b)=z^1.
$$

Corresponding to the nonlinear system (1), we shall consider also the linear system

$$
\begin{cases}
z^\Delta(t)=A(t)z(t)+B(t)u(t), & t\in[t_0,b]_{\mathbb{T}}\\
z(t_0)=z^0.
\end{cases}
\tag{4}
$$

The solution of (4) is given by

$$z(t) = e_A(t, t_0)z^0 + \int_{t_0}^{t} e_A(t, \sigma(s))B(s)u(s)\Delta s. \tag{5}$$

Definition 4.3 ([27]). *For the linear system* (4) *we define the following concepts:*
1) The controllability map is defined as $\mathcal{B}^b : L^2_\Delta((t_0, b]_\mathbb{T}, \mathbb{R}^m) \to \mathbb{R}^n$ *by*

$$\mathcal{B}^b u = \int_{t_0}^{b} e_A(t, \sigma(s))B(s)u(s)\Delta s. \tag{6}$$

2) The Grammian map is defined by $\mathcal{L}_{\mathcal{B}^b} = \mathcal{B}^b\mathcal{B}^{b*}$.

Proposition 4.1 ([27]). *The adjoint* $\mathcal{B}^{b*} : \mathbb{R}^n \longrightarrow L^2_\Delta([t_0, b]_\mathbb{T}, \mathbb{R}^m)$ *of the operator* $\mathcal{B}^b$ *is given by*

$$(\mathcal{B}^{b*}z)(t) = B^*(t)e^*_A(b, \sigma(t))z$$

and

$$\mathcal{L}_{\mathcal{B}^b} z = \int_{t_0}^{b} e_A(b, \sigma(s))B(s)B^*(s)e^*_A(b, \sigma(s))z\Delta s.$$

Theorem 4.1 ([27]). *System* (4) *is controllable on* $[t_0, b]_\mathbb{T}$ *if and only if one of the following statement holds:*
1) $\mathrm{Rang}(\mathcal{B}^b) = \mathbb{R}^n$,
2) There exists $\gamma > 0$ *such that* $\langle L_{\mathcal{B}^b} z, z\rangle > 0$, *for every* $z \in \mathbb{R}^n \setminus \{0\}$,
3) There exists $\gamma > 0$ *such that* $\left\|\mathcal{B}^{b*}z\right\|_{L^2_\Delta} \geq \gamma \|z\|$ *for every* $z \in \mathbb{R}^n$,
4) $\mathcal{L}_{\mathcal{B}^b}$ *is invertible. Moreover,* $\mathcal{G} = \mathcal{B}^b\mathcal{L}^{-1}_{\mathcal{B}^b}$ *is a right inverse of* $\mathcal{B}^b$, *and the control* $u \in L^2_\Delta([t_0, b]_\mathbb{T}, \mathbb{R}^m)$ *steering the system from the initial state* z^0 *to a final state* z^1 *is given by*

$$u = \mathcal{B}^{b*}\mathcal{L}^{-1}_{\mathcal{B}^b}(z^1 - e_A(b, t_0)z^0). \tag{7}$$

5. Approximate controllability

In this section we will give condition to get the approximate controllability of system (1).

Theorem 5.1. *Suppose that* b *is left-dense,* $|f(t, \varphi)| \leq M_f$, $|h(t, \varphi)| \leq M_h$ *for* $t \in [\tau(t_0), b]_\mathbb{T}$, $\varphi \in \mathcal{PC}_p$, *and* (4) *is controllable on* $[\delta, b]_\mathbb{T}$ *for each* $\delta \in [t_0, b]_\mathbb{T}$, *then system* (1) *is approximate controllable on* $[t_0, b]_\mathbb{T}$.

Proof. Given $\phi \in \mathcal{PC}_{rd}$, a final state z^1 and $\varepsilon > 0$, we want to find a control $u^\varepsilon \in L^2_\Delta([t_0, b]_\mathbb{T}; \mathbb{R}^m)$ steering the system (1) to an ε-neighborhood of z^1 and on time b. Indeed, consider a control $u \in L^2_\Delta([t_0, b]_\mathbb{T}; \mathbb{R}^m)$ arbitrary but fixed and the corresponding solution $z(t) = z(t, t_0, \phi, u)$ of system (1). Since b is left-dense then given $\varepsilon > 0$ there exists $\delta_\varepsilon \in [t_0, b]_\mathbb{T}$ such that $b - \delta_\varepsilon < \frac{\varepsilon}{M(\|A\|M_f + M_h)}$.

We define the control $u^\varepsilon \in L^2_\Delta([t_0, b]_\mathbb{T}; \mathbb{R}^m)$ as follow:

$$u^\varepsilon(t) = \begin{cases} u(t) & \text{if} \quad t \in [t_0, \delta_\varepsilon]_\mathbb{T}, \\ \tilde{u}(t) & \text{if} \quad t \in [\delta_\varepsilon, b]_\mathbb{T}, \end{cases} \tag{8}$$

where $\tilde{u}(t) = B^*(t)e^*_A(b, \sigma(t))\mathcal{L}^{-1}_{\mathcal{B}^b}(z^1 - e_A(b, \delta_\varepsilon)z(\delta_\varepsilon))$ is the control steering the system (4) from the initial state $z(\delta_\varepsilon)$ to the final state z^1 on $[\delta_\varepsilon, b]_\mathbb{T}$.

Then the corresponding solution $z^{\delta_\varepsilon}(t) = z^{\delta_\varepsilon}(t, t_0, \phi, u^\varepsilon)$ of the problem (1) at time b can be expressed by

$$\begin{aligned}
z^{\delta_\varepsilon}(b) &= f(b, z_\tau^{\delta_\varepsilon}(b)) + e_A(b,t_0)\big[g(z^{\delta_\varepsilon})(t_0) + \phi(t_0) - f(t_0, g_\tau(z^{\delta_\varepsilon})(t_0) + \phi_\tau(t_0))\big] \\
&+ \int_{t_0}^{b} e_A(b,\sigma(s))A(s)f(s, z_\tau^{\delta_\varepsilon}(s))\Delta s + \int_{t_0}^{b} e_A(b,\sigma(s))B(s)u^\varepsilon(s)\Delta s \\
&+ \int_{t_0}^{b} e_A(b,\sigma(s))h(s, z_\tau^{\delta_\varepsilon}(s))\Delta s + \sum_{0<t_k<\delta} e_A(b,t_k)J_k(t_k, z^{\delta_\varepsilon}(t_k)) \\
&= f(b, z_\tau^{\delta_\varepsilon}(b)) + e_A(b,\delta_\varepsilon)\Big\{ e_A(\delta_\varepsilon, t_0)\big[g(z^{\delta_\varepsilon})(t_0) + \phi(t_0) - f(t_0, g_\tau(z^{\delta_\varepsilon})(t_0) + \phi_\tau(t_0))\big] \\
&+ \int_{t_0}^{\delta_\varepsilon} e_A(\delta_\varepsilon,\sigma(s))A(s)f(s, z_\tau^{\delta_\varepsilon}(s))\Delta s + \int_{t_0}^{\delta_\varepsilon} e_A(\delta_\varepsilon,\sigma(s))B(s)u(s)\Delta s \\
&+ \int_{t_0}^{\delta_\varepsilon} e_A(\delta_\varepsilon,\sigma(s))h(s, z_\tau^{\delta_\varepsilon}(s))\Delta s + \sum_{0<t_k<\delta_\varepsilon} e_A(\delta_\varepsilon,t_k)J_k(t_k, z^{\delta_\varepsilon}(t_k)) \Big\} \\
&+ \int_{\delta_\varepsilon}^{b} e_A(b,\sigma(s))A(s)f(s, z_\tau^{\delta_\varepsilon}(s))\Delta s + \int_{\delta_\varepsilon}^{b} e_A(b,\sigma(s))B(s)\tilde{u}(s)\Delta s \\
&+ \int_{\delta_\varepsilon}^{b} e_A(b,\sigma(s))h(s, z_\tau^{\delta_\varepsilon}(s))\Delta s \\
&= f(b, z_\tau^{\delta_\varepsilon}(b)) + e_A(b,\delta_\varepsilon)\big[z^{\delta_\varepsilon}(\delta_\varepsilon) - f(\delta_\varepsilon, z_\tau^{\delta_\varepsilon}(\delta_\varepsilon))\big] + \int_{\delta_\varepsilon}^{b} e_A(b,\sigma(s))A(s)f(s, z_\tau^{\delta_\varepsilon}(s))\Delta s \\
&+ \int_{\delta_\varepsilon}^{b} e_A(b,\sigma(s))B(s)\tilde{u}(s)\Delta s + \int_{\delta_\varepsilon}^{b} e_A(b,\sigma(s))h(s, z_\tau^{\delta_\varepsilon}(s))\Delta s.
\end{aligned}$$

On the other hand, the corresponding solution $y(t) = y(t, \delta_\varepsilon, y(\delta_\varepsilon), \tilde{u})$ of the initial value problem (4) at time $t = b$, is given by

$$y(b) = e_A(b,\delta_\varepsilon)y(\delta_\varepsilon) + \int_{\delta_\varepsilon}^{b} e_A(b,\sigma(s))B(s)\tilde{u}(s)\Delta s = z^1.$$

Taking $y(\delta_\varepsilon) = e_A(\delta_\varepsilon, b)f(b, z_\tau^{\delta_\varepsilon}(b)) - f(\delta_\varepsilon, z_\tau^{\delta_\varepsilon}(\delta_\varepsilon)) + z^{\delta_\varepsilon}(\delta_\varepsilon)$, we have that

$$y(b) = f(b, z_\tau^{\delta}(b)) + e_A(b,\delta_\varepsilon)\big[z^{\delta_\varepsilon}(\delta_\varepsilon) - f(\delta, z_\tau^{\delta_\varepsilon}(\delta))\big] + \int_{\delta}^{b} e_A(b,\sigma(s))B(s)\tilde{u}(s)\Delta s,$$

so

$$z^{\delta_\varepsilon}(b) = z^1 + \int_{\delta_\varepsilon}^{b} e_A(b,\sigma(s))A(s)f(s, z_\tau^{\delta_\varepsilon}(s))\Delta s + \int_{\delta_\varepsilon}^{b} e_A(b,\sigma(s))h(s, z_\tau^{\delta_\varepsilon}(s))\Delta s.$$

Hence

$$\begin{aligned}
\left| z^{\delta_\varepsilon}(b) - z^1 \right| &\leq \int_{\delta_\varepsilon}^{b} \|e_A(b,\sigma(s))\| \, \|A(s)\| \left| f(s, z_\tau^{\delta_\varepsilon}(s)) \right| \Delta s + \int_{\delta_\varepsilon}^{b} \|e_A(b,\sigma(s))\| \left| h(s, z_\tau^{\delta_\varepsilon}(s)) \right| \Delta s \\
&\leq M\big(\|A\| \, M_f + M_h \big)(b - \delta_\varepsilon) < \varepsilon.
\end{aligned}$$

This conclude the proof. □

6. Exact controllability

In this section we will prove that under certain conditions and by using the Banach fixed point Theorem, the system (1) is exactly controllable.

Theorem 6.1. *Suppose that system* (4) *is exactly controllable and*

$$\left(1+\|B\|^2 M^2 \left\|\mathcal{L}_{\mathcal{B}^b}^{-1}\right\| b\right)\left[L_f + ML_g + ML_fL_g + M\|A\| L_f b + ML_h b + M\sum_{k=1}^{p} d_k\right] < 1.$$

Then problem (1) *is exactly controllable.*

Proof. Suppose for a moment that system (1) is exactly controllable. So, for every $\phi \in \mathcal{PC}_{rd}$ and for every $z^1 \in \mathbb{R}^n$ there exist $u \in L^2_\Delta([t_0,b]_\mathbb{T};\mathbb{R}^m)$ such that the corresponding solution $z(t) = z(t,\phi,u)$ of (1) satisfies $z(b) = z^1$, where

$$\begin{aligned} z(b) &= f(b, z_\tau(b)) + e_A(b,t_0)\big[g(z)(t_0) + \phi(t_0) - f(t_0, g_\tau(z)(t_0) + \phi_\tau(t_0))\big] \\ &+ \int_{t_0}^{b} e_A(b,\sigma(s))A(s)f(s,z_\tau(s))\Delta s + \int_{t_0}^{b} e_A(t,\sigma(s))B(s)u(s)\Delta s \\ &+ \int_{t_0}^{b} e_A(b,\sigma(s))h(s,z_\tau(s))\Delta s + \sum_{0<t_k<b} e_A(b,t_k)J_k(t_k,z(t_k)). \end{aligned}$$

Consider the controllability operator given by (6), then

$$\begin{aligned} \mathcal{B}^b u &= z_1 - f(b,z_\tau(s)) - e_A(b,t_0)\big[g(z)(t_0) + f(t_0, g_\tau(z)(t_0) + \phi_\tau(t_0))\big] \\ &- \int_{t_0}^{b} e_A(b,\sigma(s))A(s)f(s,z_\tau(s))\Delta s - \int_{t_0}^{b} e_A(b,\sigma(s))h(s,z_\tau(s))\Delta s \\ &- \sum_{0<t_k<b} e_A(b,t_k)J_k(t_k,z(t_k)). \end{aligned}$$

If we define $\mathscr{L} : \mathcal{PC}_p([\tau(t_0),b]_\mathbb{T};\mathbb{R}^n) \longrightarrow \mathbb{R}^n$ by

$$\begin{aligned} \mathscr{L}(z) &= z_1 - f(b,z_\tau(s)) - e_A(b,t_0)\big[g(z)(t_0) + f(t_0, g_\tau(z)(t_0) + \phi_\tau(t_0))\big] \\ &- \int_{t_0}^{b} e_A(b,\sigma(s))A(s)f(s,z_\tau(s))\Delta s - \int_{t_0}^{b} e_A(b,\sigma(s))h(s,z_\tau(s))\Delta s \\ &- \sum_{0<t_k<b} e_A(b,t_k)J_k(t_k,z(t_k)), \end{aligned}$$

we get that

$$u = \mathcal{G}\mathscr{L}(z).$$

Therefore, the controllability of system (1) will be equivalent to find a fixed points for the following operator

$$\mathscr{T} : \mathcal{PC}_p([\tau(t_0),t_0]_\mathbb{T};\mathbb{R}^n) \longrightarrow \mathcal{PC}_p([\tau(t_0),t_0]_\mathbb{T};\mathbb{R}^n)$$

given by

$$
\begin{aligned}
(\mathscr{T}z)(t) = & f(t, z_\tau(t)) + e_A(t,t_0)\big[g(z)(t_0) + \phi(t_0) - f(t_0, g_\tau(z)(t_0) + \phi_\tau(t_0))\big] \\
& + \int_{t_0}^{t} e_A(t,\sigma(s))A(s)f(s,z_\tau(s))\Delta s + \int_{t_0}^{t} e_A(t,\sigma(s))\mathcal{G}\mathscr{L}(z)(s)\Delta s \\
& + \int_{t_0}^{t} e_A(t,\sigma(s))h(s,z_\tau(s))\Delta s + \sum_{0<t_k<t} e_A(t,t_k)J_k(t_k,z(t_k)).
\end{aligned}
$$

Now, let $z, \tilde{z} \in \mathcal{PC}_p$, then

$$
\begin{aligned}
|(\mathscr{T}z)(t) - (\mathscr{T}z)(t)| \leq & |f(t,z_\tau(t)) - f(t,\tilde{z}_\tau(t))| + \|e_A(t,t_0)\| \left[\, |g(z)(t_0) - g(\tilde{z})(t_0)| \right. \\
& \left. + |f(t_0, g_\tau(z)(t_0) + \phi_\tau(t_0)) - f(t_0, g_\tau(\tilde{z})(t_0) + \phi_\tau(t_0))| \,\right] \\
& + \int_{t_0}^{t} \|e_A(t,\sigma(s))\| \, \|A(s)\| \, |f(s,z_\tau(s)) - f(s,\tilde{z}_\tau(s))| \, \Delta s \\
& + \int_{t_0}^{t} \|e_A(t,\sigma(s))\| \, \|B(s)\| \, |\mathcal{G}\mathscr{L}(z)(s) - \mathcal{G}\mathscr{L}(\tilde{z})(s)| \, \Delta s \\
& + \int_{t_0}^{t} \|e_A(t,\sigma(s))\| \, |h(s,z_\tau(s)) - h(s,\tilde{z}_\tau(s))| \, \Delta s \\
& + \sum_{0<t_k<t} \|e_A(t,t_k)\| \, |J_k(t_k,z(t_k)) - J_k(t_k,\tilde{z}(t_k))|
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
|\mathcal{G}\mathscr{L}(z)(t) - \mathcal{G}\mathscr{L}(\tilde{z})(t)| & \leq \|B\| \, M \left\|\mathcal{L}_{\mathcal{B}^b}^{-1}\right\| |\mathscr{L}(z)(t) - \mathscr{L}(\tilde{z})(t)| \\
& \leq \|B\| \, M \left\|\mathcal{L}_{\mathcal{B}^b}^{-1}\right\| \Bigg(L_f + ML_g + ML_fL_g + M\|A\| \, bL_f \\
& \qquad + ML_hb + M\sum_{k=1}^{p} d_k \Bigg) \|z - \tilde{z}\|_{\mathcal{PC}_p}.
\end{aligned}
$$

So

$$
\begin{aligned}
\|\mathscr{T}z - \mathscr{T}\tilde{z}\|_{\mathcal{PC}_p} \leq & \left(1 + \|B\|^2 M^2 \left\|W^{-1}\right\| b\right) \Bigg(L_f + ML_g + ML_fL_g + M\|A\| \, L_fb \\
& + ML_hb + M\sum_{k=1}^{p} d_k \Bigg) \|z - \tilde{z}\|_{\mathcal{PC}_p}.
\end{aligned}
$$

Since

$$
\left(1 + \|B\|^2 M^2 \left\|\mathcal{L}_{\mathcal{B}^b}^{-1}\right\| b\right) \left[L_f + ML_g + ML_fL_g + M\|A\| \, L_fb + ML_hb + M\sum_{k=1}^{p} d_k \right] < 1,
$$

then the operator $\mathscr{T}$ has a unique fixed point, that is to say

$$
\mathscr{T}z = z.
$$

Due to $u = \mathcal{G}\mathscr{L}(z)$, we get that

$$\begin{aligned}\mathcal{B}^b u = \mathscr{L}(z) =& z_1 - f(b, z_\tau(s)) - e_A(b, t_0)\big[g(z)(t_0) + f(t_0, g_\tau(z)(t_0) + \phi_\tau(t_0))\big] \\ & - \int_{t_0}^{b} e_A(b, \sigma(s))A(s)f(s, z_\tau(s))\Delta s - \int_{t_0}^{b} e_A(b, \sigma(s))h(s, z_\tau(s))\Delta s \\ & - \sum_{0<t_k<b} e_A(b, t_k)J_k(t_k, z(t_k)).\end{aligned}$$

and so the system (1) is exactly controllable. This conclude the proof. □

7. An example

Let us consider the following control system

$$\begin{cases} \Big[z(t) - \Big(1 + \frac{\tanh(z(\frac{t}{5}))}{8(t+10)^2}\Big)\Big]^{\Delta} = z(t) + 2u(t) + e^{-\frac{z(\frac{t}{5})}{10(t+5)^3}}, \quad t \in [1, 5]_{\mathbb{T}} \\ z(s) = \Big(1 + \frac{\sin(z)}{30^2}\Big)(s) + \phi(s), \quad s \in [\frac{1}{5}, 1]_{\mathbb{T}} \\ z(t_k^+) = z(t_k^-) + 1 + \frac{\cos(z(t_k^-))}{4(t_k+8)^4}, k = 1, 2. \end{cases} \tag{9}$$

Here $t_0 = 1$, $t_1 = \frac{5}{2}$, $t_2 = \frac{9}{2}$ and $b = 5$, $\tau(t) = \frac{t}{5}$. Define the functions $f(t, z) = 1 + \frac{\tanh(z)}{8(t+10)^2}$, $h(t, z) = e^{-\frac{z}{10(t+5)^3}}$, $g(z) = 1 + \frac{\sin(z)}{30^2}$, $J_k(t, z) = 1 + \frac{\cos(z)}{4(t+8)^4}$, $A(t) = 1$ and $B(t) = 2$. Then we have,

$$\begin{aligned} |f(t, z) - f(t, \tilde{z})| &= \tfrac{1}{8(t+10)^2} |\tanh(z) - \tanh(\tilde{z})| \le \tfrac{1}{8 \cdot 10^2} |z - \tilde{z}|, \\ |h(t, z) - h(t, \tilde{z})| &= \left| e^{-\frac{z}{10(t+5)^3}} - e^{-\frac{\tilde{z}}{10(t+5)^3}} \right| \le \tfrac{1}{10 \cdot 5^3} |z - \tilde{z}|, \\ |J_k(t, z) - J_k(t, \tilde{z})| &= \tfrac{1}{4(t+8)^4} |\cos(z) - \cos(\tilde{z})| \le \tfrac{1}{4 \cdot 8^4} |z - \tilde{z}|, \\ |g(z) - g(\tilde{z})| &= \tfrac{1}{30^2} |\sin(z) - \sin(\tilde{z})| \le \tfrac{1}{30^2} |z - \tilde{z}|, \end{aligned}$$

and $L_f + M[L_g + L_f L_g + \|A\| L_f b + L_h b + d_1 + d_2] \le 0.63$, therefore the conditions (H1)-(H4) are satisfied. On the other hand, the operator

$$\begin{aligned}\mathcal{L}_{\mathcal{B}^5} =& \int_{t_0}^{5} e_1(5, \sigma(s))B(s)B^*(s)e_1^*(5, \sigma(s))\Delta s = 4\int_{t_0}^{5} e_1(5, \sigma(s))e_1(5, \sigma(s))\Delta s \\ =& 4\int_{t_0}^{5} e_{1\oplus 1}(5, \sigma(s))\Delta s \neq 0,\end{aligned}$$

so it is invertible, and hence the linear system

$$\begin{cases} z(t)^{\Delta} = z(t) + 2u(t), \quad t \in [t_0, 5]_{\mathbb{T}} \\ z(t_0) = z^0, \end{cases} \tag{10}$$

is controllable.

Therefore, if for example we take $[0, 5]_{\mathbb{T}} = [0, 5]_{\mathbb{R}}$, or $[0, 5]_{\mathbb{T}} = [\frac{1}{5}, 1] \cup [2, 3] \cup [4, 5]$, then by Theorem 5.1, system (9) is approximately controllable.

8. Final Remark

In this paper we study a control system governed by a neutral differential equation on time scales with impulses and nonlocal conditions. Specifically, first of all, we prove the existence and uniqueness of solutions. After that, we prove the approximate controllability of the system assuming that the associated linear control problem on time scales is exactly controllable on $[\delta, b]_{\mathbb{T}}$, for any $\delta \in (t_0, b)_{\mathbb{T}}$ with b being a left-dense point. Next, assuming certain conditions on the nonlinear term, we prove the exact controllability applying Banach Fixed Point Theorem. Finally, we consider an example to illustrate the applicability of our results. It is good to mention that, our technique can be extended to the infinite dimensional case, where the operator $A(t) = A$ generates a compact strongly continuous semigroup on time scales. Bashirov in [11] presented a new and simple technique to study exact controllability without using fixed point theorems, which is based on a piecewise construction of steering controls that allows to prove the exact controllability of semilinear systems. However, we think that to use this technique we need certain additional conditions on the time scale. Nevertheless, this will be analyzed in a forthcoming papers; the continuous and the time scales cases.

Acknowledgement. The authors are very grateful to anonymous referees for careful reading of this work, which permitted them to improve the final presentation of this manuscript.

References

[1] Agarwall R., O'Regan D. and Saker S., (2014), Dynamic Inequalities on Time Scales, Springer.

[2] Agarwal, R., Otero-Espinar, V., Perera, K., Vivero, D. (2007), Basic properties of Sobolev's spaces on bounded time scales, Adv. Differ. Equ., Vol. 331, pp. 1263-1274.

[3] Akhmet M.U. and Turan M., (2006), The differential equations on time scales through impulsive differential equations, Nonlinear Analysis, 65, pp. 2043-2060.

[4] Atici, F.M., Biles, D.C. and Lebedinsky A., (2006), An application of time scale to econommics, Math. Comput. Modeling, 43, pp. 718-726.

[5] Bartosiewicz, Z. and E. Pawluszewicz, (2004), Linear control systems on time scale: Unification of continuous and discrete, Proc. 10th IEEE Int. Conf. Methods Models Automation Robotics MMAR'04, Miedzyzdroje, Poland, pp. 263-266.

[6] Bartosiewicz, Z. and E. Pawluszewicz, (2004), Unification of countinuous-time and discrete-time systems: The linear case, Proc. 16Th Int. Symp. Math. Theory Netw. Syst. MNTS'2004, Leuven, Belgium, pp. 1–8.

[7] Bartosiewicz Z., (2013), Linear positive control systems on time scales: controllability, Math. Control Signals Syst., 25 (3), pp. 327-343.

[8] Bashirov, A.E. and Ghahramanlou, N., (2014), Systems. Cogent Engineering, 1, On Partial Approximate Controllability of Semilinear, doi: 10.1080/23311916.2014.965947.

[9] Bashirov, A.E. and Ghahramanlou, N., (2013), On partial complete controllability of semilinear systems. Abstract and Applied Analysis, Article ID 52105, 8 pages.

[10] Bashirov, A.E., Mahmudov, N., Semi, N. and Etikan H., (2007), On partial controllability concepts. International Journal of Control, 80 (1), pp. 1-7.

[11] Bashirov, A.E., (2020), On exact controllability of semilinear systems. Math. Meth. Appl. Sci. pp. 1-8.

[12] Benchohra M., Henderson J., Ntouyas S.K. and Ouahab A., (2006), Impulsive functional dynamic equations on time scales, Dynamic Systems and Applications, 15, pp. 43-52.

[13] Bohner, M., Fan, M. and Zhang, J., (2007), Periodicity of scalar dynamic equations and applications to population models, J. Math. Anal. Appl., 300 (1), pp. 1-9.

[14] Bohner, M., Fan, M. and Zhang, J., (2006), Existence of periodic solutions in predator-prey and competition dynamic systems, Nonlinear Analysis: Real world and applications, 7, pp. 1193-1204.

[15] Bohner M., Rao V.S.H. and Sanyal S., (2011), Global stability of complex-valued neural networks on time scales, Diff. Eqns. and Dyn. Systs., 19 (1 & 2), pp. 3-11.

[16] Bohner, M. and Peterson, A., (2001), Dynamics Equations on Time Scales. An Introduction with Applications, Birkhuser Boston Inc., Boston MA.

[17] Bohner, M. and Peterson, A., (2003), Advances in Dynamic Equations on Time Scales, Birkhuser Boston Inc., Boston MA.

[18] Bohner M. and Wintz N., (2012), Controllability and observability of linear systems on time scales, Math. Bohem 137 (2), pp. 149-163.

[19] Bohner M. and Wintz N., (2013), The Kalman filter for linear systems on time scales, J. Math. Anal. Appl. 406 (2), pp. 419-436.

[20] Brigo, D. and Mercurio, F., (2000), Discrete time vs continuous time stock-price dynamics and implications for option pricing, Finance and Stochastics, 4, pp. 147-159.

[21] Burlica M., Necula M., Rosu D. and Vrabie I., (2016), Delay differential evolution subjected to nonlocal initial conditions, Monographs and Research Notes in Mathematics. Taylor and Francis Group, LLC.

[22] Byszewski L. and Lakshmikantham V., (1990), Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, Appl. Anal., 40, pp. 11-19.

[23] Byszewski L., (1990), Existence and uniqueness of solutions of nonlocal problems for hiperbolic equation $u_{xt} = F(x, t, u, u_x)$, J. of App. Math. and Stoch. Anal. 3 (3), pp. 163-168.

[24] Byszewski L., (1991), Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, J. of Math. Anal. and Appl., 162, pp. 494-505.

[25] Chabrowski J., (1984), On nonlocal problems for parabolic equations, Nagoya Math. J., 93, pp. 109-131.

[26] Chang Y.K. and Li W.T, (2006), Existence results for impulsive dynamic equations on time scales with nonlocal initial conditions, Mathematical and Computer Modelling, 43, pp. 377-384.

[27] Duque C., Uzcátegui J. and Leiva H., (2019), Approximate controllability of semilinear dynamic equations on time scale, Asian Journal of Control, 21 (5), pp. 1-7.

[28] Duque C., Uzcátegui J. and Leiva H., (2017), Controllability of semilinear dynamic systems on time scales, Journal of Abstract Differential Equations and Applications, 8 (2), pp. 70-80.

[29] Federson M., Mesquita J.G. and Slavík A., (2013), Basic results for functional differential and dynamic equations involving impulses, Math. Nachr., 286, (2-3), pp. 181-204.

[30] Georgiev G., (2016), Integral equations on time scales, Atlantis Studies in Dynamic Systems, 5.

[31] Georgiev S., (2019), Functional dynamic equations on time scales, Springer.

[32] Grow, D. and Wintz, N., (2021), Bilinear state systems on an unbounded time scale, Appl. Math. and Comput., 397, 125917.

[33] Hilger S., (1988), Ein Maßkettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten, PhD thesis, Universität Würzburg.

[34] Hilger S., (1997), Differential and difference calculus-unified!, Nonlinear Anal., 30 (5), pp. 2683-2694.

[35] Janglajew K. and Pawluszewicz E., (2015), Constrained local controllability of dynamic systems on times scales, Advances in Difference Equations. 89. https://doi.org/10.1186/s13662-015-0422-x

[36] Klappler, I. and Qian, I., (1998), Remarks on discrete and continuous large-scale models of DNA dynamics, Biophys. J., 74, pp. 2504-2514.

[37] Lakshmikantham V., Bainov D.D. and Simeonov P.S., (1989), Theory of impulsive differential equations, Series in Modern Applied Mathematics, 6, World Scientific.

[38] Malik M. and Kumar V., (2020), Existence, stability and controllability results of a Volterra integro-dynamic system with non-instantaneous impulses on time scales, IMA J. Math. Control and Information. 37 (1), pp. 276-299.

[39] Malik M. and Kumar V., (2019), Controllability of neutral differential equations with impulses on time scales Differential Equations and Dynamical Systems, 29, pp. 211-225.

[40] Martynyuk A.A., (2016), Stability theory for dynamic equations on time scales. Birkhäuser.

[41] Tisdell, C.C. and Zaidi, A., (2008), Basic qualitative and quantitative results for solutions to nonlinear dynamic equations on time scales with an application to economic modelling, Nonlinear Anal. 69 (11), pp. 3504-3524.

[42] Vrabie I.I., (2013), Delay evolution equations with mixed nonlocal plus local initial conditions, Comm. in Cont. Math. 1350035, 22 pages.

[43] Wang C., Agarwall R., O'Regan D. and Sakthivel R., (2020), Theory of translations closedness for time scales. With applications in translation functions and dynamic equations. Springer, 62.

[44] Zada A., Alam L., Kuman P., Kuman W., Ali G. and Alzabut J., (2020), Controllability of Impulsive Non-Linear Delay Dynamic Systems on Time Scale, IEEE Access doi 10.1109/Access.2020.2995328.

Cosme Duque received his B.S. degree in Mathematics from Universidad de Los Andes, Mérida, Venezuela, in 1999, and both his Masters and Doctor's degree in Mathematics from Universidad de los Andes, Mérida, Venezuela, in 2001 and 2009, respectively. He is currently a professor in the Department of Mathematics at Universidad de Los Andes, Venezuela. His research interests include differential and difference equations, biomathematics, control theory and dynamic equations on time scales.

Hugo Leiva completed his Ph.D. in Georgia Tech, USA in 1995. He was a Full Professor in Universidad de Los Andes, Mérida, Venezuela, from 1996 to 2015, and Visiting Professor in Gatech, USA, in 2001-2003, UNAM, México in 2009 and Louisiana State University, USA, in 2015-2016. He is currently a professor at Universidad Yachay Tech, Ecuador.His research interests are differential equations, control theory and functional analysis.