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OPTIMAL CONTROL OF FIRST-ORDER UNDIVIDED INCLUSIONS

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Abstract. The article is devoted to the optimization of first-order evolution inclusions (DFI) with undivided conditions. Optimality conditions are formulated in terms of locally adjoint mappings (LAMs). The construction of "duality relations" is an indispensable approach for the differential inclusions. In this case, the presence of discrete-approximate problems is a bridge between discrete and continuous problems. At the end of the article, as an example, we consider duality in optimization problems with linear discrete and first-order polyhedral DFIs.

Keywords: Endpoint and state constraints, infimal convolution, necessary and sufficient, duality, conjugate, Euler-Lagrange.

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1. Introduction

We consider the following problem with undivided constraints:

$$\text{minimize } f(x(0), x(T)), \tag{1}$$

$$\text{(PC)} \qquad x'(t) \in F(x(t), t), \text{ a.e. } t \in [0, T], \tag{2}$$

$$(x(0), x(T)) \in S. \tag{3}$$

$$x(t) \in D(t), \forall t \in [0, T]. \tag{4}$$

Here $F(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, f is continuous cost functional $f : \mathbb{R}^{2n}, \to \mathbb{R}^1, S \subseteq \mathbb{R}^{2n}$ and $D(t) \subseteq \mathbb{R}^n, \forall t \in [0, T]$ are nonempty subsets. It is required to find an absolutely-continuous solution of the first-order DFIs (2)-(4) minimizing (1).

The qualitative theory of set-valued mappings and DFIs and their optimization, are studied in papers [1–9, 11, 13–18, 20, 21]. In [9] on the basis of the apparatus of LAMs, a sufficient condition of optimality is derived for the non-convex problem with a first-order partial DFIs are proved. A sufficient condition for an extremum is an extremal relation for the primal and dual problem. In the paper of Mahmudov [14], an approximation of the Bolza problem of optimal control theory with a fixed time interval given by convex and

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nonconvex DFIs of the second-order is studied, where the main goal is to obtain necessary and sufficient optimal conditions for the Cauchy problem of the second-order. One of the most characteristic features of such approaches with second-order DFIs is the presence of LAM equivalence relations. We hope that all these improvements will serve for the further development of the theory of duality theory. The paper [6] considers a class of second-order evolution differential inclusions in Hilbert spaces. The article considers approximate controllability for one class of second-order control systems. First, it is established a set of sufficient conditions for approximate controllability for a class of second-order evolutionary differential inclusions in Hilbert spaces. Further, the result is extended to the study of the concept of approximate controllability with nonlocal conditions.

In the present paper to solve the main problem (PC), an auxiliary problem denoted as (PD), is used

$$\text{(PD)} \qquad \begin{aligned} &\text{minimize } f(x_0, x_T), && (5)\\ &x_{t+1} \in F(x_t, t),\ t = 0, \dots, T-1, && (6)\\ &(x_0, x_T) \in S, && (7)\\ &x_t \in D_t, t = 1, \dots, T-1, && (8) \end{aligned}$$

where f, S, F, D_t are the same function, set and set-valued mappings as in problem (PC), respectively, T is fixed natural number. A sequence $\{x_t\}_{t=0}^{T} = \{x_t : t = 0, 1, \dots, T\}$ is called a feasible trajectory for the stated problem (5)-(8). It is required find a trajectory $\{x_t\}_{t=0}^{T}$ to a problem (PD) for the first-order discrete-time problem, satisfying (6)-(8) and minimizing $f(x_0, x_T)$. Note that the reasons for adopting discrete modelling are as follows: First, statistics are collected at discrete times (day, week, month, or year). Thus, discrete-time models can be described in a more straightforward, more accurate and timely manner than continuous-time models. Secondly, the use of discrete-time models avoids some mathematical complexities such as the choice of the function space and the regularity of the solution. Third, the numerical simulation of continuous-time models is obtained through discretization.

The present paper deals with the theory of optimal control for the DSI and DFI problems with undivided endpoints and state constraints.

The rest of the work is organized as follows:

Section 2 from the book of Mahmudov [10] presents the basic concepts of convex analysis, convex upper approximation, and the corresponding subdifferential notion.

In Section 3, the problem for the first-order DSI (PD) is reduced to a standard programming problem and, using the tangent direction cone method, necessary and sufficient optimality conditions are formulated for it.

In Section 4 , using a discretized method for a discrete analogue of the problem (PC) in terms of the new introduced set-valued mapping G, we formulate necessary and sufficient conditions for optimality, consisting of the Euler-Lagrange inclusion expressed by LAM G^* and the transversality condition. Then, in order to pass from LAM G^* to LAM F^*, the so-called equivalence result is proved separately.

In Section 5, using the limit procedure in Section 4, we formulate the Euler - Lagrange inclusion and the transversality condition for the problem (PC). In the following sections, we will show that the constructed Euler - Lagrange inclusion is an extremal relation for problem (PC) and dual problem (PC*).

In Section 6, we prove that if ρ and ρ^* are the values of primal and dual problems, respectively, then $\rho \geq \rho^*$.

In Section 7, on the basis of the previous section, we construct a dual problem for a discrete-approximate problem related to a continuous problem (PC). Moreover, since the formulated dual problem is expressed through the "support function" M_G of above mentioned set-valued mapping G, our main problem is to find a connection between the M_G and the M_F functions.

Section 8 is devoted to establishing the duality theorem for the main problem (PC). First of all, to construct the dual problem to the problem (PC), we again use the limiting process in the dual problem of the previous section as the partition grid tends to zero. Then the appearing Riemann integral sums are replaced by integrals of M_F and W_{D_t}. A dual control problem is formulated in terms of functions with bounded variations, so as to allow for jumps caused by the presence of state constraints in the primal problem.

Section 9 of the paper demonstrates two optimal control models: a linear discrete model and a model with a polyhedral DFI.

2. Needed facts and problem statement

All the necessary concepts can be found in Mahmudov's book [10]. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping and

$$H_F(x, y^*) = \sup_y \{\langle y, y^*\rangle : y \in F(x)\}, y^* \in \mathbb{R}^n,$$
$$F_A(x, y^*) = \{y \in F(x) : \langle y, y^*\rangle = H_F(x, y^*)\}.$$

A set-valued mapping $F^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$F^*(y^*; (x, y)) := \{x^* : (x^*, -y^*) \in K_F^*(x, y)\},$$

is the LAM to a set-valued F at a point $(x, y) \in gphF$, where $K_F^*(x, y)$ is the dual cone of tangent directions $K_{gphF}(x, y) \equiv K_F(x, y)$. The LAM can be determined using the Hamilton function

$$F^*(y^*; (x, y)) := \{x^* : H_F(x_1, y^*) - H_F(x, y^*) \le \langle x^*, x_1 - x\rangle,$$
$$\forall (x_1, y_1) \in \mathbb{R}^{2n}\}, \ (x, y) \in gphF, \ z \in F_A(x, y^*).$$

Definition 2.1. *A function $f(x, y)$ is closed if its epigraph epi $f = \{(x^0, x, y) : x^0 \ge f(x, y)\}$ closed.*

Definition 2.2. *The function $f^*(x^*, y^*) = \sup_{x,y}\{\langle x, x^*\rangle + \langle y, y^*\rangle - f(x, y)\}$ conjugate to f.*

Denote

$$M_F(x^*, y^*) = \inf_{x,y}\{\langle x, x^*\rangle - \langle y, y^*\rangle : (x, y) \in \text{gph}\, F\}.$$

or

$$M_F(x^*, y^*) = \inf_x \{\langle x, x^*\rangle - H_F(x, y^*)\}$$

Definition 2.3. *The infinal convolution of functions $f_i, i = 1, 2$ is defined as follows*

$$(f_1 \oplus f_2)(u) = \inf\{f_1(u^1) + f_2(u^2) : u^1 + u^2 = u\}, u^i \in \mathbb{R}^n, i = 1, 2.$$

Definition 2.4. *It is said that for the convex problem (5) - (8) the nondegeneracy condition is satisfied, if either (i) or (ii) for the points $x_t \in \mathbb{R}^n$ is true:*
(i) $(x_t, x_{t+1}) \in \text{ri}\,\text{gph}\, F(\cdot, t)\, (t = 0, \ldots, T-1)$, $x_t \in \text{ri}\, D_t\, (t = 1, \ldots, T-1)$, $(x_0, x_T) \in \text{ri}\, S$, $(x_0, x_T) \in \text{ridom}\, f$;
(ii) $(x_t, x_{t+1}) \in$ *int* $\text{gph}\, F(\cdot, t)\, (t = 0, \ldots, T-1)$, $x_t \in \text{int}\, D_t\, (t = 1, \ldots, T-1)$, $(x_0, x_T) \in \text{int}\, S$,

Definition 2.5. *With respect to [14] $h(\bar{x},x)$ is called a CUA of the function $g:\mathbb{R}^n\rightarrow\mathbb{R}^1\{\pm\infty\}$ at a point $x\in \operatorname{domg}=\{x:|g(x)|<+\infty\}$ if $h(\bar{x},x)\geq V(\bar{x},x)$ for all $\bar{x}\neq 0$ and $h(\cdot,x)$ is a convex closed positive homogeneous function, where*

$$V(\bar{x},x)=\sup_{r(\cdot)}\limsup_{\gamma\downarrow 0}(1/\gamma)[g(x+\gamma\bar{x}+r(\gamma))-g(x)],\gamma^{-1}r(\gamma)\rightarrow 0.$$

Definition 2.6. *[10] A set $\partial h(0,x)=\{x^*\in\mathbb{R}^n:h(\bar{x},x)\geq\langle\bar{x},x\rangle,\bar{x}\in\mathbb{R}^n\}$ is called a subdifferential of the function g at a point x and is denoted by $\partial g(x)$.*

Condition A [10] Let in the problem (5)-(8) $F(\cdot,t)$ the cones of tangent directions $K_{G(,t)}\left(\tilde{x}_t,\tilde{x}_{t+1}\right)$ be local tents, where $\tilde{x}_t$ be the points of the trajectory $\{\tilde{x}_t\}_{t=0}^T$. Moreover, let f admit a continuous CUA $h\left(\cdot,\tilde{x}_0,\tilde{x}_T\right)$ at the point $(\tilde{x}_0,\tilde{x}_T)$, which implies that the subdifferential $\partial f\left(\tilde{x}_0,\tilde{x}_T\right)=\partial h\left(0,\tilde{x}_0,\tilde{x}_T\right)$ is defined.

3. Optimization of first order DSIs

Define in the space $\mathbb{R}^{n(T+1)}$ the sets

$$M_t=\left\{u=(x_0,\ldots,x_T):(x_t,x_{t+1})\in gphF(\cdot,t)\right\},t=0,\ldots,T-1;$$
$$P=\left\{u=(x_0,\ldots,x_T):(x_0,x_T)\in S\right\};\Phi_t=\left\{u=(x_0,\ldots,x_T):x_t\in D_t\right\},t=1,\ldots,T-1.$$

Clearly, denoting $\varphi(u)=f\left(x_0,x_T\right)$ we can reduce this problem to a mathematical programming problem; it is not hard to see that (5)-(8) is equivalent to the following one

$$\text{minimize } \varphi(u) \text{ subject to } u\in\Omega=\left(\bigcap_{t=0}^{T-1}M_t\right)\cap\left(\bigcap_{t=1}^{T-1}\Phi_t\right)\cap P, \tag{9}$$

where Ω is the convex set.
In the sense of first order DSI terminology [10,19], we give necessary and sufficient conditions for problem (5) - (8), which play an important role in the following improvements.

Theorem 3.1. *Suppose $f:\mathbb{R}^{2n}\rightarrow\mathbb{R}^1\cup\{+\infty\}$ is a convex function, $F(\cdot,t):\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is set-valued mapping and $S\subseteq\mathbb{R}^{2n},D_t\subseteq\mathbb{R}^n(t=1,\ldots,T-1)$. Then, for the trajectory $\{\tilde{x}\}_{t=0}^T$ to be optimal it is necessary that there exist vectors $x_t^*,t=0,\ldots,T$ and a scalar $\lambda\in\{0,1\}$, not all equal to zero, satisfying inclusion (i) and the condition (ii):*
(i) $x_t^*\in F^*\left(x_{t+1}^*;\left(\tilde{x}_t,\tilde{x}_{t+1}\right),t\right)+K_{D_t}^*\left(\tilde{x}_t\right),t=1,\ldots,T-1,$
(ii) $(x_0^*,-x_T^*)\in\lambda\partial f\left(\tilde{x}_0,\tilde{x}_T\right)-K_S^*\left(\tilde{x}_0,\tilde{x}_T\right).$
Moreover, under the nondegeneracy condition $\lambda=1$ and these conditions are also sufficient for optimality.

Proof. In view of Theorems 1.30 and 1.11 [10] we obtain

$$K_\Omega^*(\tilde{u})=\sum_{t=0}^{T-1}K_{M_t}^*(\tilde{u})+\sum_{t=1}^{T-1}K_{\Phi_t}^*(\tilde{u})+K_P^*(\tilde{u}),\tilde{u}=(\tilde{x}_0,\ldots,\tilde{x}_N)$$

By the necessary optimality conditions [10] for the problem (9) there exist not all zero vectors $u^*(t)\in K_{M_t}^*(\tilde{u}),t=0,1,\ldots,T-1,u_0^*\in K_P^*(\tilde{u}),\ \bar{u}^*(t)\in K_{\Phi_t}^*(\tilde{u}),t=1,\ldots,T-1,$ and the scalar $\lambda\in\{0,1\}$, such that

$$\lambda\hat{u}^*=\sum_{t=0}^{T-1}u^*(t)+\sum_{t=1}^{T-1}\bar{u}^*(t)+u_0^*,\hat{u}^*\in\partial_u\varphi(\tilde{u}). \tag{10}$$

From the form of the function φ it is easy to see that

$$\partial_u\varphi(\tilde{u})=\left\{(\hat{x}_0^*,0,\ldots,0,\hat{x}_T^*):(\hat{x}_0^*,\hat{x}_T^*)\in\partial f\left(\tilde{x}_0,\tilde{x}_T\right)\right\},$$

whence it follows that the vector $\hat{u}^* \in \partial_u \varphi(\tilde{u})$ has the form $\hat{u}^* = (\hat{x}_0^*, \underbrace{0, \dots, 0}_{T-1}, \hat{x}_T^*), (\hat{x}_0^*, \hat{x}_T^*) \in \partial f(\tilde{x}_0, \tilde{x}_T)$. We should compute the cone of tangent directions $K_F(\tilde{x}_t, \tilde{x}_{t+1})$ to $\operatorname{gph} F(\cdot, t)$; for sufficiently small scalar $\gamma > 0$ we have

$$K_{M_t}(\tilde{u}) = \{\bar{u} = (\bar{x}_0, \dots, \bar{x}_T) : (\tilde{x}_t + \gamma \bar{x}_t, \tilde{x}_{t+1} + \gamma \bar{x}_{t+1}) \in \operatorname{gph} F(\cdot, t)\}$$
$$= \{\bar{u} = (\bar{x}_0, \dots, \bar{x}_T) : (\bar{x}_t, \bar{x}_{t+1}) \in K_F(\tilde{x}_t, \tilde{x}_{t+1}), \bar{x}_k \in \mathbb{R}^n, k \neq t, t+1\}, t = 0, \dots, T-1. \tag{11}$$

It is also not hard to calculate the cone of tangent $K_{\Phi_t}(\tilde{u})$:

$$K_{\Phi_t}(\tilde{u}) = \{\bar{u} = (\bar{x}_0, \dots, \bar{x}_T) : \bar{x}_t \in K_{D_t}(\tilde{x}_t), \bar{x}_k \in \mathbb{R}^n, k \neq t\}, t = 1, \dots, T-1. \tag{12}$$

Further, for sufficiently small scalar $\gamma > 0$ we have

$$K_P(\tilde{u}) = \{\bar{u} = (\bar{x}_0, \dots, \bar{x}_T) : \tilde{u} + \gamma \bar{u} \in P\}$$
$$= \{\bar{u} = (\bar{x}_0, \dots, \bar{x}_T) : (\tilde{x}_0 + \gamma \bar{x}_0, \tilde{x}_T + \gamma \bar{x}_T) \in S, \bar{x}_k \in \mathbb{R}^n, k \neq 0, T\}$$
$$= \{\bar{u} = (\bar{x}_0, \dots, \bar{x}_T) : (\bar{x}_0, \bar{x}_T) \in K_S(\tilde{x}_0, \tilde{x}_T), \bar{x}_k \in \mathbb{R}^n, k \neq 0, T\}. \tag{13}$$

Taking into account that in formulas (11)-(13) $\bar{x}_k$ are arbitrary vectors, we can easily compute the dual cones of tangent directions, correspondingly:

$$K_{M_t}^*(\tilde{u}) = \left\{u^*(t) = (x_0^*(t), \dots, x_T^*(t)) : \left(x_t^*(t), x_{t+1}^*(t)\right)\right.$$
$$\left. \in K_F^*(\tilde{x}_t, \tilde{x}_{t+1}), x_k^*(t) = 0, k \neq t, t+1\right\}, t = 0, \dots, T-1,$$
$$K_{\Phi_t}^*(\tilde{u}) = \left\{\bar{u}^*(t) = \left(0, \bar{x}_1^*(t), \dots, \bar{x}_{T-1}^*(t), 0\right) : \bar{x}_t^*(t) \in K_{D_t}^*(\tilde{x}_t), \bar{x}_k(t) = 0, k \neq t\right\}, \tag{14}$$
$$t = 1, \dots, T-1;$$
$$K_P^*(\tilde{u}) = \{u_0^* = (x_{00}^*, \dots, x_{0T}^*) : (x_{00}^*, x_{0T}^*) \in K_S^*(\tilde{x}_0, \tilde{x}_T), x_{0k}^* = 0, k \neq 0, T\}.$$

Let us now compute the sum of vectors $\sum_{t=0}^{T-1} u^*(t)$ and $\sum_{t=1}^{T-1} \bar{u}^*(t)$. Using the structure of vectors $u^*(t) = \left(0, \dots, 0, x_t^*(t), x_{t+1}^*(t), 0, \dots, 0\right), t = 0, \dots, T-1$ we have

$$\sum_{t=0}^{T-1} u^*(t) = \begin{cases} x_0^*(0), & t = 0, \\ x_t^*(t-1) + x_t^*(t), & t = 1, \dots, T-1, \\ x_T^*(T-1), & t = T. \end{cases}$$

Analogously, since $\bar{u}^*(t) = \left(0, \bar{x}_1^*(t), \dots, \bar{x}_{T-1}^*(t), 0\right), \bar{x}_k(t) = 0, k \neq t$ we have

$$\sum_{t=1}^{T-1} \bar{u}^*(t) = \left(0, \bar{x}_1^*(1), \bar{x}_2^*(2), \dots, \bar{x}_{T-1}^*(T-1), 0\right), \bar{x}_t^*(t) \in K_{\Phi_t}^*(\tilde{x}_t), t = 1, \dots, T-1.$$

Moreover, it is clear that

$$u_0^* = (x_{00}^*, \underbrace{0, \dots, 0}_{T-1}, x_{0T}^*), (x_{00}^*, x_{0T}^*) \in K_S^*(\tilde{x}_0, \tilde{x}_T).$$

On the other hand, using the componentwise representation (10), we obtain that

$$\begin{aligned} &\lambda \hat{x}_0^* = x_0^*(0) + x_{00}^*, && t = 0, \\ &0 = x_t^*(t-1) + x_t^*(t) + \bar{x}_t^*(t), && t = 1, \dots, T-1, \\ &\lambda \hat{x}_T^* = x_T^*(T-1) + x_{0T}^*, && t = T, \end{aligned} \tag{15}$$

where

$$(\hat{x}_0^*, \hat{x}_T^*) \in \partial f(\tilde{x}_0, \tilde{x}_T), \bar{x}_t^*(t) \in K_{D_t}^*(\tilde{x}_t), t = 1, \dots, T-1; (x_{00}^*, x_{0T}^*) \in K_S^*(\tilde{x}_0, \tilde{x}_T). \tag{16}$$

From the second relation of (15) by definition of LAM we deduce that

$$x_t^*(t) \in F^*\left(-x_{t+1}^*(t); (\tilde{x}_t, \tilde{x}_{t+1})\right), t = 1, \ldots, T-1. \tag{17}$$

Now with new notation $-x_{t+1}^*(t) \equiv x_{t+1}^*, t = 1, \ldots, T-1$ in the formula (17) we obtain

$$x_t^* \in F^*\left(x_{t+1}^*; (\tilde{x}_t, \tilde{x}_{t+1})\right) + \bar{x}_t^*(t), t = 1, \ldots, T-1$$

or

$$x_t^* \in F^*\left(x_{t+1}^*; (\tilde{x}_t, \tilde{x}_{t+1})\right) + K_{D_t}^*(\tilde{x}_t), t = 1, \ldots, T-1 \tag{18}$$

In addition, it is clear that setting $x_0^*(0) \equiv x_0^*$ the first and third relations of (15) can be combined as follows

$$(x_0^*, x_T^*(T-1)) = \lambda(\hat{x}_0^*, \hat{x}_T^*) - (x_{00}^*, x_{0T}^*)$$

or, in view of first and third formulas of (3.8) and notation $-x_T^*(T-1) \equiv x_T^*$

$$(x_0^*, -x_T^*) \in \lambda \partial f(\tilde{x}_0, \tilde{x}_T) - K_S^*(\tilde{x}_0, \tilde{x}_T). \tag{19}$$

As a result, taking into account formulas (18), (19), the necessary condition is proved. Besides, by Theorem 3.4 [10, p.99], under the nondegeneracy condition (10) holds with scalar $\lambda = 1$, where $u^* \in \partial_u \varphi(\tilde{u}) \cap K_\Omega^*(\tilde{u})$. □

4. Optimization of first order discrete-approximate inclusions

Suppose that $h = T/N$ is the step along the t-axis and $x(t) \equiv x_h(t)$ is a grid function on a uniform grid on $[0, T]$, where N is a sufficiently large positive integer. We introduce the following first order difference operator $\Delta x(t) \quad t = 0, h, \ldots, T-h$

$$\Delta x(t) = \frac{1}{h}[x(t+h) - x(t)],$$

and put in accordance with the problem (PC) a discrete-approximate first order problem with undivided endpoints and state conditions:

$$\begin{gathered} \text{minimize } f(x(0), x(T)), \\ \Delta x(t) \in F(x(t), t), t = 0, h, \ldots, T-h, \\ (x(0), x(T)) \in S; x(t) \in D(t), \quad t = h, \ldots, T-h. \end{gathered} \tag{20}$$

Obviously, Theorem 3.1 cannot be applied to problem (20) in its current form. Therefore, we must reduce problem (20) to a problem of the form (5) - (8) or (PD) with endpoints and state constraints. Introducing an auxiliary multivalued mapping $G(x, t) = x + hF(x, t)$, we reduce problem (20) to the following form:

$$\begin{gathered} \text{minimize } f(x(0), x(T)), \\ (PDA) \qquad x(t+h) \in G(x(t), t), t = 0, h, \ldots, T-h, \\ (x(0), x(T)) \in S; x(t) \in D(t), t = h, \ldots, T-h. \end{gathered} \tag{21}$$

Obviously, now problems (PD) and (21) have the same form. Thus, we can apply Theorem 3.1 to problem (21) that is there exist $x^*(t), t = 0, \ldots, T$ and a scalar $\lambda \in \{0, 1\}$, satisfying the following adjoint inclusion and the transversality condition:

$$\begin{gathered} x^*(t) \in G^*\left(x^*(t+h); (\tilde{x}(t), \tilde{x}(t+h)), t\right) + K_{D(t)}^*(\tilde{x}(t)), t = 1, \ldots, T-h, \\ (x^*(0), -x^*(T)) \in \lambda \partial f(\tilde{x}(0), \tilde{x}(T)) - K_S^*(\tilde{x}(0), \tilde{x}(T)). \end{gathered} \tag{22}$$

On the other hand, we must be able to express LAM G^* through LAM F^*. On this path, the following equivalence result turns out to be extremely important.

Proposition 4.1. *Suppose $F(\cdot,t):\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping and that $G(\cdot,t):\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined as $G(x,t)=x+hF(x,t)$. Then the inclusions are equivalent*

(1) $x^* \in G^*\left(y^*;(x,y),t\right), \quad y \in G_A\left(x;y^*,t\right),$

(2) $\dfrac{x^*-y^*}{h} \in F^*\left(y^*;\left(x,\dfrac{y-x}{h}\right),t\right), \dfrac{y-x}{h} \in F_A\left(x;y^*,t\right), \quad y^* \in \mathbb{R}^n.$

Proof. Let $K_G(x,y),(x,y)\in \operatorname{gph} G(\cdot,t)$ be cone of tangent directions. It is easy to verify that, the inclusions

$$(\bar{x},\bar{y}) \in K_G(x,y) \tag{23}$$

and

$$(\bar{x},(\bar{y}-\bar{x})/h) \in K_F(x,(y-x)/h) \tag{24}$$

are equivalent. In fact, satisfaction of (24) means that for small $\gamma>0$

$$\bar{x}=\gamma(\tilde{x}-x), \frac{\bar{y}-\bar{x}}{h}=\gamma\left(\frac{\tilde{y}-\tilde{x}}{h}-\frac{y-x}{h}\right),$$

and for all $(\tilde{x},\tilde{y})$ such that $\frac{\tilde{y}-\tilde{x}}{h}\in F(\tilde{x},t)$ or equivalently, $\tilde{y}\in G(\tilde{x})$. Simplifying the latter relations, we have $\bar{x}=\gamma(\tilde{x}-x), \bar{y}-\bar{x}=\gamma(\tilde{y}-\tilde{x})-\gamma(y-x)$. Now, we have $\bar{y}=\gamma(\tilde{y}-y)$, i.e. $(\bar{x},\bar{y})\in K_G(x,y)$. Suppose now $x^*\in G^*\left(y^*;(x,y),t\right), \quad y\in G_A\left(x;y^*,t\right)$, that is

$$\langle \bar{x},x^*\rangle - \langle \bar{y},y^*\rangle \geq 0, \quad (\bar{x},\bar{y})\in K_G(x,y). \tag{25}$$

It means that

$$\langle \bar{x},a\rangle - \left\langle \frac{\bar{y}-\bar{x}}{h},y^*\right\rangle \geq 0,$$

for which (24) is satisfied. Here the vector a should be appropriately defined. It is not hard to see that the latter inequality is equivalent to the inequality

$$\langle \bar{x},ha+y^*\rangle - \langle \bar{y},y^*\rangle \geq 0. \tag{26}$$

Then comparing it with (25) and (26), we find that $a=\left(x^*-y^*\right)/h$. Then from the equivalence of (23) and (24) we have

$$\frac{x^*-y^*}{h} \in F^*\left(y^*;\left(x,\frac{y-x}{h}\right),t\right).$$

Moreover, it is easy to see that

$$G^*\left(y^*;(x,y),t\right)\neq\varnothing \text{ if } y\in G_A\left(x;y^*,t\right) \text{ and}$$
$$F^*\left(y^*;\left(x,\frac{y-x}{h}\right),t\right)\neq\varnothing \text{ if } \frac{y-x}{h}\in F_A\left(x;y^*\right),$$

respectively. The theorem is proved. □

Remark 4.1. *Note that it is possible to weaken the condition imposed on $G(x,y,t)=x+hF(x,t)$ assuming that cone of tangent directions $K_G(x,y),(x,y)\in gphG(\cdot,t)$ is a local tent. Then the corresponding inclusions (1) and (2) of Proposition 4.1 concerning LAM will again be equivalent. In particular, it is known that for a convex multivalued mapping a local tent always exists [10, p.120].*

Theorem 4.1. *Let $F(\cdot,t):\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ and $f:\mathbb{R}^{2n}\to\mathbb{R}^1\cup\{+\infty\}$ be a proper convex function, $S\subseteq\mathbb{R}^{2n}, D_t\subseteq\mathbb{R}^n(t=h,\dots,T-h)$ be convex subsets. Then for optimality of $\{\tilde{x}(t)\}, t=0,\dots,T$ in the problem (20) it is necessary that there exist a scalar $\lambda\in\{0,1\}$ and vectors $x^*(t), t=0,\dots,T$ such that:*

(i) $-\Delta x^*(t)\in F^*\left(x^*(t+h);(\tilde{x}(t),\Delta\tilde{x}(t)),t\right)+K^*_{D(t)}(\tilde{x}(t))$

(ii) $\Delta\tilde{x}(t)\in F_A\left(\tilde{x}(t);x^*(t+h),t\right), t=1,\dots,T-h,$

(iii) $(x^*(0), -x^*(T)) \in \lambda \partial f(\tilde{x}(0), \tilde{x}(T)) - K_s^*(\tilde{x}(0), \tilde{x}(T))$.
In addition, if the nondegeneracy condition is satisfied, these conditions are also sufficient for optimality of $\{\tilde{x}(t)\}, t = 0, \ldots, T$.

Proof. Obviously, it suffices to show the validity of Euler-Lagrange's inclusion (i). Using Proposition 4.1 it follows from the Euler-Lagrange inclusion in (22), that

$$\frac{x^*(t) - x^*(t+h)}{h} \in F^*\left(x^*(t+h); (\tilde{x}(t), \tilde{x}(t+h)), t\right) + \frac{1}{h} K_{D(t)}^*(\tilde{x}(t)), t = 1, \ldots, T-h. \tag{27}$$

Then taking into account, that $\frac{x^*(t) - x^*(t+h)}{h} = -\Delta x^*(t)$ and $K_{D(t)}^*(\tilde{x}(t)) \equiv \frac{1}{h} K_{D(t)}^*(\tilde{x}(t))$ ($K_{D(t)}^*(\tilde{x}(t))$ is a cone) from (27) we have the condition (i) of theorem. It remains to emphasize that by Proposition 4.1 F^* is nonempty if $y \in F_A(x, y^*, t)$, which implies that $\Delta \tilde{x}(t) \in F_A(\tilde{x}(t), x^*(t+h), t)$. □

The results of Theorem 3.1 can also be generalized to the non-convex case.

5. Sufficient condition of optimality for a continuous problem (PC)

In this section using the limit procedure in the conditions of Theorem 4.1 of Section 4 we formulate the following Euler-Lagrange inclusion and transversality condition for a problem (PC). Then considering $\lim_{h \to 0} \Delta x^*(t) = x^{*\prime}(t), \lim_{h \to 0} \Delta x(t) = x'(t)$ and setting $\lambda = 1$, we have

(a) $-x^{*\prime}(t) \in F^*\left(x^*(t); (\tilde{x}(t), \tilde{x}'(t)), t\right) + K_{D(t)}^*(\tilde{x}(t))$, a.e. $t \in [0, T]$,
(b) $\tilde{x}'(t) \in F_A(\tilde{x}(t); x^*(t), t), t \in [0, T]$
(c) $(x^*(0), -x^*(T)) \in \partial f(\tilde{x}(0), \tilde{x}(T)) - K_s^*(\tilde{x}(0), \tilde{x}(T))$.

Here and henceforth as a solution of the adjoint DFI (a) we use a function of bounded variation instead of an absolutely continuous function, to take into account the jumps caused by the presence of state constraints in the primal problem (PC). This definition is inspired by the Hamiltonian conditions, were extended from absolutely continuous trajectories to trajectories of bounded variation [12]. We recall that every function with bounded variation has almost everywhere a finite derivative and if $v(\cdot)$ is of bounded variation in $[0, T]$, then the set of discontinuities of $v(\cdot)$ can at most be denumerable. Besides, each point of discontinuity is of the first kind. Note that if a function $v(\cdot)$ is absolutely continuous on the interval $[0, T]$ then $v(\cdot)$ is of bounded variation on $[0, T]$. The space $(BV([0, T]; \mathbb{R}^n)$ of functions $v(\cdot)$ of bounded variation on the interval $[0, T]$ is a Banach space with respect to norm $\|v\|_{BV} = \|v(0)\| + V_0^T[v]$, where $V_0^T[v]$ designates the total variation of $v(\cdot)$ on $[0, T]$.

Theorem 5.1. *Suppose that* $F(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is a convex mapping,* $f(\cdot, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}^1 \cup \{+\infty\}$ *is continuous and that* $S \subseteq \mathbb{R}^{2n}, D(t) \subseteq \mathbb{R}^n, t \in [0, T]$ *are convex. Then for optimality of* $\tilde{x}(\cdot)$ *in the problem (PC) it is sufficient that there exists a function of bounded variation* $x^*(\cdot), t \in [0, T]$ *(a), (b) and condition (c).*

Proof. Using Theorem 2.1 [10, p.62] and the Euler-Lagrange type inclusion (a) we have

$$-x^{*\prime}(t) \in \partial_x H_F(\tilde{x}(t), x^*(t)) + v^*(t), v^*(t) \in K_{D(t)}^*(\tilde{x}(t)),$$

which implies that

$$H_F(x(t), x^*(t)) - H_F(\tilde{x}(t), x^*(t)) \leq -\left\langle x^{*\prime}(t) + v^*(t), x(t) - \tilde{x}(t) \right\rangle.$$

In turn, using the definition of the Hamilton function, the argmaximum set and the dual cone, the last inequality implies

$$\left\langle x^*(t), x'(t) - \tilde{x}'(t) \right\rangle + \left\langle x^{*\prime}(t), x(t) - \tilde{x}(t) \right\rangle \leq 0.$$

Integrating this inequality over the time interval $[0,T]$ we have

$$\int_0^T d\langle x^*(t), x(t)-\tilde{x}(t)\rangle \le 0$$

or

$$\langle x^*(0), x(0)-\tilde{x}(0)\rangle - \langle x^*(T), x(T)-\tilde{x}(T)\rangle \ge 0. \tag{28}$$

Further, let $(\mu^*(0),\mu^*(T)) \in K_s^*(\tilde{x}(0),\tilde{x}(T))$. Then by the transversality condition (c) we can write

$$f(x(0),x(T)) - f(\tilde{x}(0),\tilde{x}(T)) \ge \langle x^*(0)+\mu^*(0), x(0)-\tilde{x}(0)\rangle + \langle \mu^*(T)-x^*(T), x(T)-\tilde{x}(T)\rangle$$

whence

$$f(x(0),x(T)) - f(\tilde{x}(0),\tilde{x}(T)) \ge \langle x^*(0), x(0)-\tilde{x}(0)\rangle - \langle x^*(T), x(T)-\tilde{x}(T)\rangle . \tag{29}$$

Then from (28) and (29) make sure that for all possible trajectories $x(\cdot) \in AC[0,T]$

$$f(x(0),x(T)) \ge f(\tilde{x}(0),\tilde{x}(T)),$$

which completes the proof of theorem. □

6. On duality for first order DSI problems

First of all, we prove the following result useful in what follows.

Proposition 6.1. *The conjugate of* $\varphi(u) = f(x_0, x_T)$ *is*

$$\varphi^*(x_0^*,\ldots,x_T^*) = f^*(x_0^*,x_T^*);\, x_i^* = 0, i \ne 0, T.$$

Proof. The proof of the proposition follows easily from the definition of conjugate functions:

$$\begin{aligned}\varphi^*(u^*) &= \sup_u \{\langle u,u^*\rangle - \varphi(u)\} = \sup_{x_0,\ldots,x_T}\left\{\sum_{i=0}^{T}\langle x_i,x_i^*\rangle - f(x_0,x_T)\right\}\\ &= \sup_{x_0,\ldots,x_T}\left\{\sum_{i=1}^{T-1}\langle x_i,x_i^*\rangle + \langle x_0,x_0^*\rangle + \langle x_T,x_T^*\rangle - f(x_0,x_T)\right\}\\ &= \begin{cases} f(x_0^*,x_T^*), & \text{if } x_i^*=0, i=1,\ldots,T-1,\\ +\infty, & \text{otherwise.}\end{cases}\end{aligned}$$

□

We announce the following problem, denoted (PD*), a dual problem to the first order DSI problem (PD):

$$(\mathrm{PD}^*)\ \sup_{x_t^*,v_t^*,\mu_0^*,\mu_T^*}\left\{-f^*(x_0^*+\mu_0^*,\mu_T^*-x_T^*) + \sum_{t=0}^{T-1} M_F(x_t^*-v_t^*,x_{t+1}^*) - \sum_{t=1}^{T-1} W_{D_t}(-v_t^*) - W_S(-\mu_0^*,-\mu_T^*)\right\},$$

where W_{X_t}, W_S are support functions of the sets X_t, S, respectively.

Theorem 6.1. *Let (ρ, ρ^*) be a pair of the optimal values of the primal* (PD) *and dual* (PD*) *problems, respectively, then $\rho \geq \rho^*$. In particular, under the nondegeneracy condition, the presence of a solution to one problem provides the existence of a solution to another problem, $\rho = \rho^*$ and, in the finiteness of ρ, the dual problem (PD^*) has a solution.*

Proof. Recall from convex analysis at the points of continuity $u^0 \in \Omega$, of the function φ,

$$\inf_{u\in\Omega} \varphi(u) = \inf\{\varphi(u) + \delta_\Omega(u)\} = -\sup\{-\varphi(u) - \delta_\Omega(u)\}$$
$$= -(\varphi + \delta_\Omega)^*(0) = -(\varphi^* \oplus \delta_\Omega^*)(0) = \sup\{-\varphi^*(u^*) - \delta_\Omega^*(-u^*)\},$$

where $\delta_\Omega(\cdot)$ is the indicator function of Ω, i.e., $\delta_\Omega(u) = 0, u \in \Omega$ and $\delta_\Omega(u) = +\infty, u \notin \Omega$. It should be noted that the nondegeneracy condition ensures the existence of a point with this property. In general, it can be noticed that $(\varphi + \delta_\Omega)^*(0) \leq (\varphi^* \oplus \delta_\Omega^*)(0)$ and so

$$\inf_{u\in\Omega} \varphi(u) \geq \sup\{-\varphi^*(u^*) - \delta_\Omega^*(-u^*)\}.$$

Then it can be argued that the dual problem to convex programming problem (9) has the form

$$\sup\{-\varphi^*(u^*) - \delta_\Omega^*(-u^*)\}. \tag{30}$$

Besides, the supremum in (30) is attained and $\delta_\Omega = \sum_{t=0}^{T-1} \delta_{M_t} + \sum_{t=1}^{T-1} \delta_{D_t} + \delta_P$. Then, we have

$$\delta_\Omega^*(-u^*) \leq \inf\left\{\sum_{t=0}^{T-1} \delta_{M_t}^*(-u^*(t)) + \sum_{t=1}^{T-1} \delta_{\Phi_t}^*(-\bar{u}^*(t))\right.$$
$$\left. + \delta_P^*(-u_0^*) : \sum_{i=0}^{T-1} u^*(i) + \sum_{i=1}^{T-1} \bar{u}^*(i) + u_0^* = \tilde{u}^*\right\}, \tag{31}$$

where $u^*(i) = (x_0^*(i), \ldots, x_N^*(i)), i = 0, \ldots, T-1$ and $\bar{u}^*(i) = (\bar{x}_0^*(i), \ldots, \bar{x}_N^*(i)),$ $i = 1, \ldots, T-1$, $u_0^* = (x_{00}^*, \ldots, x_{0T}^*), \tilde{u}^* = (\tilde{x}_0^*, \ldots, \tilde{x}_T^*)$. Then we deduce that

$$\delta_{M_t}^*(-u^*(t)) = \begin{cases} -\inf\limits_{(x_t(t), x_{t+1}(t)) \in \operatorname{gph} F(\cdot, t)} \left[\langle x_t(t), x_t^*(t)\rangle + \langle x_{t+1}(t), x_{t+1}^*(t)\rangle\right], & x_i^*(t) = 0, \\ & i \neq t, t+1, \\ +\infty, & \text{otherwise}, \end{cases} \tag{32}$$

$$t = 0, \ldots, T-1;$$

$$\delta_{\Phi_t}^*(-\bar{u}^*(t)) = \begin{cases} \sup\limits_{x_t \in X_t} \langle x_t, -\bar{x}_t^*\rangle, & \bar{x}_i^* = 0, i \neq t, \\ +\infty, & \text{otherwise} \end{cases} \tag{33}$$

$$t = 1, \ldots, T-1;$$

$$\delta_P^*(-u_0^*) = \begin{cases} \sup\limits_{(x_0, x_T) \in S} [\langle x_0, -x_{00}^*\rangle + \langle x_T, -x_{0T}^*\rangle], & \text{if } x_{0k}^* = 0, k \neq 0, T, \\ +\infty, & \text{otherwise}. \end{cases} \tag{34}$$

Furthermore, from the formulas (31)-(34) and from the relation $\varphi^*(x_0^*, \ldots, x_T^*) = f^*(x_0^*, x_T^*)$ of Proposition 6.1, where $x_i^* = 0, i \neq 0, T$, in view of (15) with the preceding notations, we conclude that

$$\sup \{-\varphi^* (u^*) - \delta^*_\Omega (-u^*)\} = \sup \Bigg\{ -f^* (\tilde{x}^*_0, \tilde{x}^*_T) + \sum_{t=0}^{T-1} M_F \left(x^*_t(t), -x^*_{t+1}(t)\right)$$

$$- \sum_{t=1}^{T-1} W_{D_t} (-\bar{x}^*_t(t)) - W_S (-x^*_{00}, -x^*_{0T}) : x^*_0(0) + x^*_{00} = \tilde{x}^*_0, \tag{35}$$

$$x^*_t(t-1) + x^*_t(t) + \bar{x}^*_t(t) = 0 \ (t = 1, \dots, T-1), \ x^*_T(T-1) + x^*_{0T} = \tilde{x}^*_T \Bigg\},$$

where the supremum is attained, if $\alpha > -\infty$. For what follows, we denote $x^*_{t+1}(t) \equiv -x^*_{t+1}$, $t = 0, \dots, T-1$ and $\bar{x}^*_t(t) = v^*_t, x^*_{00} = \mu^*_0, x^*_{0T} = \mu^*_T$. Then, taking into account these notations, we will make sure that the right-hand side of (35) is nothing but (PD*). □

7. The dual problem for discrete approximate problem.

Here we construct the dual problem to discrete-approximate problem (20) or equivalently (21). According to (PD*) for problem (21) we have the dual problem

$$\sup_{x^*(t), v^*(t), \mu^*(0), \mu^*(T)} \Bigg\{ - f^* (x^*(0) + \mu^*(0), \mu^*(T) - x^*(T))$$

$$+ \sum_{t=0}^{T-h} M_G (x^*(t) - v^*(t), x^*(t+h)) - \sum_{t=h}^{T-h} W_{D_t} (-v^*(t)) - W_S (-\mu^*(0), -\mu^*(T)) \Bigg\}. \tag{36}$$

Here, if we are able to express the function $M_{G(\cdot,t)} \equiv M_G$ through M_F, then we can construct a dual problem to problem (20), which plays a decisive role in constructing duality to the main problem (PC). The following result turns out to be true.

Proposition 7.1. *Let $F(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex and $G(x, y, t) = x + hF(x, t)$. Then*

$$M_G (x^*, y^*) = h M_F \left(\frac{x^* - y^*}{h}, y^* \right)$$

Proof. Recall that,

$$M_G (x^*, y^*) = \inf_{x,y,z} \{\langle x, x^* \rangle - \langle y, y^* \rangle : (x, y) \in \operatorname{gph} G(\cdot, t)\}$$

$$= \inf \left\{ \langle x, x^* \rangle - \langle y, y^* \rangle : \left(x, \frac{y - x}{h} \right) \in \operatorname{gph} F(\cdot, t) \right\}.$$

Rewrite in the last curly braces in a more relevant form

$$\langle x, x^*_1 \rangle - \left\langle \frac{y - x}{h}, y^*_1 \right\rangle, \tag{37}$$

where x^*_1, y^*_1 should be determined. Rewrite (37) as follows

$$\left\langle x, x^*_1 + \frac{y^*_1}{h} \right\rangle - \left\langle y, \frac{y^*_1}{h} \right\rangle$$

and compare it with the difference of inner products $\langle x, x^* \rangle - \langle y, y^* \rangle$. Then we immediately have $x^*_1 = x^* - y^*, \quad y^*_1 = hy^*$. Substituting these expressions into (37), we have

$$\langle x, x^* - y^* \rangle + \left\langle \frac{y - x}{h}, hy^* \right\rangle. \tag{38}$$

Therefore, taking into account (38), we have the desired result:

$$M_G(x^*, y^*) = h \inf \left\{ \left\langle x, \frac{x^* - y^*}{h} \right\rangle + \left\langle \frac{y-x}{h}, y^* \right\rangle : \left(x, \frac{y-x}{h} \right) \in \operatorname{gph} F(\cdot, t) \right\} = h M_F \left(\frac{x^* - y^*}{h}, y^* \right).$$

□

Proposition 7.2. *Suppose $F(\cdot,t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping and $G(x,t) = x + hF(x,t)$. Then the function M_G can be expressed in terms of M_F as follows*

$$M_G(x^*(t) - \nu^*(t), x^*(t+h)) = h M_F(-\Delta x^*(t) - v^*(t), x^*(t+h)),$$

where $v^*(t) = \nu^*(t)/h$.

Proof. In fact, applying the Proposition 7.1 for $M_G(x^*(t) - \nu^*(t), x^*(t+h))$ in the dual problem (36) we obtain

$$M_G(x^*(t) - \nu^*(t), x^*(t+h)) = h M_F \left(\frac{x^*(t) - \nu^*(t) - x^*(t+h)}{h}, x^*(t+h) \right) = h M_F(-\Delta x^*(t) - v^*(t), x^*(t+h)).$$

□

Now, using Proposition 7.2 in the dual problem (36), we establish the following dual problem for the discrete-approximate problem (20):

$$\sup_{x^*(t), v^*(t), \mu^*(0), \mu^*(T)} \Bigg\{ -f^*(x^*(0) + \mu^*(0), \mu^*(T) - x^*(T)) + \sum_{t=0}^{T-h} h M_F(-\Delta x^*(t) - v^*(t), x^*(t+h)) - \sum_{t=h}^{T-h} h W_{D_t}(-v^*(t)) - W_S(-\mu^*(0), -\mu^*(T)) \Bigg\}. \tag{39}$$

8. The dual problem for convex DFIs

To construct the problem (PC*) we use the limit process in the problem (39), where the obtained sums are the Riemann sums of the functions M_F and W_{D_t}, respectively. But in what follows we mean that the integrals appearing in (PC*) are understood in the sense of Lebesgue

$$\sup_{x^*(t), v^*(t), \mu^*(0), \mu^*(T)} J^*[x^*(t), v^*(t), \mu^*(0), \mu^*(T)]$$

$$\text{(PC*)} \qquad J^*[x^*(t), v^*(t), \mu^*(0), \mu^*(T)] = -f^*(x^*(0) + \mu^*(0), \mu^*(T) - x^*(T)) + \int_0^T M_F\left(-x^{*\prime}(t) - v^*(t), x^*(t)\right) dt - \int_0^T W_{D_t}(-v^*(t))\, dt - W_S(-\mu^*(0), -\mu^*(T)).$$

Further, we assume that $x^*(t), v^*(t), t \in [0,T]$ are functions of bounded variations. To prove the duality theorem for (PC), we use the results of Theorem 5.1.

Theorem 8.1. *Suppose that $\tilde{x}(t)$ is an optimal solution of the primal convex problem (PC) with undivided endpoint and state constraints. Then a quadruple $\{\tilde{x}^*(\cdot), \tilde{v}^*(\cdot), \tilde{\mu}^*(0), \tilde{\mu}^*(T)\}$ is an optimal solution of the problem (PC^*) if the conditions (a)-(c) of Theorem 5.1 are satisfied.*

Proof. First of all, we should prove that for all feasible solutions $x(\cdot)$ and dual variables $\{\tilde{x}^*(\cdot), \tilde{v}^*(\cdot), \tilde{\mu}^*(0), \tilde{\mu}^*(T)\}$ of the primal (PC) and dual (PC*) problems, respectively, the following inequality holds:

$$\left.\begin{array}{l} f(x(0),x(T)) \geq J^*\left[x^*(t),v^*(t),\mu^*(0),\mu^*(T)\right] = -f^*\left(x^*(0)+\mu^*(0),\mu^*(T)-x^*(T)\right) \\ +\int_0^T M_F\left(-x^{*\prime}(t)-v^*(t),x^*(t)\right)dt - \int_0^T W_{D_t}\left(-v^*(t)\right)dt - W_S\left(-\mu^*(0),-\mu^*(T)\right) \end{array}\right\}. \tag{40}$$

For this, we use the definitions of the conjugate function f^*, the Hamilton function and the support function. Therefore, we obtain

$$\begin{array}{c} -f^*\left(x^*(0)+\mu^*(0),\mu^*(T)-x^*(T)\right) \\ \leq f(x(0),x(T)) - \langle x(0), x^*(0)+\mu^*(0)\rangle - \langle x(T), \mu^*(T)-x^*(T)\rangle. \end{array} \tag{41}$$

$$\begin{aligned} \int_0^T M_F\left(-x^{*\prime}(t)-v^*(t),x^*(t)\right)dt &\leq \int_0^T \left[\left\langle -x^{*\prime}(t)-v^*(t), x(t)\right\rangle - \left\langle x^*(t), x'(t)\right\rangle\right]dt \\ &= -\int_0^T \left[\left\langle x^{*\prime}(t), x(t)\right\rangle + \left\langle x^*(t), x'(t)\right\rangle\right]dt - \int_0^T \langle v^*(t), x(t)\rangle\, dt \\ &= -\int_0^T d\left\langle x^{*\prime}(t), x(t)\right\rangle dt - \int_0^T \langle v^*(t), x(t)\rangle\, dt \end{aligned} \tag{42}$$

$$-\int_0^T W_{D_t}\left(-v^*(t)\right)dt \leq \int_0^T \langle x(t), v^*(t)\rangle\, dt \tag{43}$$

$$-W_S\left(-\mu^*(0),-\mu^*(T)\right) \leq \langle x(0),\mu^*(0)\rangle + \langle x(T),\mu^*(T)\rangle. \tag{44}$$

Summing the inequalities (41)-(44) we deduce

$$\begin{array}{c} J^*\left[x^*(t),v^*(t),\mu^*(0),\mu^*(T)\right] \leq f(x(0),x(T)) \\ -\langle x(0), x^*(0)+\mu^*(0)\rangle - \langle x(T), \mu^*(T)-x^*(T)\rangle \\ -\int_0^T d\left\langle x^{*\prime}(t), x(t)\right\rangle dt + \langle x(0),\mu^*(0)\rangle + \langle x(T),\mu^*(T)\rangle \\ = f(x(0),x(T)) - \langle x(0),x^*(0)\rangle + \langle x(T),x^*(T)\rangle \\ -\int_0^T d\langle x^*(t),x(t)\rangle\, dt = f(x(0),x(T)) - \langle x(0),x^*(0)\rangle + \langle x(T),x^*(T)\rangle \\ + \langle x^*(0),x(0)\rangle - \langle x^*(T),x(T)\rangle = f(x(0),x(T)) \end{array}$$

and this proves the inequality (40). Further, let the quadruple $\{\tilde{x}^*(\cdot), \tilde{v}^*(\cdot), \tilde{\mu}^*(0), \tilde{\mu}^*(T)\}$ satisfy conditions $(a)-(c)$ of Theorem 5.1. Then the Euler-Lagrange type inclusion (a) and the condition (b) of Theorem 5.1 imply that

$$H_F\left(x(t),x^*(t)\right) - H_F\left(\tilde{x}(t),\tilde{x}^*(t)\right) \leq -\left\langle \tilde{x}^{*\prime}(t)+\tilde{v}^*(t), x(t)-\tilde{x}(t)\right\rangle,$$

whence by the definition of function M_F we deduce that

$$-\left\langle \tilde{x}^{*\prime}(t)+\tilde{v}^*(t),\tilde{x}(t)\right\rangle - \left\langle \tilde{x}^*(t),\tilde{x}'(t)\right\rangle - H_F\left(\tilde{x}(t),\tilde{x}^*(t)\right) = M_F\left(-\tilde{x}^{*\prime}(t)-\tilde{v}^*(t),\tilde{x}^*(t)\right). \tag{45}$$

Further, by the transversality condition (c) there is

$$-W_S\left(-\tilde{\mu}^*(0),-\tilde{\mu}^*(T)\right) = \langle \tilde{x}(0),\tilde{\mu}^*(0)\rangle + \langle \tilde{x}(T),\tilde{\mu}^*(T)\rangle; W_{D_t}\left(-\tilde{v}^*(t) = -\langle \tilde{x}(t),\tilde{v}^*(t)\rangle\right.. \tag{46}$$

Consequently, by Theorem 1.27 [10] the transversality condition (c) is equivalent to the relation

$$\begin{aligned} & -f^*\left(\tilde{x}^*(0)+\tilde{\mu}^*(0), \tilde{\mu}^*(T)-\tilde{x}^*(T)\right) \\ & =f(\tilde{x}(0), \tilde{x}(T))-\left\langle\tilde{x}(0), \tilde{x}^*(0)+\tilde{\mu}^*(0)\right\rangle-\left\langle\tilde{x}(T), \tilde{\mu}^*(T)-\tilde{x}^*(T)\right\rangle . \end{aligned} \tag{47}$$

Hence, if we take into account relations (45)-(47) in (40), then we will make sure that the equality will satisfied and for $\tilde{x}(\cdot)$, $\{\tilde{x}^*(\cdot), \tilde{v}^*(\cdot), \tilde{\mu}^*(0), \tilde{\mu}^*(T)\}$ the equality of the values of the primal and dual problems is guaranteed. Hence, $\tilde{x}(\cdot)$, $\{\tilde{x}^*(\cdot), \tilde{v}^*(\cdot), \tilde{\mu}^*(0), \tilde{\mu}^*(T)\}$ (a) - (c) is the dual relation for the primal and dual problems. Thus, we have proved that from conditions (a) - (c) it follows that $\{\tilde{x}^*(\cdot), \tilde{v}^*(\cdot), \tilde{\mu}^*(0), \tilde{\mu}^*(T)\}$ is a solution of the dual problem (PC*). The converse is proved by analogy.
Regarding (c), it suffices to recall that (47) is equivalent to (c) inscribed for a family of functions $\{\tilde{x}^*(\cdot)\tilde{v}^*(\cdot), \tilde{\mu}^*(0), \tilde{\mu}^*(T)\}$. The proof of theorem is completed. □

9. Duality in some optimal control problems.

9.1. **Linear discrete problem.**

Let us consider the problem:

$$\begin{aligned} & \text{minimize } f\left(x_0, x_T\right), \\ \text{subject to} \quad & x_{t+1}=A x_t+B u_t, u_t \in U, t=0, \ldots, T-1 \\ & \left(x_0, x_T\right) \in S, x_t \in D_t, t=1, \ldots, T-1, \end{aligned} \tag{48}$$

where A and B are $n \times n$ and $n \times r$ matrices, correspondingly, f convex function, $U \subseteq \mathbb{R}^r$, $S \subseteq \mathbb{R}^{2n}, X_t \subseteq \mathbb{R}^n(t=1, \ldots, T-1)$. Clearly, in this case $F\left(x_t\right)=A x_t+B U, t= 0, \ldots, T-1$ and

$$M_F\left(x_t^*-v_t^*, x_{t+1}^*\right)=\left\{\begin{array}{cc} -W_U\left(B^* x_{t+1}^*\right), & \text{if } x_t^*-v_t^*=A^* x_{t+1}^* \\ -\infty, & \text{otherwise.} \end{array}\right.$$

Then it is clear that, the problem dual to problem (48) has the form

$$\sup_{x_t^*, \mu_0^*, \mu_T^*}\left\{-f^*\left(x_0^*+\mu_0^*, \mu_T^*-x_T^*\right)-\sum_{t=0}^{T-1} W_U\left(B^* x_{t+1}^*\right)-\sum_{t=1}^{T-1} W_{D_t}\left(A^* x_{t+1}^*-x_t^*\right)\right. \\ \left.-W_S\left(-\mu_0^*,-\mu_T^*\right)\right\}.$$

9.2. **The polyhedral problem.**

Here we establish a dual problem $\left(\mathrm{PL}^*\right)$ to a problem with a first-order polyhedral DFI and constraints on undivided endpoints and states:

$$\begin{aligned} & \text{minimize } f(x(0), x(T)), \\ (\mathrm{PL}) \qquad & x^{\prime}(t) \in F(x(t)), \text{ a.e. } t \in[0, T], F(x)=\{y: A x-B y \leq d\}, \\ & (x(0), x(t)) \in S, \quad x(t) \in X(t), t \in[0, T] \end{aligned}$$

where F is polyhedral a mapping, A, B are $s \times n$ matrices, d is a s dimensional column-vector, $f: \mathbb{R}^{2n} \rightarrow \mathbb{R}^1$, $S \subseteq \mathbb{R}^{2n}, D(t) \subseteq \mathbb{R}^n$. We label this problem by (PL). According to the dual problem $\left(\mathrm{PC}^*\right)$, we calculate the function $M_F\left(x^*, y^*\right):$

$$M_F\left(x^*, y^*\right)=\inf \left\{\left\langle x, x^*\right\rangle-\left\langle y, y^*\right\rangle:(x, y) \in \operatorname{gph} F\right\}. \tag{49}$$

Let us denote $w=(x, y) \in \mathbb{R}^{2n}, w^*=\left(x^*,-y^*\right) \in \mathbb{R}^{2n}$. Then we have:

$$\inf \left\{\left\langle w, w^*\right\rangle: L w \leq d\right\}, \tag{50}$$

where $L = [A : -B]$ is $s \times 2n$ matrix. Then, It follows from (50) that

$$w^* = -L^*\lambda, \langle A\tilde{x} - B\tilde{y} - d, \lambda \rangle = 0, \lambda \geq 0.$$

Hence, $w^* = -L^*\lambda$ implies that $x^* = -A^*\lambda, y^* = B^*\lambda, \lambda \geq 0$. Therefore

$$M_F(x^*, y^*) = \langle \tilde{x}, -A^*\lambda \rangle + \langle \tilde{y}, B^*\lambda \rangle = -\langle A\tilde{x}, \lambda \rangle + \langle B\tilde{y}, \lambda \rangle = -\langle d, \lambda \rangle. \tag{51}$$

Then from the form of $M_F(-x^{*\prime}(t) - v^*(t), x^*(t))$ we get

$$-x^{*\prime}(t) - v^*(t) = -A^*\lambda(t), x^*(t) = B^*\lambda(t), \quad \lambda(t) \geq 0 \tag{52}$$

or

$$A^*\lambda(t) - B^*\lambda'(t) = v^*(t), \quad \lambda(t) \geq 0. \tag{53}$$

Therefore, taking into account (51)-(53) we obtain the dual problem

$$\sup_{\lambda(t)\geq 0, \mu^*(0), \mu^*(T)} \left\{ -f^*\left(B^*\lambda(0) + \mu^*(0), \mu^*(T) - B^*\lambda(T)\right) - \int_0^T \langle d, \lambda(t) \rangle dt \right.$$
$$\left. - \int_0^T W_{D_t}\left(B^*\lambda'(t) - A^*\lambda(t)\right) dt - W_S\left(-\mu^*(0), -\mu^*(T)\right) \right\}.$$

References

[1] Agarwal, R.P, Benchohra, M, Hamani S (2008) Boundary value problems for differential inclusions with fractional order, Advanced Studies in Contemporary Mathematics 16 (2), 181-196

[2] Farajzadeh, A. P., Zanganehmehr, P. & Hasanoğlu, E. (2021). A new form of Kakutani fixed point theorem and intersection theorem with applications. Appl. Comput. Math. $20(3), 430 - 435$.

[3] Grace, S.R, Agarwal, R.P, O'Regan, D(2009). A selection of oscillation criteria for second-order differential inclusions, Applied Mathematics Letters 22 (2), 153-158.

[4] Kaddouri, H., Beloul, S (2021). Fixed point theorems for multivalued Wordowski type contractions in B-metric spaces with an application to integral inclusions, TWMS J. App. Eng. Math. 11(4), 1061-1071.

[5] Loewen, P.D., Rockafellar, R.T. (1994). Optimal control of unbounded differential inclusions. SIAM J Contr. Optim. 32: 442-470.

[6] Mahmudov, N.I, Vijayakumar, V, Murugesu, R (2016) Approximate controllability of second-order evolution differential inclusions in Hilbert spaces, Mediter. J. Math. 13 (5), $3433 - 3454$.

[7] Mahmudov, E.N. (2006). Necessary and sufficient conditions for discrete and differential inclusions of elliptic type. J. Math. Anal. Appl. 323: 768-789.

[8] Mahmudov, E.N. (2007). Locally adjoint mappings and optimization of the first boundary value problem for hyperbolic type discrete and differential inclusions. Nonlinear Anal. Theory Methods Appl. 67: 2966-2981.

[9] Mahmudov, E.N. (2010). Duality in the problems of optimal control described by firstorder partial differential inclusions. Optimization. 59: 4589-599.

[10] Mahmudov, E.N. (2011) Approximation and Optimization of Discrete and Differential Inclusions. Boston, USA: Elsevier.

[11] Mahmudov, E.N (2012). Optimization of elliptic type differential inclusions and duality, TWMS Journal of applied and engineering mathematics 2 (1), 60-66.

[12] Mahmudov, E.N. (2013). Single Variable Differential and Integral Calculus. Mathematical Analysis. Paris, France: Springer.

[13] Mahmudov, E.N. (2014) Approximation and optimization of higher order discrete and differential inclusions. Nonlin. Differ. Equ. Appl. 21: 1-26.

[14] Mahmudov, E.N. (2015). Optimization of Second-Order Discrete Approximation Inclusions. Num. Funct. Anal. Optim. 36: 624-643.

[15] Mahmudov, E.N., Mastaliyeva, D.I (2022) Optimal control of higher-order retarded differential inclusions, J. Industr. Manag. Optim. doi: 10.3934/jimo.2022226, 1-14.

[16] Mahmudov, E.N. (2018). Optimization of Mayer Problem with Sturm-Liouville Type Differential Inclusions. J. Optim, Theory Appl. 177: 345-375.

[17] Mahmudov, E.N. (2019). Optimal Control of Higher Order Differential Inclusions with Functional Constraints. ESAIM: COCV, DOI: https://doi.org/10.1051/cocv/2019018.

[18] Mahmudov, E.N. (2020) Infimal Convolution and Duality in Convex Optimal Control Problems with Second Order Evolution Differential Inclusions. Evol. Equ. Contr. Theory, EECT, doi: 10.3934/eect.2020051, 1-23.

[19] Mordukhovich, B.S. (2006). Variational Analysis and Generalized Differentiation, I: Basic Theory; II: Applications, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. 330 and 331, Berlin, Heidelberg: Springer.

[20] Rockafellar, R.T. (1974). Conjugate Duality and Optimization. Philadelphia: Society for Industrial and Applied Mathematics, https://doi.org/10.1137/1.9781611970524.

[21] Van Hang, N.T., Yen, N.D. (2015). Optimality Conditions and Stability Analysis via the Mordukhovich Subdifferential. Numer. Funct. Anal. Optim. 36: 364-386.

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