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SOME ALGEBRAIC STRUCTURE OF SPHERICAL NEUTROSOPHIC MATRICES

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Abstract. In this paper, we introduce spherical neutrosophic matrices (SNMs) as generalization of intuitionistic fuzzy matrices, Pythagorean fuzzy matrices, picture fuzzy matrices and spherical fuzzy matrices. Some algebraic operations such as max-min, min-max, complement, algebraic sum and algebraic product are defined in SNMs and investigated. Further, scalar multiplication (nA) and exponentiation (A^n) operations of a spherical neutrosophic matrix A using algebraic operations are constructed, and their desirable properties are proved. Finally, define a new operation(@) on spherical neutrosophic matrices and discuss distributive law in the case where the operations of $\oplus, \otimes, \wedge$ and $\vee$ are combined each other.

Keywords: Intuitionistic fuzzy matrix, Pythagorean fuzzy matrix, Picture fuzzy matrix. Spherical fuzzy matrix, Spherical Neutrosophic Matrix, Algebraic sum, Algebraic product, Scalar multiplication, Exponentiation operations.

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1. Introduction

The concept of an intuitionistic fuzzy matrix (IFM) was introduced by Khan et al. [1] and simultaneously Im et al. [2] to generalize the concept of Thomason's [3] fuzzy matrix. Each element in an IFM is expressed by an ordered pair $\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \rangle$ with $\zeta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ and $0 \leq \zeta_{a_{ij}} + \delta_{a_{ij}} \leq 1$. Since the presence of IFM, a few analysts have significantly added to the improvement of IFM hypothesis and its applications [4, 5, 6, 7, 8, 9, 10, 11]. In such a circumstance, to accomplish a sensible result IFM falls flat. In this way, managing such circumstance, [12] established the concept of Pythagorean fuzzy matrices (PyFM) by assigning membership degree say $\zeta_{a_{ij}}$ along with non-membership degree say $\delta_{a_{ij}}$ with condition that $0 \leq \zeta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$. For further study on Pythagorean fuzzy matrix, one may refer to [13, 14, 15].

Dogra and Pal [16] construction of picture fuzzy matrices (PFM) is of exceptional reputation but decision makers are some how restricted in assigning values due to the condition on $\eta_{a_{ij}}$, $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$. In [17], some algebraic operations of Picture fuzzy matrices

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are defined and their desirable properties are proved. Spherical fuzzy matrices have its own importance in a circumstance where opinion is not only constrained to yes or no but there is some sort of abstinence or refusal. A good example of spherical fuzzy matrix could be decision making such as when four decision makers have four different categories of opinion about a candidate. Another example could be of voting where four types of voters occurs who vote in favor or vote against or refuse to vote or abstain. Spherical fuzzy matrix is a direct generalization of fuzzy matrix, intuitionistic fuzzy matrix and picture fuzzy matrix. A question arises that why we need spherical fuzzy matrix or what are the boundaries of PFMs that leads us to spherical fuzzy matrices? The main downside of PFMs is the restriction on it, i.e., $0 \leq \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leq 1$. As this condition does not allows the decision makers to assign membership values of their own consent. The decision makers are somehow limitations in a specific domain. We consider an example $\zeta_{a_{ij}} = 0.8$, $\eta_{a_{ij}} = 0.5$ and $\delta_{a_{ij}} = 0.3$ which interrupts the condition that $0 \leq \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leq 1$ but if we take the square of these values such as, $\zeta^2_{a_{ij}} = 0.64$, $\eta^2_{a_{ij}} = 0.25$ and $\delta^2_{a_{ij}} = 0.09$ where the condition $0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$ is satisfies [18].

We know the important role of matrices in science and technology. However, the classical matrix theory sometimes fails to solve the problems involving uncertainties, occurring in an imprecise environment. Kandasamy and Smarandache [19] introduced fuzzy relational maps and neutrosophic relational maps. Dhar et al.[20] define Square Neutrosophic Fuzzy Matrices whose entries are of the form $a + Ib$, where a and b are fuzzy number from $[0, 1]$ gives the definition of Neutrosophic Fuzzy Matrices multiplication.

In this paper develop the concept of spherical fuzzy matrices to spherical neutrosophic Matrices by assigning neutral membership degree say $\eta_{a_{ij}}$ along with positive and negative membership degrees say $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$ with condition that $0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 3$.

This paper is organized as follows. In section 2, we recall some preliminary definitions regarding the topic. In section 3, we define spherical neutrosophic matrices and investigates their algebraic properties. In section 4, define a new operation(@) on spherical neutrosophic matrices and investigates their desirable properties. We write the conclusion of the paper in section 5.

2. Preliminary Definitions

Here we recall some preliminary definitions regarding the topic.

Definition 2.1. [1] *An intuitionistic fuzzy matrix (IFM) of order $m \times n$ is defined as $A = \left(\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ where $\zeta_{a_{ij}} \in [0, 1]$ and $\delta_{a_{ij}} \in [0, 1]$ are the membership and non-membership values of the ij^{th} element in A satisfying the condition*

$$0 \leq \zeta_{a_{ij}} + \delta_{a_{ij}} \leq 1$$

for all i, j.

Definition 2.2. [12] *A Pythagorean fuzzy matrix (PyFM) of order $m \times n$ is defined as $A = \left(\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ where $\zeta_{a_{ij}} \in [0, 1]$ and $\delta_{a_{ij}} \in [0, 1]$ are the membership and non-membership values of the ij^{th} element in A satisfying the condition*

$$0 \leq \zeta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$$

for all i, j.

Definition 2.3. [17] *A Picture fuzzy matrix (PFM) A of the form, $A = \left(\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0, 1]$ satisfying the condition*

$$0 \leq \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leq 1$$

for all i,j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership, $\eta_{a_{ij}} \in [0,1]$ is called the degree of neutral membership and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

Definition 2.4. [18] *A Spherical fuzzy matrix (SFM) A of the form, $A = \left(\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ satisfying the condition*

$$0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$$

for all i,j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership, $\eta_{a_{ij}} \in [0,1]$ is called the degree of neutral membership and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

Definition 2.5. [19] *Let A be a neutrosophic fuzzy matrix, whose entries is of the form a+Ib (neutrosophic number), where a, b are the elements of [0,1] and I is an indeterminate such that $I^n = I$, n being a positive integer.*

3. Spherical neutrosophic matrices and their basic operations

This section, define spherical neutrosophic matrices and investigates some algebraic properties such as idempotency, commutativity, associativity, absorption distributivity, and De Morgan's laws over complement.

Now, we are going to define algebraic operations of Spherical Neutrosophic Matrices by restricting the measure of positive, neutral and negative membership but keeping their sum in the interval $[0, \sqrt[2]{3}]$.

Definition 3.1. *A Spherical neutrosophic matrix (SNM) A of the form, $A = \langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ satisfying the condition*

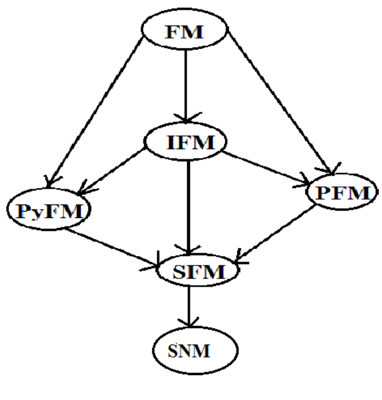
$$0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 3$$

for all i,j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership, $\eta_{a_{ij}} \in [0,1]$ is called the degree of neutral membership and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

Let $N_{m\times n}$ denote the set of all the Spherical Neutrosophic Matrices.

Example 3.1. $A = \begin{bmatrix} (0.8,0.8,0.8) & (0.2,0.4,0.2) \\ (0.3,0.4,0.2) & (0.4,0.4,0.2) \end{bmatrix}$ *is not a SFM, but it A is a SNM. The order structure of the circular fuzzy matrix is appeared in Fig. 1.*

FIGURE.1 The structure between FM, IFM, PyFM, PFM, SFM and SNM.



Each element in an PFM is expressed by an ordered pair $\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle$ with $\zeta_{a_{ij}}, \eta_{a_{ij}}$ and $\delta_{a_{ij}} \in [0,1]$ and $0 \leq \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leq 1$. It was clearly seen that $0.8+0.8+0.8 > 1$, and thus it could not be described by PFM and SFM. To describe such evaluation in this *paper we have proposed spherical neutrosophic matrix (SNM)and its algebraic operations. Each element in an SNM is expressed by an ordered pair* $\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle$ *with* $\zeta_{a_{ij}}, \eta_{a_{ij}}$ *and* $\delta_{a_{ij}} \in [0,1]$ *and* $0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 3$*. Also, we can get* $(0.8)^2 + (0.8)^2 + (0.8)^2 = 0.64 + 0.64 + 0.64 = 1.92 \leq 3$*, which is good enough to apply the SNM to control it.*

Definition 3.2. *If A and B are two spherical neutrosophic matrices, then*

- $A < B$ *iff* $\forall i, j, \zeta_{a_{ij}} \leq \zeta_{b_{ij}}, \eta_{a_{ij}} \leq \eta_{b_{ij}}$ *or* $\eta_{a_{ij}} \geq \eta_{b_{ij}}, \delta_{a_{ij}} \geq \delta_{b_{ij}}$
- $A^C = \left(\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}} \rangle\right)$
- $A \vee B = \left(\langle \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) \rangle\right)$
- $A \wedge B = \left(\langle \min\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) \rangle\right)$
- $A \oplus B = \left(\left\langle \sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$
- $A \otimes B = \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}, \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}} \right\rangle\right)$.

Definition 3.3. *The scalar multiplication operation over SNM A and is defined by*

$$nA = \left(\left\langle \sqrt{1 - [1 - \zeta^2_{a_{ij}}]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n \right\rangle\right)$$

Definition 3.4. *The exponentiation operation over SNM A and is defined by*

$$A^n = \left(\left\langle [\zeta_{a_{ij}}]^n, \sqrt{1 - [1 - \eta^2_{a_{ij}}]^n}, \sqrt{1 - [1 - \delta^2_{a_{ij}}]^n} \right\rangle\right).$$

Let $N_{m \times n}$ denote the set of all the Spherical Neutrosophic Matrices.
The following theorem relation between algebraic sum, and algebraic product of SNMs.

Theorem 3.1. *If* $A, B \in N_{m \times n}$*, then* $A \otimes B \leq A \oplus B$*.*

Proof. Let $A \oplus B = \left(\left\langle \sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$ and
$A \otimes B = \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}, \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}} \right\rangle\right)$
Assume that,

$$\zeta_{a_{ij}}\zeta_{b_{ij}} \leq \sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}$$

$(i.e)$ $\quad \zeta_{a_{ij}}\zeta_{b_{ij}} - \sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}} \geq 0$

$(i.e)$ $\quad \zeta^2_{a_{ij}}(1 - \zeta^2_{b_{ij}}) + \zeta^2_{b_{ij}}(1 - \zeta^2_{a_{ij}}) \geq 0$

which is true as $0 \leq \zeta^2_{a_{ij}} \leq 1$ and $0 \leq \zeta^2_{b_{ij}} \leq 1$
And

$$\eta_{a_{ij}}\eta_{b_{ij}} \leq \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}$$

$(i.e)$ $\quad \eta_{a_{ij}}\eta_{b_{ij}} - \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}} \geq 0$

$(i.e)$ $\quad \eta^2_{a_{ij}}(1 - \eta^2_{b_{ij}}) + \eta^2_{b_{ij}}(1 - \eta^2_{a_{ij}}) \geq 0$

which is true as $0 \leq \eta^2_{a_{ij}} \leq 1$ and $0 \leq \eta^2_{b_{ij}} \leq 1$
And

$$\delta_{a_{ij}}\delta_{b_{ij}} \leq \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}}$$

$(i.e)$ $\quad \delta_{a_{ij}}\delta_{b_{ij}} - \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}} \geq 0$

$(i.e)$ $\quad \delta^2_{a_{ij}}(1 - \delta^2_{b_{ij}}) + \delta^2_{b_{ij}}(1 - \delta^2_{a_{ij}}) \geq 0$

which is true as $0 \leq \delta^2_{a_{ij}} \leq 1$ and $0 \leq \delta^2_{b_{ij}} \leq 1$
Hence $A \otimes B \leq A \oplus B$. □

Theorem 3.2. *For any spherical neutrosophic matrix A,*
(i) $A \oplus A \geq A$,
(ii) $A \otimes A \leq A$.

Proof. (i) Let $A \oplus A = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right) \oplus \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$
$A \oplus A = \left(\left\langle \sqrt{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^2}, (\eta_{a_{ij}})^2, (\delta_{a_{ij}})^2 \right\rangle\right)$
$\sqrt{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^2} = \sqrt{\zeta_{a_{ij}} + \zeta_{a_{ij}}(1 - \zeta_{a_{ij}})} \geq \zeta_{a_{ij}}$ for all i, j
and $(\eta_{a_{ij}})^2 \leq \eta_{a_{ij}}$ for all i, j
and $(\delta_{a_{ij}})^2 \leq \delta_{a_{ij}}$ for all i, j
Hence $A \oplus A \geq A$.
Similarly, we can prove that (ii) $A \otimes A \leq A$. □

Theorem 3.3. *If* $A, B, C \in N_{m\times n}$, *then*
(i) $A \oplus B = B \oplus A$,
(ii) $A \otimes B = B \otimes B$,
(iii) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
(iv) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

Proof. (i) Let $A \oplus B$

$$= \left(\left\langle \sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$$
$$= \left(\left\langle \sqrt{\zeta_{b_{ij}}^2 + \zeta_{a_{ij}}^2 - \zeta_{b_{ij}}^2 \zeta_{a_{ij}}^2}, \eta_{b_{ij}}\eta_{a_{ij}}, \delta_{b_{ij}}\delta_{a_{ij}} \right\rangle\right)$$
$$= B \oplus A.$$

(ii) Let $A \otimes B$

$$= \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt{\eta_{a_{ij}}^2 + \eta_{b_{ij}}^2 - \eta_{a_{ij}}^2 \eta_{b_{ij}}^2}, \sqrt{\delta_{a_{ij}}^2 + \delta_{b_{ij}}^2 - \delta_{a_{ij}}^2 \delta_{b_{ij}}^2} \right\rangle\right)$$
$$= \left(\left\langle \zeta_{b_{ij}}\zeta_{a_{ij}}, \sqrt{\eta_{b_{ij}}^2 + \eta_{a_{ij}}^2 - \eta_{b_{ij}}^2 \eta_{a_{ij}}^2}, \sqrt{\delta_{b_{ij}}^2 + \delta_{a_{ij}}^2 - \delta_{b_{ij}}^2 \delta_{a_{ij}}^2} \right\rangle\right)$$
$$= B \otimes A.$$

(iii) Let $(A \oplus B) \oplus C$

$$= \left(\left\langle \left(\sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\right) \oplus \left(\zeta_{c_{ij}}, \eta_{c_{ij}}, \delta_{c_{ij}}\right) \right\rangle\right)$$
$$= \Bigg[\sqrt{\left(\sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}\right)^2 + \zeta_{c_{ij}}^2 - \left(\sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}\right)^2 \zeta_{c_{ij}}^2},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$
$$= \Bigg[\sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 + \zeta_{c_{ij}}^2 - \zeta_{a_{ij}}^2\zeta_{b_{ij}}^2\zeta_{c_{ij}}^2 - \zeta_{a_{ij}}^2\zeta_{c_{ij}}^2 - \zeta_{b_{ij}}^2\zeta_{c_{ij}}^2 + \zeta_{a_{ij}}^2\zeta_{b_{ij}}^2\zeta_{c_{ij}}^2},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$
$$= \Bigg[\sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 + \zeta_{c_{ij}}^2 - \zeta_{a_{ij}}^2\zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2\zeta_{c_{ij}}^2 - \zeta_{b_{ij}}^2\zeta_{c_{ij}}^2 + \zeta_{a_{ij}}^2\zeta_{b_{ij}}^2\zeta_{c_{ij}}^2},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$

Let $A \oplus (B \oplus C)$

$$= \Bigg[\sqrt{\zeta_{a_{ij}}^2 + \left(\sqrt{\zeta_{b_{ij}}^2 + \zeta_{c_{ij}}^2 - \zeta_{b_{ij}}^2 \zeta_{c_{ij}}^2}\right)^2 - \zeta_{a_{ij}}^2 \left(\sqrt{\zeta_{b_{ij}}^2 + \zeta_{c_{ij}}^2 - \zeta_{b_{ij}}^2 \zeta_{c_{ij}}^2}\right)^2},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$
$$= \Bigg[\sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 + \zeta_{c_{ij}}^2 - \zeta_{a_{ij}}^2\zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2\zeta_{c_{ij}}^2 - \zeta_{b_{ij}}^2\zeta_{c_{ij}}^2 + \zeta_{a_{ij}}^2\zeta_{b_{ij}}^2\zeta_{c_{ij}}^2},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$

Hence $(A \oplus B) \oplus C = A \oplus (B \oplus C)$
Similarly, we can prove that (iv)$(A \otimes B) \otimes C = A \otimes (B \otimes C)$. □

Theorem 3.4. *If $A, B \in N_{m\times n}$, then*
(i) $A \oplus (A \otimes B) \geq A$,
(ii) $A \otimes (A \oplus B) \leq A$.

Proof. (i) Let $A \oplus (A \otimes B)$

$$= \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right) \oplus \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}, \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}} \right\rangle\right)$$

$$= \Big[\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{a_{ij}}\zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}[\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}]}, \eta_{a_{ij}}[\sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}], \delta_{a_{ij}}[\sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}}]\Big]$$

$$= \Big[\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}[1 - \zeta^2_{a_{ij}}]}, \eta_{a_{ij}}\left(\sqrt{1 - [1 - \eta^2_{a_{ij}}][1 - \eta^2_{b_{ij}}]}\right), \delta_{a_{ij}}\left(\sqrt{1 - [1 - \delta^2_{a_{ij}}][1 - \delta^2_{b_{ij}}]}\right)\Big]$$

$$\geq A.$$

Hence $A \oplus (A \otimes B) \geq A$.
Similarly, we can prove that $(ii) A \otimes (A \oplus B) \leq A$. □

The following theorem is obvious.

Theorem 3.5. *If $A, B \in N_{m\times n}$, then*
(i) $A \vee B = B \vee A$,
(ii) $A \wedge B = B \wedge A$,

Theorem 3.6. *If $A, B, C \in N_{m\times n}$, then*
(i) $A \oplus (B \vee C) = (A \oplus B) \vee (A \oplus C)$,
(ii) $A \otimes (B \vee C) = (A \otimes B) \vee (A \otimes C)$,
(iii) $A \oplus (B \wedge C) = (A \oplus B) \wedge (A \oplus C)$,
(iv) $A \otimes (B \wedge C) = (A \otimes B) \wedge (A \otimes C)$.

Proof. In the following, we shall prove (i), and $(ii) - (iv)$ can be proved analogously.
(i) Let $A \oplus (B \vee C)$

$$= \Big[\sqrt{\zeta^2_{a_{ij}} + \max\left(\zeta^2_{b_{ij}}, \zeta^2_{c_{ij}}\right) - \zeta^2_{a_{ij}}.\max\left(\zeta^2_{b_{ij}}, \zeta^2_{c_{ij}}\right)}, \eta_{a_{ij}}.\max\left(\eta_{b_{ij}}, \eta_{c_{ij}}\right), \delta_{a_{ij}}.\max\left(\delta_{b_{ij}}, \delta_{c_{ij}}\right)\Big]$$

$$= \Big[\sqrt{\max\left(\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}}, \zeta_{a^2_{ij}} + \zeta^2_{c_{ij}}\right) - \max\left(\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}, \zeta^2_{a_{ij}}\zeta^2_{c_{ij}}\right)}, \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \eta_{a_{ij}}\eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \delta_{a_{ij}}\delta_{c_{ij}}\right)\Big]$$

$$= \Big[\sqrt{\max\left(\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}, \zeta^2_{a_{ij}} + \zeta^2_{c_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{c_{ij}}\right)}, \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \eta_{a_{ij}}\eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \delta_{a_{ij}}\delta_{c_{ij}}\right)\Big]$$

$$= (A \oplus B) \vee (A \oplus C).$$ □

Theorem 3.7. *If $A, B \in N_{m\times n}$, then*
(i) $(A \wedge B) \oplus (A \vee B) = A \oplus B$,
(ii) $(A \wedge B) \otimes (A \vee B) = A \otimes B$,
(iii) $(A \oplus B) \wedge (A \otimes B) = A \otimes B$,
(iv) $(A \oplus B) \vee (A \otimes B) = A \oplus B$.

Proof. In the following, we shall prove (i), and $(ii)-(iv)$ can be proved analogously.
(i) Let $(A\wedge B)\oplus(A\vee B)$

$$=\left[\sqrt{\min\left(\zeta^2_{a_{ij}},\zeta^2_{b_{ij}}\right)+\max\left(\zeta^2_{a_{ij}},\zeta^2_{b_{ij}}\right)-\min\left(\zeta^2_{a_{ij}},\zeta^2_{b_{ij}}\right).\max\left(\zeta^2_{a_{ij}},\zeta^2_{b_{ij}}\right)},\right.$$
$$\left.\max\left(\eta_{a_{ij}},\eta_{b_{ij}}\right).\min\left(\eta_{a_{ij}},\eta_{b_{ij}}\right),\quad \max\left(\delta_{a_{ij}},\delta_{b_{ij}}\right).\min\left(\delta_{a_{ij}},\delta_{b_{ij}}\right)\right]$$
$$=\left(\left\langle\sqrt{\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}},\eta_{a_{ij}}\eta_{b_{ij}},\delta_{a_{ij}}\delta_{b_{ij}}\right\rangle\right)$$
$$=A\oplus B.$$
□

In the following theorems, the operator complement obey th De Morgan's laws for the operation $\oplus,\otimes,\vee,\wedge$.

Theorem 3.8. *If $A,B\in N_{m\times n}$, then*
(i) $(A\oplus B)^C=A^C\otimes B^C$,
(ii) $(A\otimes B)^C=A^C\oplus B^C$,
(iii) $(A\oplus B)^C\leq A^C\oplus B^C$,
(iv) $(A\otimes B)^C\geq A^C\otimes B^C$.

Proof. We shall prove $(iii),(iv)$, and $(i),(ii)$ are straightfforward.
(iii) Let $(A\oplus B)^C=\left(\left\langle\delta_{a_{ij}}\delta_{b_{ij}},\sqrt{\eta^2_{a_{ij}}+\eta^2_{b_{ij}}-\eta^2_{a_{ij}}\eta^2_{b_{ij}}},\sqrt{\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}\right\rangle\right)$.
$A^C\oplus B^C=\left(\left\langle\sqrt{\delta^2_{a_{ij}}+\delta^2_{b_{ij}}-\delta^2_{a_{ij}}\delta^2_{b_{ij}}},\eta_{a_{ij}}\eta_{b_{ij}},\zeta_{a_{ij}}\zeta_{b_{ij}}\right\rangle\right)$.
Since $\delta_{a_{ij}}\delta_{b_{ij}}\leq\sqrt{\delta^2_{a_{ij}}+\delta^2_{b_{ij}}-\delta^2_{a_{ij}}\delta^2_{b_{ij}}}$
$\sqrt{\eta^2_{a_{ij}}+\eta^2_{b_{ij}}-\eta^2_{a_{ij}}\eta^2_{b_{ij}}}\geq\eta_{a_{ij}}\eta_{b_{ij}}$
$\sqrt{\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}\geq\zeta_{a_{ij}}\zeta_{b_{ij}}$
Hence $(A\oplus B)^C\leq A^C\oplus B^C$.
(iv) Let $(A\otimes B)^C=\left(\left\langle\sqrt{\delta^2_{a_{ij}}+\delta^2_{b_{ij}}-\delta^2_{a_{ij}}\delta^2_{b_{ij}}},\eta_{a_{ij}}\eta_{b_{ij}},\zeta_{a_{ij}}\zeta_{b_{ij}}\right\rangle\right)$.
$A^C\otimes B^C=\left(\left\langle\delta_{a_{ij}}\delta_{b_{ij}},\sqrt{\eta^2_{a_{ij}}+\eta^2_{b_{ij}}-\eta^2_{a_{ij}}\eta^2_{b_{ij}}},\sqrt{\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}\right\rangle\right)$.
Since $\sqrt{\delta^2_{a_{ij}}+\delta^2_{b_{ij}}-\delta^2_{a_{ij}}\delta^2_{b_{ij}}}\geq\delta_{a_{ij}}\delta_{b_{ij}}$
$\eta_{a_{ij}}\eta_{b_{ij}}\leq\sqrt{\eta^2_{a_{ij}}+\eta^2_{b_{ij}}-\eta^2_{a_{ij}}\eta^2_{b_{ij}}}$
$\zeta_{a_{ij}}\zeta_{b_{ij}}\leq\sqrt{\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}$
Hence $(A\otimes B)^C\geq A^C\otimes B^C$. □

Theorem 3.9. *If $A,B\in N_{m\times n}$, then*
(i) $(A^C)^C=A$,
(ii) $(A\vee B)^C=A^C\wedge B^C$,
(iii) $(A\wedge B)^C=A^C\vee B^C$.

Proof. We shall prove (ii) only, (i) is obvious.
$A\vee B=\left(\left\langle\max\left(\zeta_{a_{ij}},\zeta_{b_{ij}}\right),\min\left(\eta_{a_{ij}},\eta_{b_{ij}}\right),\min\left(\delta_{a_{ij}},\delta_{b_{ij}}\right)\right\rangle\right)$
$(A\vee B)^C=\left(\left\langle\min\left(\delta_{a_{ij}},\delta_{b_{ij}}\right),\min\left(\eta_{a_{ij}},\eta_{b_{ij}}\right),\max\left(\zeta_{a_{ij}},\zeta_{b_{ij}}\right)\right\rangle\right)$
$\Rightarrow A^C=\left(\left\langle\delta_{a_{ij}},\eta_{a_{ij}},\zeta_{a_{ij}}\right\rangle\right)$
$B^C=\left(\left\langle\delta_{b_{ij}},\eta_{b_{ij}},\zeta_{b_{ij}}\right\rangle\right)$
$\Rightarrow A^C\wedge B^C=\left(\left\langle\min\left(\delta_{a_{ij}},\delta_{b_{ij}}\right),\min\left(\eta_{a_{ij}},\eta_{b_{ij}}\right),\max\left(\zeta_{a_{ij}},\zeta_{b_{ij}}\right)\right\rangle\right)$

Hence $(A \vee B)^C = A^C \wedge B^C$,
Similarly, we can prove that $(iii)(A \wedge B)^C = A^C \vee B^C$. □

Based on the Definition 3.2, 3.3 and 3.4., we shall next prove the algebraic properties of spherical neutrosophic matrices under the operations of scalar multiplication and exponentiation.

Theorem 3.10. *For $A, B \in N_{m\times n}$ and $n, n_1, n_2 > 0$, we have*
(i) $n(A \oplus B) = nA \oplus nB$,
(ii) $n_1A \oplus n_2A = (n_1 + n_2)A$,
(iii) $(A \otimes B)^n = A^n \otimes B^n$,
(iv) $A_1^n \otimes A_2^n = A^{(n_1+n_2)}$.

Proof. For the two SNMs A and B, and $n, n_1, n_2 > 0$, according to definition, we can obtain
(i) Let $n(A \oplus B)$

$$= n\left(\left\langle \sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2\zeta_{b_{ij}}^2}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$$
$$= \left(\left\langle \sqrt{1-[1-\zeta_{a_{ij}}^2]^n[1-\zeta_{a_{ij}}^2]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$
$$= \left(\left\langle \sqrt{1-[1-\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2\zeta_{b_{ij}}^2]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$

$nA \oplus nB$

$$= \left(\left\langle \left(\sqrt{1-[1-\zeta_{a_{ij}}^2]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n\right) \oplus \left(\sqrt{1-[1-\zeta_{b_{ij}}^2]^n}, [\eta_{b_{ij}}]^n, [\delta_{b_{ij}}]^n\right) \right\rangle\right)$$
$$= \Big[\sqrt{\left(1-[1-\zeta_{a_{ij}}^2]^n + 1-[1-\zeta_{a_{ij}}^2]^n\right) - \left(1-[1-\zeta_{a_{ij}}^2]^n\right)\left(1-[1-\zeta_{b_{ij}}^2]^n\right)},$$
$$[\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n\Big]$$
$$= \left(\left\langle \sqrt{1-[1-\zeta_{a_{ij}}^2]^n[1-\zeta_{b_{ij}}^2]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$
$$= \left(\left\langle \sqrt{1-[1-\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2\zeta_{b_{ij}}^2]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$
$$= n(A \oplus B).$$

(ii)Let $n_1A \oplus n_2B$

$$= \left(\left\langle \left(\sqrt{1-[1-\zeta_{a_{ij}}^2]^{n_1}}, [\eta_{a_{ij}}]^{n_1}, [\delta_{a_{ij}}]^{n_1}\right) \oplus \left(\sqrt{1-[1-\zeta_{a_{ij}}^2]^{n_2}}, [\eta_{a_{ij}}]^{n_2}, [\delta_{a_{ij}}]^{n_2}\right) \right\rangle\right)$$
$$= \Big[\sqrt{1-[1-\zeta_{a_{ij}}^2]^{n_1} + 1-[1-\zeta_{a_{ij}}^2]^{n_2} - \left(1-[1-\zeta_{a_{ij}}^2]^{n_1}\right)\left(1-[1-\zeta_{a_{ij}}^2]^{n_2}\right)},$$
$$[\eta_{a_{ij}}]^{n_1}[\eta_{a_{ij}}]^{n_2}, [\delta_{a_{ij}}]^{n_1}[\delta_{a_{ij}}]^{n_2}\Big]$$
$$= \left(\left\langle \sqrt{1-[1-\zeta_{a_{ij}}^2]^{n_1+n_2}}, [\eta_{a_{ij}}]^{n_1+n_2}, [\delta_{a_{ij}}]^{n_1+n_2} \right\rangle\right)$$
$$= (n_1 + n_2)A.$$

(iii) Let $(A \otimes B)^n$

$$= \Big[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt{1-[1-\eta_{a_{ij}}^2 + \eta_{b_{ij}}^2 - \eta_{a_{ij}}^2\eta_{b_{ij}}^2]^n}, \sqrt{1-[1-\delta_{a_{ij}}^2 + \delta_{b_{ij}}^2 - \delta_{a_{ij}}^2\delta_{b_{ij}}^2]^n}\Big]$$
$$= \Big[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt{1-[1-\eta_{a_{ij}}^2]^n[1-\eta_{b_{ij}}^2]^n, 1-[1-\delta_{a_{ij}}^2]^n[1-\delta_{b_{ij}}^2]^n}\Big]$$

$A^n \otimes B^n$

$$= \Big[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt{1-[1-\eta_{a_{ij}}^2]^n + 1-[1-\eta_{b_{ij}}^2]^n - \left(1-[1-\eta_{a_{ij}}^2]^n\right)\left(1-[1-\eta_{b_{ij}}^2]^n\right)},$$
$$\sqrt{1-[1-\delta_{a_{ij}}^2]^n + 1-[1-\delta_{b_{ij}}^2]^n - \left(1-[1-\delta_{a_{ij}}^2]^n\right)\left(1-[1-\delta_{b_{ij}}^2]^n\right)}\Big]$$
$$= \left(\left\langle \left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt{1-[1-\eta_{a_{ij}}^2]^n[1-\eta_{b_{ij}}^2]^n}, \sqrt{1-[1-\delta_{a_{ij}}^2]^n[1-\delta_{b_{ij}}^2]^n} \right\rangle\right)$$

$= (A \otimes B)^n$.

(iv) Let $A^{n_1} \otimes A^{n_2}$

$$= \Bigg[\left(\zeta_{a_{ij}}\right)^{n_1+n_2},$$

$$\sqrt{1-[1-\eta^2_{a_{ij}}]^{n_1} + 1-[1-\eta^2_{a_{ij}}]^{n_2} - \left(1-[1-\eta^2_{a_{ij}}]^{n_1}\right)\left(1-[1-\eta^2_{a_{ij}}]^{n-2}\right)},$$

$$\sqrt{1-[1-\delta^2_{a_{ij}}]^{n_1} + 1-[1-\delta^2_{a_{ij}}]^{n-2} - \left(1-[1-\delta^2_{a_{ij}}]^{n_1}\right)\left(1-[1-\delta^2_{a_{ij}}]^{n_2}\right)} \Bigg]$$

$$= \left(\left\langle \left(\zeta_{a_{ij}}\right)^{n_1+n_2}, \sqrt{1-[1-\eta^2_{a_{ij}}]^{n_1+n_2}}, \sqrt{1-[1-\delta^2_{a_{ij}}]^{n_1+n_2}} \right\rangle\right)$$

$= A^{(n_1+n_2)}$.

Hence proved. □

Theorem 3.11. *For $A, B \in N_{m\times n}$ and $n > 0$, we have*
(i) $nA \le nB$,
(ii) $A^n \le B^n$.

Proof. (*i*) Let $A \le B$

$\Rightarrow \zeta_{a_{ij}} \le \zeta_{b_{ij}}$ and $\eta_{a_{ij}} \ge \eta_{b_{ij}}$ and $\delta_{a_{ij}} \ge \delta_{b_{ij}}$ for all i, j.

$\Rightarrow \sqrt{1-[1-\zeta^2_{a_{ij}}]^n} \le \sqrt{1-[1-\zeta^2_{b_{ij}}]^n}$,

$[\eta_{a_{ij}}]^n \ge [\eta_{b_{ij}}]^n$ and

$[\delta_{a_{ij}}]^n \ge [\delta_{b_{ij}}]^n$. for all i, j.

(*ii*) Also, $[\zeta_{a_{ij}}]^n \ge [\zeta_{b_{ij}}]^n$,

$\sqrt{1-[1-\eta^2_{a_{ij}}]^n} \le \sqrt{1-[1-\eta^2_{b_{ij}}]^n}$,

$\sqrt{1-[1-\delta^2_{a_{ij}}]^n} \le \sqrt{1-[1-\delta^2_{b_{ij}}]^n}$, for all i, j. □

Theorem 3.12. *For $A, B \in N_{m\times n}$ and $n > 0$, we have*
(i) $n(A \wedge B) = nA \wedge nB$,
(ii) $n(A \vee B) = nA \vee nB$.

Proof. (*i*) Let $n(A \wedge B)$

$$= \left[\sqrt{1-[1-\min\left(\zeta^2_{a_{ij}}, \zeta^2_{b_{ij}}\right)]^n}, \max\left([\eta_{a_{ij}}]^n, [\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n, [\delta_{b_{ij}}]^n\right)\right]$$

$$= \left[\sqrt{1-[\max\left(1-\zeta^2_{a_{ij}}, 1-\zeta^2_{b_{ij}}\right)]^n}, \max\left([\eta_{a_{ij}}]^n, [\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n, [\delta_{b_{ij}}]^n\right)\right]$$

$$= \left[\sqrt{1-\left(\max\left([1-\zeta^2_{a_{ij}}]^n, [1-\zeta^2_{b_{ij}}]^n\right)\right)}, \max\left([\eta_{a_{ij}}]^n, [\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n, [\delta_{b_{ij}}]^n\right)\right]$$

$$= \Bigg[\max\left(\sqrt{1-[1-\zeta^2_{a_{ij}}]^n}, \sqrt{1-[1-\zeta^2_{b_{ij}}]^n}\right), \max\left([\eta_{a_{ij}}]^n, [\eta_{b_{ij}}]^n\right),$$

$$\max\left([\delta_{a_{ij}}]^n, [\delta_{b_{ij}}]^n\right)\Bigg]$$

$= nA \wedge nB$.

Hence $n(A \wedge B) = nA \wedge nB$,

Similarly, we can prove that $(ii) n(A \vee B) = nA \vee nB$. □

Theorem 3.13. *For $A, B \in N_{m\times n}$ and $n > 0$, we have*
(i) $(A \wedge B)^n = A^n \wedge B^n$,
(ii) $(A \vee B)^n = A^n \vee B^n$.

Proof. (*i*) Let $(A \wedge B)^n$

$$= \Big[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \sqrt{1-[\max\left(1-\eta^2_{a_{ij}}, 1-\eta^2_{b_{ij}}\right)]^n}, \sqrt{1-[\max\left(1-\delta^2_{a_{ij}}, 1-\delta^2_{b_{ij}}\right)]^n}\Big]$$

$$= \Big[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \sqrt{1-\left(\min\left([1-\eta^2_{a_{ij}}]^n, [1-\eta^2_{b_{ij}}]^n\right)\right)}, \sqrt{1-\left(\min\left([1-\delta^2_{a_{ij}}]^n, [1-\delta^2_{b_{ij}}]^n\right)\right)}\Big]$$

$$= \Big[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \max\left(\sqrt{1-[1-\eta^2_{a_{ij}}]^n}, \sqrt{1-[1-\eta^2_{b_{ij}}]^n}\right), \max\left(\sqrt{1-[1-\delta^2_{a_{ij}}]^n}, \sqrt{1-[1-\delta^2_{b_{ij}}]^n}\right)\Big]$$

$A^n \wedge B^n$

$$= \Big[\left([\zeta_{a_{ij}}]^n, \sqrt{1-[1-\eta^2_{a_{ij}}]^n}, \sqrt{1-[1-\delta^2_{a_{ij}}]^n}\right) \bigwedge \left([\zeta_{b_{ij}}]^n, \sqrt{1-[1-\eta^2_{b_{ij}}]^n}, \sqrt{1-[1-\delta^2_{b_{ij}}]^n}\right)\Big]$$

$$= \Big[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \max\left(\sqrt{1-[1-\eta^2_{a_{ij}}]^n}, \sqrt{1-[1-\eta^2_{b_{ij}}]^n}\right), \max\left(\sqrt{1-[1-\delta^2_{a_{ij}}]^n}, \sqrt{1-[1-\delta^2_{b_{ij}}]^n}\right)\Big]$$

$$= (A \wedge B)^n.$$

Hence $(A \wedge B)^n = A^n \wedge B^n$,
Similarly, we can prove that $(ii)(A \vee B)^n = A^n \vee B^n$. □

Theorem 3.14. *For $A, B \in N_{m\times n}$ and $n > 0$, we have* $(A \oplus B)^n \neq A^n \oplus B^n$.

Proof. Let $(A \oplus B)^n$

$$= \Big[\left(\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}\right)^n, \sqrt{1-[1-\eta^2_{a_{ij}}\eta^2_{b_{ij}}]^n}, \sqrt{1-[1-\delta^2_{a_{ij}}\delta^2_{b_{ij}}]^n}\Big]$$

$$A^n = \left(\left\langle [\zeta_{a_{ij}}]^n, \sqrt{1-[1-\eta^2_{a_{ij}}]^n}, \sqrt{1-[1-\delta^2_{a_{ij}}]^n}\right\rangle\right)$$

$$B^n = \left(\left\langle [\zeta_{b_{ij}}]^n, \sqrt{1-[1-\eta^2_{b_{ij}}]^n}, \sqrt{1-[1-\delta^2_{b_{ij}}]^n}\right\rangle\right)$$

$$A^n \oplus B^n = \Big[\sqrt{[\zeta^n_{a_{ij}}]^2 + [\zeta^n_{b_{ij}}]^2 - [\zeta^n_{a_{ij}}]^2[\zeta^n_{b_{ij}}]^2}, \left(\sqrt{1-[1-\eta^2_{a_{ij}}]^n}\right)^n \cdot \left(\sqrt{1-[1-\eta^2_{b_{ij}}]^n}\right)^n, \left(\sqrt{1-[1-\delta^2_{a_{ij}}]^n}\right)^n \cdot \left(\sqrt{1-[1-\delta^2_{a_{ij}}]^n}\right)^n\Big]$$

Hence $(A \oplus B)^n \neq A^n \oplus B^n$. □

4. New operation (@) on spherical neutrosophic matrices

In this section, we define a new operation(@) on spherical neutrosophic matrices and prove their desirable properties.

Definition 4.1. *If A and B are two Spherical Neutrosophic Matrices, then*

$$A@B = \left(\left\langle \sqrt{\frac{\zeta_{a_{ij}}^2+\zeta_{b_{ij}}^2}{2}}, \sqrt{\frac{\eta_{a_{ij}}^2+\eta_{b_{ij}}^2}{2}}, \sqrt{\frac{\delta_{a_{ij}}^2+\delta_{b_{ij}}^2}{2}} \right\rangle\right).$$

Remark 4.1. *Obviously, for every two spherical neutrosophic matrices A and B, then $A@B$ is a spherical neutrosophic matrix.*

Simple illustration given: For $A@B$,

$$\begin{aligned}0 &\le \frac{\zeta_{a_{ij}}+\zeta_{b_{ij}}}{2} + \tfrac{\eta_{a_{ij}}+\eta_{b_{ij}}}{2} + \frac{\delta_{a_{ij}}+\delta_{b_{ij}}}{2}\\ &\le \frac{\zeta_{a_{ij}}+\eta_{a_{ij}}+\delta_{a_{ij}}}{2} + \frac{\zeta_{b_{ij}}+\eta_{b_{ij}}+\delta_{b_{ij}}}{2} \le \frac{1}{2}+\frac{1}{2} = 1.\end{aligned}$$

Theorem 4.1. *For any spherical neutrosophic matrix A, $A@A = A$.*

Proof. Let
$$\begin{aligned}A@A &= \left(\left\langle \sqrt{\frac{\zeta_{a_{ij}}^2+\zeta_{a_{ij}}^2}{2}}, \sqrt{\frac{\eta_{a_{ij}}^2+\eta_{a_{ij}}^2}{2}}, \sqrt{\frac{\delta_{a_{ij}}^2+\delta_{a_{ij}}^2}{2}} \right\rangle\right)\\
&= \left(\left\langle \left(\sqrt{\frac{\zeta_{a_{ij}}^2+\zeta_{a_{ij}}^2}{2}}\right)^2, \left(\sqrt{\frac{\eta_{a_{ij}}^2+\eta_{a_{ij}}^2}{2}}\right)^2, \left(\sqrt{\frac{\delta_{a_{ij}}^2+\delta_{a_{ij}}^2}{2}}\right)^2 \right\rangle\right)\\
&= \left(\left\langle \frac{2\zeta_{a_{ij}}^2}{2}, \frac{2\eta_{a_{ij}}^2}{2}, \frac{2\delta_{a_{ij}}^2}{2} \right\rangle\right)\\
&= \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right). \text{ Since } \zeta_{a_{ij}}^2 \le \zeta_{a_{ij}}, \eta_{a_{ij}}^2 \le \eta_{a_{ij}}, \delta_{a_{ij}}^2 \le \delta_{a_{ij}}\\
&= A.\end{aligned}$$
□

Remark 4.2. *If $a, b \in [0,1]$, then $ab \le \frac{a+b}{2}, \frac{a+b}{2} \le a+b-ab$.*

Theorem 4.2. *If $A, B \in N_{m\times n}$, then*
(i) $(A \oplus B) \vee (A@B) = A \oplus B$,
(ii) $(A \otimes B) \wedge (A@B) = A \otimes B$,
(iii) $(A \oplus B) \wedge (A@B) = A@B$,
(iv) $(A \otimes B) \vee (A@B) = A@B$.

Proof. we shall prove (i) and (iii), (ii) and (iv) can be proved analogously.
(i) Let $(A \oplus B) \vee (A@B)$
$$\begin{aligned}&= \left[\max\left(\sqrt{\zeta_{a_{ij}}^2+\zeta_{b_{ij}}^2-\zeta_{a_{ij}}^2\zeta_{b_{ij}}^2}, \sqrt{\frac{\zeta_{a_{ij}}^2+\zeta_{b_{ij}}^2}{2}}\right), \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \sqrt{\frac{\eta_{a_{ij}}^2+\eta_{b_{ij}}^2}{2}}\right),\right.\\
&\qquad \left.\min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \sqrt{\frac{\delta_{a_{ij}}^2+\delta_{b_{ij}}^2}{2}}\right)\right]\\
&= \left(\left\langle \sqrt{\zeta_{a_{ij}}^2+\zeta_{b_{ij}}^2-\zeta_{a_{ij}}^2\zeta_{b_{ij}}^2}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)\\
&= A \oplus B.\end{aligned}$$
(iii) $(A \oplus B) \wedge (A@B)$
$$= \left[\min\left(\sqrt{\zeta_{a_{ij}}^2+\zeta_{b_{ij}}^2-\zeta_{a_{ij}}^2\zeta_{b_{ij}}^2}, \sqrt{\frac{\zeta_{a_{ij}}^2+\zeta_{b_{ij}}^2}{2}}\right), \max\left(\eta_{a_{ij}}\eta_{b_{ij}}, \sqrt{\frac{\eta_{a_{ij}}^2+\eta_{b_{ij}}^2}{2}}\right),\right.$$

$$\max\left(\delta_{a_{ij}}\delta_{b_{ij}}, \sqrt{\frac{\delta^2_{a_{ij}}+\delta^2_{b_{ij}}}{2}}\right)\Bigg]$$

$$= \left(\left\langle \sqrt{\frac{\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}}{2}}, \sqrt{\frac{\eta^2_{a_{ij}}+\eta^2_{b_{ij}}}{2}}, \sqrt{\frac{\delta^2_{a_{ij}}+\delta^2_{b_{ij}}}{2}} \right\rangle\right)$$

$= A@B$,

Hence proved. □

Remark 4.3. *The spherical neutrosophic matrix forms a semilattice, associativity, commutativity, idempotency under the spherical neutrosophic matrix operation of algebraic sum and algebraic product. The distributive law also holds for* $\oplus, \otimes$ *and* $\wedge, \vee, @$ *are combined each other.*

5. Conclusion

In this paper, constructed spherical neutrosophic matrix and algebraic operations are defined. Spherical neutrosophic matrix is the direct extension of Pythagorean fuzzy matrix, we seen that how we put neutral membership, $\eta_{a_{ij}} = 0$ in SNMs to reduce in Pythagorean fuzzy matrices. Also seen that how SNMs is extension of picture fuzzy matrix by taking squares of the membership degrees we obtain the spherical neutrosophic matrices. In this paper, the order structure of the circular fuzzy matrix is appeared in Fig.1. We developed some properties such as, idempotency, commutativity, associativity, absorption law, distributivity and De Morgan's laws over complement are proved. Finally, defined a new operation(@) on spherical neutrosophic matrices and discussed distributive laws. In the future, the application of the proposed aggregating operators of SNMs needs to be explored in the decision making, risk analysis and many other uncertain and fuzzy environment.

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I. Silambarasan for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, N.2.
