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# STABILIZATION OF HYPERBOLIC FITZHUGH-NAGUMO EQUATIONS WITH ONE INTERIOR FEEDBACK CONTROLLER

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Abstract. We show that the hyperbolic modification of FitzHugh-Nagumo system defined in a bounded domain $G \subset \mathbb{R}^n$ is global exponetally stablizable by a feedback controller action in an open subdomian $\omega \subset G$ such that $G \setminus \omega$ is sufficiently thin.

Keywords: Hyperbolic FitzHugh-Nagumo equations, feedback controller, stabilization.

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## 1. Introduction

We study the problem of interior feedback stabilization of the following coupled system of equations

$$\begin{cases} \tau\partial_t^2 u + \partial_t u - \Delta u + f(u) + v = 1_\omega(x)w, \ x \in G, t > 0, \\ \partial_t v + dv - bu = 0, \ x \in G, t > 0, \\ u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), v(x,0) = v_0(x), \ x \in G, \\ u(x,t) = 0, \ x \in \partial G, t > 0, \end{cases} \tag{1}$$

where $G \subset \mathbb{R}^N (N \leq 3)$ is a bounded domain with sufficiently smooth boundary $\partial G$, $\tau > 0, b > 0, d > 0$ are given numbers, $u_0, u_1, v_0$ are given functions, $w$ is the control function, $1_\omega(x)$ is a characteristic function of the domain $\omega$, $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function that satisfies the conditions

$$f(s)s - \mathcal{F}(s) \geq -r_1 s^2, \ \forall s \in \mathbb{R}, \tag{2}$$

$$a_1|s|^{p+2} - r_2 s^2 \leq \mathcal{F}(s) \leq a_2|s|^{p+2} + r_3, \ \forall s \in \mathbb{R}, \tag{3}$$

where $a_1, a_2, r_1, r_2, r_3$ are given positive numbers, $p \geq 2$ if $N = 1, 2$ and $p \in [2, 4]$ if $N = 3$ are given numbers and

$$\mathcal{F}(s) = \int_0^s f(y)dy.$$

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The system of equations (1) is one of the mathematical models describing the transmission of electrical impulses through a nerve axon.
The parabolic FitzHugh Nagumo equations, i.e. the system of equations

$$\begin{cases} \partial_t u - \Delta u + f(u) + v = 0, \ x \in G, t > 0, \\ \partial_t v + dv - bu = 0, \ x \in G, t > 0, \end{cases} \tag{4}$$

with $f(u) = u(a-u)(b-u)$, where $a > 0, b > 0, c > 0$ are given paprameters, arised as a model describing the signal transmission across axons. There are many publications devoted to the mathematical analysis of the Cauchy problem and initial boundary value problems for parabolic FitzHugh-Nagumo equations (4), where the authors obtained results on global existence of solutions of the Caucy problem and initial boundary value problems for this system. It is shown that under some natural restrictions on the nonlinear term $f(u)$ (icluding, the nonlinearity of the form $f(u) = u(a-u)(b-u)$, the semigroup generated by the initial boundary value problem for (4) possesses a finite dimensional global attractor (see, e.g., [7], [8], [13] and references therein.)
In [15], the authors proved that the parabolic FitzHugh-Nagumo system can be exponentially stabilized by a feedback controller acting on subdomain in the reaction - diffusion equation. It is necessary to note that there are many publications devoted to the study of internal feedback stablization of various parabolic equations and systems (see, e.g., [1], [2] and referencis therein).
A number of papers are devoted to the study of modification of the FitzHug-Nagumo equations taking into account the effects of relaxation (see, e.g., [14], [9], [12] ).
In these papers the authors studied the problems of existence of solitary waves, existece and uniqueness of solution to the Cauchy problem and intial boubdary value problems, existence of bouded solutions of considered systems and some other qualitative properties of the system.
Our main goal in this short note is to show that the hyperbolic FitzHugh-Nagumo system also can be exponentially stabilized by a feedback controller acting on subdomain in the nonlinear damped wave equation.

We will be using the following notations:
$Q_T = G \times (0,T)$; $L^p(G), 1 \leq p \leq \infty$, and $H^s(G)$, $s > 0$, are the usual Lebesgue and Sobolev spaces respectively. With $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the inner product and norm of $L^2(G)$ .

We will need below the following inequalities:

*Young's inequality*

$$ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q\epsilon^{1/(p-1)}} b^q, \text{ for all } a, b, \epsilon > 0, \text{ with } q = p/(p-1), 1 < p < \infty. \tag{5}$$

.
*Intepolation inequality*

$$\|\nabla u\|^2 \leq \|\nabla u\| \|\Delta u\|, \quad \forall u \in H^2(G) \cap H_0^1(G). \tag{6}$$

*Poincaré inquality*

$$\|u\|^2 \leq \lambda_1^{-1} \|\nabla u\|^2, \quad \forall u \in H_0^1(G), \tag{7}$$

where $\lambda_1$ is the first eigenvalue of the Laplace oparator $-\Delta$ under the homogeneous Dirichlet's boundary condidition.

*Sobolev inequality*

$$\|u\|_{L^p(G)} \le c_0\|\nabla u\|, \quad \forall u \in H_0^1(G), \tag{8}$$

where $p \in [2, \frac{2n}{n-2}]$ if $N > 2$ and $p > 0$ is arbitrary if $N = 1, 2$, $c_0$ is an absolute constant for $p = \frac{2n}{n-2}$ and depends on $G$ otherwise.
Finally let us give the definition of a weak solution of the problem (1).

**Definition 1.1.** *A pair of functions $[u, v]$ is called a weak solution of the problem*(1) *if*

$$u \in C(0, T; H_0^1(G), \quad v \in C(0, T; L^2(G)), \quad \forall T > 0$$

*and the equations* (11),(12) *are satisfied in the sense of distributions.*

$1_\omega(x)$ is the characteristic function of the subdomain $\omega \subset G$ with smooth boundary and $\overline{\omega} \subset G$.
Let us denote by $\lambda_1(G_\omega)$ the first eigenvalue of the problem

$$-\Delta v = \lambda v, \ x \in G_\omega; \ v = 0, \ x \in \partial G_\omega,$$

where $G_\omega := G \setminus \overline{\omega}$. We will need the following Lemma in the proof of the main result of this section:

**Lemma 1.1.** *(see, e.g.,*[15]*) For each $\gamma > 0$ there exists a number $\mu_0(\gamma) > 0$ such that the following inequality holds true*

$$\int_G \left(|\nabla v(x)|^2 + \mu 1_\omega(x)v^2(x)\right) dx \ge (\lambda_1(G_\omega) - \gamma) \int_G v^2(x)dx, \ \forall v \in H_0^1(G), \tag{9}$$

*whenever $\mu > \mu_0$.*

## 2. Uniform estimates

To study the stabilization of the system, we apply the feedback controller

$$w = -\mu u \tag{10}$$

with $\mu > 0$ and get the following closed-loop system:

$$\tau\partial_t^2 u + \partial_t u - \Delta u + f(u) + v = -\mu 1_\omega(x)u, \ x \in G, t > 0, \tag{11}$$

$$\partial_t v + dv - bu = 0, \ x \in G, t > 0, \tag{12}$$

$$u(x, 0) = u_0(x), \partial_t u(x, 0) = u_1(x), v(x, 0) = v_0(x), \ x \in G, \tag{13}$$

$$u(x, t) = 0, \ x \in \partial G, t > 0, \tag{14}$$

First we multiply the equation (11) by $\partial_t u + \epsilon u$, $\epsilon > 0$, and integrate the obtained relation over the domain $G$

$$\frac{d}{dt}E_\epsilon(t) + (1 - \epsilon\tau)\|\partial_t u(t)\|^2 + \epsilon\|\nabla u(t)\|^2 + \epsilon(f(u), u) + \epsilon(v, u) - (\partial_t v, u) = -\mu\epsilon\int_\omega u^2(x, t)dx, \tag{15}$$

where

$$E_\epsilon(t) := \frac{\tau}{2}\|\partial_t u\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + (\mathcal{F}(u), 1) + \frac{\epsilon}{2}\|u(t)\|^2 + \epsilon\tau(\partial_t u, u) + (u, v) + \frac{\mu}{2}\int_\omega u^2 dx$$

and $\epsilon > 0$ is a parameter to be chosen below. Now we multiply the eqation (12) by $\partial_t v + \frac{\epsilon}{b}v$ and obtain the equality

$$\|\partial_t v\|^2 + (\frac{d}{2} + \frac{\epsilon}{2b})\frac{d}{dt}\|v(t)\|^2 - b(\partial_t v, u) + \frac{\epsilon d}{b}\|v\|^2 - \epsilon(u, v) = 0 \tag{16}$$

Adding (15) and (16) then using the inequality

$$(1+b)|(\partial_t v, u)| \le \|\partial_t v\|^2 + \frac{1}{4}(1+b)^2\|u\|^2$$

we get

$$\frac{d}{dt}\Phi_\epsilon(t) + (1-\epsilon\tau)\|\partial_t u(t)\|^2 + \epsilon\|\nabla u(t)\|^2 + \epsilon(f(u),u) + \frac{d\epsilon}{b}\|v\|^2 - \frac{1}{4}(1+b)^2\|u\|^2 \le -\mu\epsilon\int_\omega u^2(x,t)dx, \quad (17)$$

where

$$\Phi_\epsilon(t) := E_\epsilon(t) + (\frac{d}{2} + \frac{\epsilon}{2b})\|v(t)\|^2.$$

By using the condition(3) and the inequalities

$$\epsilon\tau|(\partial_t u, u)| \le \frac{\tau}{4}\|\partial_t u\|^2 + \tau\epsilon^2\|u\|^2,$$

$$|(u,v)| \le \frac{d}{2}\|v\|^2 + \frac{1}{2d}\|u\|^2$$

we get the following estimate from below for $\Phi_\epsilon(t)$ :

$$\Phi_\epsilon(t) \ge \frac{\tau}{4}\|\partial_t u\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + a_1\int_G |u(x,t)|^{p+2}dx + (\frac{\epsilon}{2} - r_2 - \tau\epsilon^2 - \frac{1}{2d})\|u\|^2 + \frac{\epsilon}{2b}\|v\|^2 + \frac{\mu}{2}\int_\omega u^2(x,t)dx. \quad (18)$$

Adding to the left hand side of (17) $\delta\Phi_\epsilon(t) - \delta\Phi_\epsilon(t)$ with some $\delta \in (0,\epsilon)$ we can rewrite it in the following form

$$\frac{d}{dt}\Phi_\epsilon(t) + \delta\Phi_\epsilon(t) + (1-\epsilon\tau-\frac{\delta\tau}{2})\|\partial_t u\|^2 + (\epsilon-\frac{\delta}{2})\|\nabla u\|^2 + (\frac{d\epsilon}{b} - \frac{\delta d}{2} - \frac{\delta\epsilon}{2b})\|v\|^2 + \epsilon(f(u),u) - \delta(\mathcal{F}(u),1) - \left(\frac{1}{4}(1+b)^2 + \frac{\delta\epsilon}{2}\right)\|u\|^2 - \epsilon\delta\tau(\partial_t u, u) - \delta(u,v) = -(\mu\epsilon - \frac{\delta\mu}{2})\int_\omega u^2(x,t)dx \quad (19)$$

By using the inequality (20)

$$\epsilon(f(u),u) - \delta(\mathcal{F}(u),1) \ge -\epsilon(r_1+r_2)\|u\|^2 \quad (20)$$

that follows from (2) and the inequalities

$$\delta\epsilon\tau|(\partial_t u, u)| \le \frac{\delta\epsilon\tau}{2}\|\partial_t u\|^2 + \frac{\delta\epsilon\tau}{2}\|u\|^2,$$

$$\delta|(u,v)| \le \frac{\delta}{2\lambda_1}\|\nabla u\|^2 + \frac{\delta}{2}\|v\|^2.$$

we obtain from (19) that

$$\frac{d}{dt}\Phi_\epsilon(t) + \delta\Phi_\epsilon(t) + \left(1-\epsilon\tau-\frac{\delta\tau}{2}-\frac{\delta\epsilon\tau}{2}\right)\|\partial_t u\|^2 + \left(\epsilon-\frac{\delta}{2}-\frac{\delta}{2\lambda_1}(\epsilon\tau+1)\right)\|\nabla u\|^2 + \left(\frac{\epsilon d}{b} - \delta(\frac{d}{2}+\frac{\epsilon}{2b}) - \frac{\delta}{2}\right)\|v\|^2 - \epsilon(r_1+r_2)\|u\|^2 \le -\mu(\epsilon-\frac{\delta}{2})\int_\omega u^2(x,t)dx$$

Choosing $\delta$ small enough we can get form the last inequality that

$$\frac{d}{dt}\Phi_\epsilon(t) + \delta\Phi_\epsilon(t) + \frac{1}{4\tau}\|\nabla u\|^2 - \frac{1}{2\tau}(r_1 + r_2 + \frac{\tau}{4})\|u\|^2 \leq -\frac{\mu}{4\tau}\int_\omega u^2(x,t)dx. \quad (21)$$

According to the Lemma 1.1 if $G \setminus \omega$ is sufficiently thin, we can choose $\mu$ big enough such that

$$\|\nabla u\|^2 + \mu\int_\omega u(x,t)dx \geq 2(r_1 + r_2 + \frac{\tau}{4})\|u\|^2.$$

and

$$\|\nabla u\|^2 + 2\mu\int_\omega u(x,t)dx \geq (r_2 + \frac{1}{2d})\|u\|^2.$$

By using these inequalities we get

$$\frac{d}{dt}\Phi_\epsilon(t) + \delta\Phi_\epsilon(t) \leq 0. \quad (22)$$

and

$$\Phi_\epsilon(t) \geq \frac{\tau}{4}\|\partial_t u\|^2 + \frac{1}{4}\|\nabla u\|^2 + a_1\int_G |u(x,t)|^{p+2}dx + \frac{1}{4\tau b}\|v\|^2 \quad (23)$$

Taking into account (23) we deduce from (22) the follwing exponential decay estimate

$$\frac{\tau}{4}\|\partial_t u\|^2 + \frac{1}{4}\|\nabla u\|^2 + a_1\int_G |u(x,t)|^{p+2}dx + \frac{1}{4\tau b}\|v\|^2 \leq \Phi_\epsilon(0)e^{-\delta t}, \ \forall t > 0.$$

So we have proved the following

**Theorem 2.1.** *If the nonlinear term satisfies the conditions* (2), (3) *and* $mes(G \setminus \omega)$ *is small enough, then solution of the problem*(11)-(14) *tend to zero as* $t \to \infty$ *with an exponential rate.*

**Remark 2.1.** *Let us note that the existence and uniquennes of a solution to the problem can be done by using the standard Faedo - Galerkin method (see, e.g.,* [10]*).*

## 3. Conclusions

We considered a modification of the FitzHug-Nagumo equations taking into account the effects of relaxation and proved that the considered system can be exponentially stabilized by a feedback controller acting on subdomain $\omega \subset G$ such that $G \setminus \omega$ is sufficiently thin.

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