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STRONG ASYMPTOTIC STABILITY FOR A COUPLED SYSTEM OF DEGENERATE WAVE EQUATIONS WITH ONLY ONE FRACTIONAL FEEDBACK

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Abstract. We prove the well-posedness and study the strong asymptotic stability of a coupled system of degenerate wave equations with a fractional feedback acting on one end only.

Keywords: System of coupled degenerate wave equqtions, strong asymptotic stability, fractional boundary feedback.

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1. Introduction

In this paper, we consider a system of coupled wave equations in the presence of boundary control of nonlocal type:

$$\begin{cases} u_{tt}(x,t) - (x^\delta u_x)_x(x,t) + \alpha(u-v) = 0 & \text{in } (0,L)\times(0,+\infty), \\ v_{tt}(x,t) - (x^\delta v_x)_x(x,t) + \alpha(v-u) = 0 & \text{in } (0,L)\times(0,+\infty), \\ u(L,t) = u(0,t) = v(L,t) = 0 & \text{on } (0,+\infty), \\ (x^\delta v_x)(0,t) - \rho\partial_t^{\tau,\omega} v(0,t) = 0 & \text{on } (0,+\infty), \\ \begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) \end{cases} & \text{on } (0,L), \end{cases} \qquad (P)$$

where $\delta \in (0,1), \alpha$ is a strictly positive constants $\rho > 0$ and the initial data (u_0, u_1, v_0, v_1) belong to a suitable function space. The notation $\partial_t^{\tau,\omega}$ stands for the generalized Caputo's fractional derivative of order τ, $0 < \tau < 1$, with respect to the time variable (see [3]). It

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is defined as follows

$$\partial_t^{\tau,\omega} h(t) = \frac{1}{\Gamma(1-\tau)} \int_0^t (t-s)^{-\tau} e^{-\omega(t-s)} \frac{dh}{ds}(s)\, ds, \quad \omega \geq 0.$$

Physically, u and v may represent the displacements of two vibratings objects measured from their equilibrium positions, the coupling terms $\pm\alpha(u-v)$ are the distributed springs linking the two vibrating objects.

The exponential stability of the system (P) has been established by Najafi et al [5] in the linear and nonlinear boundary feedback.

In [3], Kerdache et al. investigated the decay rate of the energy of the coupled wave equations with two boundary nonlocal controls, that is,

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha(u-v) = 0 & \text{in } (0,L)\times(0,+\infty), \\ v_{tt}(x,t) - v_{xx}(x,t) + \alpha(v-u) = 0 & \text{in } (0,L)\times(0,+\infty), \\ u(0,t) = v(0,t) = 0 & \text{on } (0,+\infty), \\ u_x(L,t) + \delta_1 \partial_t^{\tau,\omega} u_t(L,t) = 0 & \text{on } (0,+\infty), \\ v_x(L,t) + \delta_2 \partial_t^{\tau,\omega} v_t(L,t) = 0 & \text{on } (0,+\infty), \\ \begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x) \end{cases} & \text{on } (0,L). \end{cases} \qquad (PBF)$$

Using semigroup theory, they prove an optimal polynomial type decay rate.

The question we are interested in this paper is what are the stability properties of our system (P).

To our best knowledge, this is the first attempt to study the asymptotic stability of solutions for a coupled system of degenerate wave equations with only one boundary fractional feedback. we will remark that there is a price paid compared with the hypothesis assumed by Najafi et al and by Kerdache et al.

The organization of this paper is as follows. In section **2**, first we reformulate the system (P) into classical in-put out-put dynamic systems and we deduce the well-posedness property of the problem by the semigroup approach. Second, using a criteria of Arendt-Batty and Lyubich-Vu we show that the augmented model is strongly stable.

2. Well-posedness and strong stability

2.1. **Well-Posedness.** In this subsection, We reformulate system (P) into an augmented system. Indeed, by using Theorem 2.1 in [3], system (P) becomes

$$\begin{cases} u_{tt}(x,t) - (x^\delta u_x)_x(x,t) + \alpha(u-v) = 0 & \text{in } (0,L)\times(0,+\infty), \\ v_{tt}(x,t) - (x^\delta v_x)_x(x,t) + \alpha(v-u) = 0 & \text{in } (0,L)\times(0,+\infty), \\ \partial_t \vartheta(\xi,t) + (\xi^2+\omega)\vartheta(\xi,t) - \mu(\xi) v_t(0,t) = 0 & \text{in } (-\infty,\infty)\times(0,+\infty), \\ u(0,t) = u(L,t) = v(L,t) = 0 & \text{on } (0,+\infty) \\ (x^\delta v_x)(0,t) = \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\vartheta(\xi,t)\, d\xi & \text{on } (0,+\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{on } (0,L), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x) & \text{on } (0,L), \\ \vartheta(\xi,0) = 0, & \text{on } (-\infty,+\infty), \end{cases} \qquad (P')$$

where $\zeta = \rho(\pi)^{-1}\sin(\tau\pi)$ and $\mu(\xi) = |\xi|^{\frac{2\tau-1}{2}}$. For a solution (u,v,ϑ) of (P'), we define the energy

$$E(t) = \frac{1}{2}\int_0^L (|u_t|^2 + |v_t|^2 + |x^{\delta/2}u_x|^2 + |x^{\delta/2}v_x|^2 + \alpha|u-v|^2)\, dx + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\vartheta|^2\, d\xi. \quad (1)$$

The energy of the system is decreasing, in fact for smooth solution, a direct computation gives

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \omega)|\vartheta(\xi,t)|^2 \, d\xi \leq 0. \tag{2}$$

We now discuss the well-posedness of (P'). For this purpose, we introduce the following spaces:

$\mathbf{H}^1_\delta(0,L) = \left\{ u \text{ is locally absolutely continuous in } (0,L] : x^{\delta/2}u_x \in L^2(0,L) \right\},$
$H^1_{0,\delta}(0,L) = \left\{ u \text{ is locally absolutely continuous in } (0,L] : x^{\delta/2}u_x \in L^2(0,L)/\ u(L) = 0 \right\},$
$H^1_\delta(0,L) = \left\{ u \in H^1_{0,\delta}(0,L)/\ u(0) = 0 \right\},$
$H^2_\delta(0,L) = \left\{ u \in L^2(0,L)/x^{\delta/2}u_x \in L^2(0,L), x^\delta u_x \in H^1(0,L) \right\}.$

We then reformulate (P') into a semigroup setting. Let $\tilde{u} = u_t, \tilde{v} = v_t$, and set

$$\mathcal{H} = H^1_\delta(0,L) \times L^2(0,L) \times H^1_{0,\delta}(0,L) \times L^2(0,L) \times L^2(\mathbb{R})$$

equipped with the inner product

$$\begin{aligned}\langle U, U_1 \rangle_{\mathcal{H}} = &\int_0^L \left(\tilde{u}\overline{\tilde{u}}_1 + x^\delta u_x \overline{u}_{1x} \right) dx + \int_0^L \left(\tilde{v}\overline{\tilde{v}}_1 + x^\delta v_x \overline{v}_{1x} \right) dx \\ &+\alpha \int_0^L (u-v)\overline{(u_1 - v_1)} \, dx + \zeta \int_{-\infty}^{+\infty} \vartheta \overline{\vartheta}_1 \, d\xi\end{aligned} \tag{3}$$

for any $U = (u, \tilde{u}, v, \tilde{v}, \vartheta)^T$ and $U_1 = (u_1, \tilde{u}_1, , v_1, \tilde{v}_1, \vartheta_1)^T$. We use $\|U\|_{\mathcal{H}}$ to denote the corresponding norm.

Let $U = (u, \tilde{u}, v, \tilde{v}, \vartheta)^T$ and rewrite (P') as

$$U' = \mathcal{A}U, \quad U(0) = U_0 = (u_0, u_1, v_0, v_1, 0), \tag{4}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}(u, \tilde{u}, v, \tilde{v}, \vartheta) = \Big(\tilde{u}, (x^\delta u_x)_x - \alpha(u-v), \tilde{v}, (x^\delta v_x)_x - \alpha(v-u), -(\xi^2+\omega)\vartheta + \mu(\xi)\tilde{v}(0) \Big). \tag{5}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, \tilde{u}, v, \tilde{v}, \vartheta)^T \text{ in } \mathcal{H} : u \in H^2_\delta(0,L) \cap H^1_\delta(0,L), \tilde{u} \in H^1_\delta(0,L), \\ v \in H^2_\delta(0,L) \cap H^1_{0,\delta}(0,L), \tilde{v} \in H^1_{0,\delta}(0,L), \\ (x^\delta v_x)(0) - \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\vartheta(\xi) \, d\xi = 0, \\ -(\xi^2+\omega)\vartheta + \mu(\xi)\tilde{v}(0) \in L^2(-\infty,+\infty), |\xi|\vartheta \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{6}$$

Remark 2.1.
• *Notice that if* $u \in H^2_\delta(0,L), \delta \in [1,2)$,*we have* $(x^\delta v_x)(0) \equiv 0$. *Indeed, if* $x^\delta v_x(x) \to l$ *when* $x \to 0$, *then* $x^\delta |v_x(x)|^2 \sim l/x^\delta$ *and therefore* $l = 0$ *otherwise* $v \notin \mathbf{H}^1_\delta(0,L)$. *So, one cannot consider the case* $\delta \geq 1$.

(••) *If we define*

$$|u|_{H^1_{0,\delta}(0,L)} = \left(\int_0^L x^\delta |u_x(x)|^2 \, dx \right)^{1/2} \qquad \forall u \in \mathbf{H}^1_\delta(0,L).$$

Then

$$\|u\|^2_{L^2(0,L)} \leq C_* |u|^2_{H^1_{0,\delta}(0,L)} \quad \forall u \in H^1_{0,\delta}(0,L). \tag{7}$$

Indeed, let $u \in H^1_{0,\delta}(0,L)$*. For any* $x \in]0,L]$ *we have that*

$$|u(x)| = \left| \int_x^L u_x(s)\, ds \right| \le |u|_{H^1_{0,\delta}(0,L)} \left\{ \int_0^L \frac{1}{x^\delta}\, ds \right\}^{1/2}.$$

Therefore

$$\int_0^L |u(x)|^2\, dx \le \frac{L^{1-\delta}}{1-\delta} |u|^2_{H^1_{0,\delta}(0,L)}.$$

($\bullet\bullet\bullet$) *Moreover, For every* $u \in H^1_{0,\delta}(0,L)$*,* u *is absolutely continuous in* $[0,L]$*. Indeed, as*

$$u'(x) = \frac{1}{x^{\delta/2}} x^{\delta/2} u'(x) \qquad \forall x \in]0,L].$$

then

$$\begin{aligned} \int_0^L |u'(x)|\, dx &\le \left(\int_0^L \frac{1}{x^\delta}\, dx \right)^{1/2} |u(x)|_{H^1_{0,\delta}(0,L)} \\ &= \frac{L^{(1-\delta)/2}}{\sqrt{1-\delta}} |u(x)|_{H^1_{0,\delta}(0,L)} \end{aligned}$$

u' *is summable over* $(0,L)$*. So* u *is absolutely continuous in* $[0,L]$*. Hence in the definition of* $D(\mathcal{A})$*, it makes sense to consider the value of* $\tilde{v}$ *at* 0*.*

Lemma 2.1 (see [3])**.** *If* $\lambda \in D_\omega = \mathbb{C} \backslash]-\infty, -\omega]$ *then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \omega + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \omega)^{\alpha - 1}.$$

The well-posedness of problem (P') is ensured by the following theorem.

Theorem 2.1 (Existence and uniqueness)**.** *Let* $U_0 \in \mathcal{H}$*, then there exists a unique solution* $U \in C([0,+\infty), \mathcal{H})$*, of problem (4), Moreover if* $U_0 \in D(\mathcal{A})$*, then* $U \in C([0,+\infty), D(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H})$*.*

Proof of Theorem 2.1. We show that $\mathcal{A}$ is a maximal monotone. First, it follows from (2) that

$$\Re \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \omega) |\vartheta(\xi)|^2\, d\xi \le 0. \tag{8}$$

Then $\mathcal{A}$ is monotone. For the maximality, let $G = (g_1, g_2, g_3, g_4, g_5)^T \in \mathcal{H}$ and look for $U = (u, \tilde{u}, v, \tilde{v}, \vartheta, \tilde{\vartheta})^T \in D(\mathcal{A})$ satisfying $\lambda U - \mathcal{A}U = G$ for $\lambda > 0$, that is,

$$\begin{cases} \lambda u - \tilde{u} = g_1, \\ \lambda \tilde{u} - (x^\delta u_x)_x + \alpha(u - v) = g_2, \\ \lambda v - \tilde{v} = g_3, \\ \lambda \tilde{v} - (x^\delta v_x)_x + \alpha(v - u) = g_4, \\ \lambda \vartheta + (\xi^2 + \omega)\vartheta - \mu(\xi)\tilde{v}(0) = g_5. \end{cases} \tag{9}$$

From $(9)_3$ and $(9)_5$, we get

$$\vartheta = \frac{g_5(\xi)}{\xi^2 + \omega + \lambda} + \frac{\lambda v(0)\mu(\xi)}{\xi^2 + \omega + \lambda} - \frac{g_3(0)\mu(\xi)}{\xi^2 + \omega + \lambda}. \tag{10}$$

Inserting $(9)_1$ and $(9)_3$ in $(9)_2$ and $(9)_4$, we get

$$\begin{aligned} \lambda^2 u - (x^\delta u_x)_x + \alpha(u - v) = g_2 + \lambda g_1, \\ \lambda^2 v - (x^\delta v_x)_x + \alpha(v - u) = g_4 + \lambda g_3. \end{aligned} \tag{11}$$

Multiplying Equations $(11)_1$ and $(11)_2$ by $w \in H^1_\delta(0,L)$ and $\chi \in H^1_{0,\delta}(0,L)$ respectively, integrate over $(0,L)$, then using by parts integration, we get

$$\begin{cases} \lambda^2 \int_0^L (u\overline{w} + v\overline{\chi})\, dx + \int_0^L \left(x^\delta u_x \overline{w_x} + x^\delta v_x \overline{\chi_x} \right) dx + \rho\lambda(\lambda+\omega)^{\tau-1} v(0)\overline{\chi}(0) \\ +\alpha \int_0^L (u-v)(\overline{w}-\overline{\chi})\, dx = \int_0^L ((g_2+\lambda g_1)\overline{w} + (g_4+\lambda g_3)\overline{\chi})\, dx \\ \qquad -\zeta\overline{\chi}(0) \int_{-\infty}^{+\infty} \dfrac{\mu(\xi) g_5(\xi)}{\lambda+\xi^2+\omega} d\xi + \rho(\lambda+\omega)^{\tau-1} g_3(0)\overline{\chi}(0), \end{cases} \tag{12}$$

where we have used the fact that $\int_{-\infty}^{+\infty} \mu^2(\xi)/(\xi^2+\lambda+\omega)\, d\xi = \frac{\pi}{\sin\tau\pi}(\lambda+\omega)^{\tau-1}$. Consequently, problem (12) is equivalent to the problem

$$a((u,v),(w,\chi)) = b(w,\chi), \tag{13}$$

where the sesquilinear form $a : [H^1_\delta(0,L) \times H^1_{0,\delta}(0,L)]^2 \to \mathbb{C}$ and the antilinear form $b : H^1_\delta(0,L) \times H^1_{0,\delta}(0,L) \to \mathbb{C}$ are defined by

$$a((u,v),(w,\chi)) = \lambda^2 \int_0^L (u\overline{w} + v\overline{\chi})\, dx + \int_0^L \left(x^\delta u_x \overline{w_x} + x^\delta v_x \overline{\chi_x} \right) dx$$
$$+\rho\lambda(\lambda+\omega)^{\tau-1} v(0)\overline{\chi}(0) + \alpha \int_0^L (u-v)(\overline{w}-\overline{\chi})\, dx$$

and

$$b(w,\chi) = \int_0^L ((g_2+\lambda g_1)\overline{w} + (g_4+\lambda g_3)\overline{\chi})\, dx - \zeta\overline{\chi}(0) \int_{-\infty}^{+\infty} \frac{\mu(\xi) g_5(\xi)}{\lambda+\xi^2+\omega} d\xi$$
$$+\rho(\lambda+\omega)^{\tau-1} g_3(0)\overline{\chi}(0).$$

the sesquilinear form $a(.,.)$ is a bounded since for any $(u,v),(w,\chi) \in \wedge = H^1_\delta(0,L) \times H^1_{0,\delta}(0,L)$

$$a((u,v),(w,\chi)) \le \lambda^2(\|u\|_{L^2(0,L)}\|w\|_{L^2(0,L)} + \|v\|_{L^2(0,L)}\|\chi\|_{L^2(0,L)}) + |u|_{H^1_{0,\delta}(0,L)}|w|_{H^1_{0,\delta}(0,L)}$$
$$+|v|_{H^1_{0,\delta}(0,L)}|\chi|_{H^1_{0,\delta}(0,L)} + \rho\lambda(\lambda+\omega)^{\tau-1}|v(0)||\overline{\chi}(0)| + +\alpha\|u-v\|_{L^2(0,L)}\|\overline{w}-\overline{\chi}\|_{L^2(0,L)}$$
$$\le M\|(u,w)\|_\wedge \|(v,\chi)\|_\wedge,$$

where we have used the Sobolev Poincaré's inequality and (•••) in Remark 2.1. Moreover $a(.,.)$ is coercive because

$$a((u,v),(u,v)) \ge \|(u,v)\|^2_\wedge.$$

Moreover b is continuous. Therefore, Lax-Milgram says that, $\exists!\ (u,v) \in H^1_\delta(0,L) \times H^1_{0,\delta}(0,L)$ satisfying

$$a((u,v),(w,\chi)) = b(w,\chi), \forall (w,\chi) \in \wedge.$$

In particular, taking $w \in \mathcal{D}(0,L)$ and $\chi \equiv 0$ in (13), we obtain

$$\lambda^2 u - (x^\gamma u_x)_x + \alpha(u-v) = g_2 + \lambda g_1 \text{ in } \mathcal{D}'(0,L). \tag{14}$$

As $g_2 + \lambda g_1 \in L^2(0,L)$, using (14), we deduce that

$$\lambda^2 u - (x^\delta u_x)_x + \alpha(u-v) = g_2 + \lambda g_1 \text{ in } L^2(0,L). \tag{15}$$

Due to the fact that $u \in H^1_\delta(0,L)$ and $v \in H^1_{0,\delta}(0,L)$, we obtain $(x^\gamma u_x)_x \in L^2(0,L)$, then $u \in H^2_\delta(0,L) \cap H^1_\delta(0,L)$. Similarly, taking $\chi \in \mathcal{D}(0,L)$ and $w \equiv 0$ in (13), we obtain

$$\lambda^2 v - (x^\delta v_x)_x + \alpha(v-u) = g_3 + \lambda g_4 \text{ in } L^2(0,L). \tag{16}$$

Due to the fact that $u \in H^1_\delta(0,L)$ and $v \in H^1_{0,\delta}(0,L)$ we obtain $(x^\gamma u_x)_x \in L^2(0,L)$. So $v \in H^2_\delta(0,L) \cap H^1_{0,\delta}(0,L)$.

Multiplying both sides of the conjugate of equalities (15) and (16) by $w \in H^1_\delta(0,L)$ and $\chi \in H^1_{0,\delta}(0,L)$, integrating by parts on $(0,L)$, and comparing with (13) we obtain

$$\begin{aligned}&-(x^\gamma v_x)(0)\overline{\chi}(0) + \rho\lambda(\lambda+\omega)^{\tau-1}v(0)\overline{\chi}(0)\\&+\zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\omega+\lambda}g_5(\xi)\,d\xi\overline{w}(0) \;-\rho(\lambda+\omega)^{\tau-1}g_3(0)\overline{\chi}(0)=0.\end{aligned}$$

Consequently, defining $\tilde{u} = \lambda u - g_1$ and $\tilde{v} = \lambda v - g_3$ and ϑ by (10), we deduce that

$$-(x^\gamma v_x)(0) + \zeta\int_{-\infty}^{+\infty}\mu(\xi)\vartheta(\xi)\,d\xi = 0.$$

In order to complete the existence of $U \in D(\mathcal{A})$, we need to prove ϑ and $|\xi|\vartheta \in L^2(-\infty,\infty)$. From (10), we get

$$\int_{\mathbb{R}}|\vartheta(\xi)|^2\,d\xi \le 3\int_{\mathbb{R}}\frac{|g_5(\xi)|^2}{(\xi^2+\omega+\lambda)^2}\,d\xi + 3(\lambda^2|v(0)|^2+|g_3(0)|^2)\int_{\mathbb{R}}\frac{|\xi|^{2\tau-1}}{(\xi^2+\omega+\lambda)^2}\,d\xi.$$

Using Lemma 2.1, it easy to see that

$$\int_{\mathbb{R}}\frac{|\xi|^{2\tau-1}}{(\xi^2+\omega+\lambda)^2}\,d\xi = (1-\tau)\frac{\pi}{\sin\tau\pi}(\lambda+\omega)^{\tau-2}.$$

On the other hand, using the fact that $g_5 \in L^2(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}}\frac{|g_5(\xi)|^2}{(\xi^2+\omega+\lambda)^2}\,d\xi \le \frac{1}{(\omega+\lambda)^2}\int_{\mathbb{R}}|g_5(\xi)|^2\,d\xi < +\infty.$$

It follows that $\vartheta \in L^2(\mathbb{R})$. Next, using (10), we get

$$\int_{\mathbb{R}}|\xi\vartheta(\xi)|^2\,d\xi \le 3\int_{\mathbb{R}}\frac{|\xi|^2|g_5(\xi)|^2}{(\xi^2+\omega+\lambda)^2}\,d\xi + 3(\lambda^2|v(0)|^2+|g_3(0)|^2)\int_{\mathbb{R}}\frac{|\xi|^{2\tau+1}}{(\xi^2+\omega+\lambda)^2}\,d\xi.$$

Using again Lemma 2.1, it easy to see that

$$\int_{\mathbb{R}}\frac{|\xi|^{2\tau+1}}{(\xi^2+\omega+\lambda)^2}\,d\xi = \tau\frac{\pi}{\sin\tau\pi}(\lambda+\omega)^{\tau-1}.$$

Now, using the fact that $g_5 \in L^2(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}}\frac{|\xi|^2|g_5(\xi)|^2}{(\xi^2+\omega+\lambda)^2}\,d\xi \le \frac{1}{(\omega+\lambda)}\int_{\mathbb{R}}|g_5(\xi)|^2\,d\xi < +\infty.$$

It follows that $|\xi|\vartheta \in L^2(\mathbb{R})$. Finally, since $\vartheta \in L^2(\mathbb{R})$, we get

$$-(\xi^2+\omega)\vartheta + \tilde{v}(0)\mu(\xi) = \lambda\vartheta(\xi) - g_5(\xi) \in L^2(\mathbb{R}).$$

Then $U \in D(\mathcal{A})$ and Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. At last, the result of Theorem 2.1 follows from the Hille-Yosida theorem (see [6]).

□

2.2. **Strong stability of the system.** We use a general criteria of Arendt-Batty [2] and Lyubich-Vu [4], following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements. Our main result is the following theorem.

Theorem 2.2. *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$ if and only if the coefficient α satisfies*

$$\alpha \neq \left(\frac{2-\delta}{2}\right)^2 \frac{L^{-(2-\delta)}}{2}(j^2_{\nu_\delta,k} - j^2_{\nu_\delta,m}), \quad k,m \in \mathbb{N}, \tag{C}$$

where $\nu_\delta = (1-\delta)/(2-\delta)$ and $j_{\nu,1} < j_{\nu,2} < \ldots < j_{\nu,k} < \ldots$ denote the sequence of positive zeros of the Bessel function of first kind and of order ν.

For the proof of Theorem 2.2, we need the following two lemmas.

Lemma 2.2. *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof. We make a distinction between $i\lambda = 0$ and $i\lambda \neq 0$.
Step 1. Solving for $\mathcal{A}U = 0$ leads to the following system

$$\begin{cases} -\tilde{u} = 0, \\ -(x^\delta u_x)_x + \alpha(u-v) = 0, \\ -\tilde{v} = 0, \\ -(x^\delta v_x)_x + \alpha(v-u) = 0, \\ (\xi^2+\omega)\vartheta - \mu(\xi)\tilde{v}(0) = 0, \end{cases} \tag{17}$$

Then, from (8), $(17)_1$ and $(17)_3$ we have

$$\vartheta \equiv 0, \ \tilde{u} \equiv 0 \text{ and } \tilde{v} \equiv 0. \tag{18}$$

Let set $\Phi = u + v$ and $\Psi = u - v$. Then Φ and Ψ satisfy

$$\begin{cases} -(x^\delta \Phi_x)_x = 0, \\ -(x^\delta \Psi_x)_x + 2\alpha\Psi = 0. \end{cases} \tag{19}$$

Hence $(x^\delta \Phi_x)(x) = c$, where c is a constant. Then $\Phi(x) = \dfrac{c}{\delta+1}x^{1-\delta}$. Using the fact that $\Phi(L) = 0$, we deduce that $c = 0$ and hence $\Phi \equiv 0$. Hence $(x^\delta\Psi_x)(0) = 0$ and $\Psi(0) = 0$. Multiplying equation $(19)_2$ by $\overline{\Psi}$, we get $\int_0^L x^\delta |\Psi_x|^2\,dx + 2\alpha\int_0^L |\Psi|^2\,dx = 0$. Thus $\Psi \equiv 0$. Therefore $U = 0$, thanks to the boundary conditions in (6). Hence, $i\lambda = 0$ is not an eigenvalue of $\mathcal{A}$.
Step 2. We will argue by contradiction. Let us suppose that there $\lambda \in \mathbb{R}, \lambda \neq 0$ and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda u - \tilde{u} = 0, \\ i\lambda\tilde{u} - (x^\delta u_x)_x + \alpha(u-v) = 0, \\ i\lambda v - \tilde{v} = 0, \\ i\lambda\tilde{v} - (x^\delta v_x)_x + \alpha(v-u) = 0, \\ i\lambda\vartheta + (\xi^2+\omega)\vartheta - \mu(\xi)\tilde{v}(0) = 0. \end{cases} \tag{20}$$

Then, from (8) we have $\vartheta \equiv 0$. Hence From $(20)_5$, we have $\tilde{v}(0) = 0$. Then, from $(20)_3$ and $(6)_2$ we obtain $u(0) = u(L) = v(0) = v(L) = (x^\delta v_x)(0) = 0$.

Inserting $(20)_1, (20)_3$ into $(20)_2$ and $(20)_4$, we get

$$\begin{cases} -\lambda^2 u - (x^\delta u_x)_x + \alpha(u-v) = 0, \\ -\lambda^2 v - (x^\delta v_x)_x + \alpha(v-u) = 0. \end{cases} \tag{21}$$

Then $\Phi = u + v$ and $\Psi = u - v$ satisfy

$$\begin{cases} \lambda^2\Phi + (x^\delta\Phi_x)_x = 0, \\ (\lambda^2 - 2\alpha)\Psi + (x^\delta\Psi_x)_x = 0. \end{cases} \tag{22}$$

The solution of the equation (22) is given by

$$\begin{cases}\Phi(x) = c_1\Phi_+(x) + c_2\Phi_-(x),\\ \Psi(x) = \tilde{c}_1\Phi_{++}(x) + \tilde{c}_2\Phi_{--}(x),\end{cases}$$

where

$$\begin{cases}\Phi_+(x) = x^{\frac{1-\delta}{2}} J_{\nu_\delta}\left(\frac{2}{2-\delta}\lambda x^{\frac{2-\delta}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\delta}{2}} J_{-\nu_\delta}\left(\frac{2}{2-\delta}\lambda x^{\frac{2-\delta}{2}}\right),\\ \Phi_{++}(x) = x^{\frac{1-\delta}{2}} J_{\nu_\delta}\left(\frac{2}{2-\delta}\sqrt{\lambda^2-2\alpha}x^{\frac{2-\delta}{2}}\right),\\ \Phi_{--}(x) = x^{\frac{1-\delta}{2}} J_{-\nu_\delta}\left(\frac{2}{2-\delta}\sqrt{\lambda^2-2\alpha}x^{\frac{2-\delta}{2}}\right),\end{cases} \tag{23}$$

where

$$J_\nu(y) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{y}{2}\right)^{2m+\nu} = \sum_{m=0}^{\infty} c^+_{\nu,m} y^{2m+\nu}, \tag{24}$$

$$J_{-\nu}(y) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m-\nu+1)}\left(\frac{y}{2}\right)^{2m-\nu} = \sum_{m=0}^{\infty} c^-_{\nu,m} y^{2m-\nu}, \tag{25}$$

togheter with the boundary conditions

$$\Phi(0) = \Phi(L) = \Psi(0) = \Psi(L) = 0, \quad (x^\delta v_x)(0) = 0.$$

As $\Phi(0) = \Psi(0) = 0$, then $c_2 = \tilde{c}_2 = 0$. As $v(x) = \dfrac{1}{2}(\Phi(x) - \Psi(x))$, we deduce that

$$(x^\delta \Phi_x)(0) = (x^\delta \Psi_x)(0).$$

Then

$$c_1(1-\delta)\tilde{\tilde{c}}^+_{\nu_{\delta,0}} = \tilde{c}_1(1-\delta)\tilde{\tilde{c}}^{++}_{\nu_{\delta,0}},$$

where

$$\tilde{\tilde{c}}^+_{\nu_{\delta,0}} = c^+_{\nu_{\delta,0}}\left(\frac{2}{2-\delta}\lambda\right)^{\nu_\delta}, \tilde{\tilde{c}}^{++}_{\nu_{\delta,0}} = c^+_{\nu_{\delta,0}}\left(\frac{2}{2-\delta}\sqrt{\lambda^2-2\alpha}\right)^{\nu_\delta}\left(\text{where } c^+_{\nu_{\delta,0}} = \frac{1}{\Gamma(\nu_\delta+1)2^{\nu_\delta}}\right).$$

Moreover $\Phi(L) = \Psi(L) = 0$. Then

$$\begin{cases}c_1 J_{\nu_\delta}\left(\dfrac{2}{2-\delta}\lambda L^{\frac{2-\delta}{2}}\right) = 0, \quad \tilde{c}_1 J_{\nu_\delta}\left(\dfrac{2}{2-\delta}\sqrt{\lambda^2-2\alpha}L^{\frac{2-\delta}{2}}\right) = 0,\\ c_1(1-\delta)\tilde{\tilde{c}}^+_{\nu_{\delta,0}} = \tilde{c}_1(1-\delta)\tilde{\tilde{c}}^{++}_{\nu_{\delta,0}}\end{cases}$$

If Bessel are zero then

$$\frac{2}{2-\delta}\lambda L^{\frac{2-\delta}{2}} = j_{\nu_\delta,k} \text{ and } \frac{2}{2-\delta}\sqrt{\lambda^2-2\alpha}L^{\frac{2-\delta}{2}} = j_{\nu_\delta,m}$$

for some integers k and m. Hence, eigenvalues on $i\mathbb{R}$ exist iff

$$\alpha = \left(\frac{2-\delta}{2}\right)^2\frac{L^{-(2-\delta)}}{2}(j^2_{\nu_\delta,k} - j^2_{\nu_\delta,m}).$$

Hence, if condition (C) is satisfied we deduce that $c_1 = 0$ or $\tilde{c}_1 = 0$ and consequently $u = v = 0$.

Therefore $U = 0$. Consequently, $\mathcal{A}$ does not have purely imaginary eigenvalues. □

Lemma 2.3.
If $\lambda \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.
If $\lambda = 0$ and $\omega \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.

Proof. **Case 1: $\lambda \neq 0$.** Let $G = (g_1, g_2, g_3, g_4, g_5)^T \in \mathcal{H}$ be given, and let $U = (u, \tilde{u}, v, \tilde{v}, \vartheta)^T \in D(\mathcal{A})$ be such that

$$(i\lambda I - \mathcal{A})U = G. \tag{26}$$

Equivalently, we have

$$\begin{cases} i\lambda u - \tilde{u} = g_1, \\ i\lambda \tilde{u} - (x^\delta u_x)_x + \alpha(u - v) = g_2, \\ i\lambda v - \tilde{v} = g_3, \\ i\lambda \tilde{v} - (x^\delta v_x)_x + \alpha(v - u) = g_4, \\ i\lambda \vartheta + (\xi^2 + \omega)\vartheta - \mu(\xi)\tilde{v}(0) = g_5. \end{cases} \tag{27}$$

Inserting $(27)_1, (27)_3$ into $(27)_2$ and $(27)_4$, we get

$$\begin{cases} -\lambda^2 u - (x^\delta u_x)_x + \alpha(u - v) = (g_2 + i\lambda g_1), \\ -\lambda^2 v - (x^\delta v_x)_x + \alpha(v - u) = (g_4 + i\lambda g_3). \end{cases} \tag{28}$$

Solving system (28) is equivalent to finding $(u, v) \in H^2_\delta \cap H^1_\delta(0, L) \times H^2_\delta(0, L) \cap H^1_{0,\delta}(0, L)$ such that

$$\begin{cases} \displaystyle\int_0^L (-\lambda^2 u\overline{w} - (x^\delta u_x)_x \overline{w} + \alpha(u - v)\overline{w})\, dx = \int_0^L (g_2 + i\lambda g_1)\overline{w}\, dx, \\ \displaystyle\int_0^L (-\lambda^2 v\overline{\chi} - (x^\delta v_x)_x \overline{\chi} + \alpha(v - u)\overline{\chi})\, dx = \int_0^L (g_4 + i\lambda g_3)\overline{\chi}\, dx \end{cases} \tag{29}$$

for all $(w, \chi) \in H^1_\delta(0, L) \times H^1_{0,\delta}(0, L)$. By using $(27)_3$ and $(27)_5$ the functions u and v satisfying the following system

$$\begin{cases} \displaystyle -\lambda^2 \int_0^L (u\overline{w} + v\overline{\chi})\, dx + \int_0^L \left(x^\delta u_x \overline{w_x} + x^\delta v_x \overline{\chi_x}\right) dx + i\rho\lambda(i\lambda + \omega)^{\tau-1} v(0)\overline{\chi}(0) \\ \displaystyle +\alpha \int_0^L (u - v)\,(\overline{w} - \overline{\chi})\, dx = \int_0^L ((g_2 + i\lambda g_1)\overline{w} + (g_4 + i\lambda g_3)\overline{\chi})\, dx \\ \displaystyle \qquad -\zeta\overline{\chi}(0) \int_{-\infty}^{+\infty} \frac{\mu(\xi) g_5(\xi)}{i\lambda + \xi^2 + \omega} d\xi + \rho(i\lambda + \omega)^{\tau-1} g_3(0)\overline{\chi}(0). \end{cases} \tag{30}$$

We can rewrite (30) as

$$\mathcal{B}((u, v), (w, \chi)) = \mathcal{L}(w, \chi), \quad \forall (w, \chi) \in H^1_\delta(0, L) \cap H^1_{0,\delta}(0, L), \tag{31}$$

where

$$\mathcal{B}((u, v), (w, \chi)) = \mathcal{B}_1((u, v), (w, \chi)) + \mathcal{B}_2((u, v), (w, \chi))$$

with

$$\begin{cases} \displaystyle \mathcal{B}_1((u, v), (w, \chi)) = \int_0^L \left(x^\delta u_x \overline{w_x} + x^\delta v_x \overline{\chi_x}\right) dx + \alpha \int_0^L (u - v)(\overline{w} - \overline{\chi})\, dx \\ \qquad\qquad + i\rho\lambda(i\lambda + \omega)^{\tau-1} v(0)\overline{\chi}(0), \\ \displaystyle \mathcal{B}_2((u, v), (w, \chi)) = -\int_0^L \lambda^2 u\overline{w}\, dx - \int_0^L \lambda^2 v\overline{\chi}\, dx \end{cases} \tag{$*$}$$

and

$$\mathcal{L}(w, \chi) = \int_0^L ((g_2 + i\lambda g_1)\overline{w} + (g_4 + i\lambda g_3)\overline{\chi})\, dx - \zeta\overline{\chi}(0) \int_{-\infty}^{+\infty} \frac{\mu(\xi) g_5(\xi)}{i\lambda + \xi^2 + \omega} d\xi \\ + \rho(i\lambda + \omega)^{\tau-1} g_3(0)\overline{\chi}(0).$$

Let $(H^1_\delta(0,L)\times H^1_{0,\delta}(0,L))'$ be the dual space of $H^1_\delta(0,L)\times H^1_{0,\delta}(0,L)$. Let us define the following operators

$$\begin{array}{c} B: H^1_\delta(0,L)\times H^1_{0,\delta}(0,L)\to (H^1_\delta(0,L)\times H^1_{0,\delta}(0,L))' \\ (u,v)\mapsto B(u,v) \\ B_i: H^1_\delta(0,L)\times H^1_{0,\delta}(0,L)\to (H^1_\delta(0,L)\times H^1_{0,\delta}(0,L))' \quad i\in\{1,2\} \\ u\mapsto B_i(u,v) \end{array} \tag{$**$}$$

such that

$$\begin{cases}(B(u,v))(w,\chi)=\mathcal{B}((u,v)),(w,\chi)),\ \forall(w,\chi)\in H^1_\delta(0,L)\times H^1_{0,\delta}(0,L),\\ (B_i(u,v))(w,\chi)=\mathcal{B}_i((u,v)),(w,\chi),\ \forall(w,\chi)\in H^1_\delta(0,L)\times H^1_{0,\delta}(0,L), i\in\{1,2\}.\end{cases} \tag{$***$}$$

It is easy to see that $\mathcal{B}_1$ is sesquilinear, continuous and coercive form on $(H^1_\delta(0,L)\times H^1_{0,\delta}(0,L))^2$. Then, from $(**)$ and Lax-Milgram theorem, the operator B_1 is an isomorphism. Moreover, using the compact embedding from $H^1_{0,\delta}(0,L)$ to $L^2(0,L)$ we deduce that B_2 is a compact operator. Therefore, from the above steps, we obtain that the operator $B=B_1+B_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we still need to prove that the operator B is injective to obtain that the operator B is an isomorphism. Let $(u,v)\in ker(B)$, then

$$\mathcal{B}((u,v)),(w,\chi)=0 \quad \forall(w,\chi)\in H^1_\delta(0,L)\times H^1_{0,\delta}(0,L). \tag{32}$$

In particular for $(w,\chi)=(u,v)$, it follows that

$$\lambda^2\left[\|u\|^2_{L^2(0,L)}+\|v\|^2_{L^2(0,L)}\right]-i\rho\lambda(i\lambda+\omega)^{\tau-1}|v(0)|^2=\|x^{\delta/2}u_x\|^2_{L^2(0,L)} +\|x^{\delta/2}v_x\|^2_{L^2(0,L)}+\alpha\|u-v\|^2_{L^2(0,L)}.$$

Hence, we obtain $v(0)=0$ and From (32), we have $(x^\delta v_x)(0)=0$. Then, according to Lemma 2.2, we deduce that $(u,v)=(0,0)$ and consequently $Ker(B)=\{0\}$. Finally, from Fredholm alternative, we deduce that the operator B is isomorphism. It is easy to see that the operator $\mathcal{L}$ is a antilinear and continuous form on $H^1_\delta(0,L)\times H^1_{0,\delta}(0,L)$. Consequently, (31) admits a unique solution $(u,v)\in H^1_\delta(0,L)\times H^1_{0,\delta}(0,L)$. Hence $i\lambda-\mathcal{A}$ is surjective for all $\lambda\in\mathbb{R}^*$.

Case 2: $\lambda=0$ and $\omega\neq 0$. Using Lax-Milgram Lemma, we obtain the result.

□

3. Conclusions

We have studied the boundary stabilization of the coupled system of degenerate wave equations with only one dissipation law of fractional derivative type acting at a degenerate point. If α is outside a discrete set of exceptional values, using Arendt-Batty and Lyubich-Vu criteria, we proved the strong stability of the system. We will be investigated in the future the non-uniform stability by spectral analysis.

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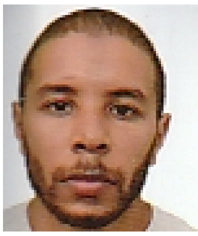

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