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ON QUASI STATISTICAL CONVERGENCE IN GRADUAL NORMED LINEAR SPACES

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Abstract. During the last few years, enormous works by different researchers have been carried out in summability theory by linking different notions of convergence of sequences. In the present paper, we introduce the concept of quasi statistical convergence and strong quasi statistical summability in gradual normed linear spaces. We investigate some of its basic properties and the interrelationship between the newly introduced notions. It has been observed that every gradual quasi statistical convergent sequence is gradual quasi statistical bounded but not necessarily gradual bounded. Finally, we introduce the concept of gradual quasi statistical Cauchy sequences and show that every gradual quasi statistical convergent sequence is a gradual quasi statistical Cauchy sequence.

Keywords: Gradual number, gradual normed linear space, quasi-density, gradual quasi statistical convergence.

AMS Subject Classification (2020): 03E72, 40A35, 40G15.

1. Introduction

In 1965, the concept of fuzzy sets [39] was introduced by Zadeh as one of the extensions of the classical set-theoretical concept. Nowadays it has wide applications in different branches of science and engineering. The term "fuzzy number" plays a vital role in the study of fuzzy set theory. Fuzzy numbers were essentially the generalization of intervals, not numbers. Indeed fuzzy numbers do not obey a couple of algebraic properties of the classical numbers. So the term "fuzzy number" is debatable to many researchers due to its different behavior. The term "fuzzy intervals" is often used by many authors instead of fuzzy numbers. To overcome the confusion among the researchers, in 2008, Fortin et.al. [14] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual

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real numbers are mainly known by their respective assignment function which is defined in the interval $(0,1]$. So in some sense, every real number can be viewed as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have been used in computation and optimization problems.

In 2011, Sadeqi and Azari [31] first introduced the concept of gradual normed linear space. They studied various properties of the space from both the algebraic and topological points of view. Further investigation in this direction has been occurred due to Ettefagh et. al. [11, 12], Choudhury and Debnath [7], and many others. For an extensive study on gradual real numbers, one may refer to [1, 9, 25, 37] where many more references can be found.

On the other hand, in 1951 Fast [13] and Steinhaus [36] introduced the idea of statistical convergence independently in connection with summability. Later on, it was further investigated from the sequence space point of view by Fridy [15, 16], Salat [32], and many mathematicians across the globe. Following their work several investigations and generalizations have been made by Altinok and Kucukaslan [2, 3], Hazarika and Esi [20], Mursaleen [27], Savas and Gurdal [35], Tripathy [38], and many others [4, 5, 6, 8, 10, 17, 18, 19, 21, 22, 23, 24, 26, 28, 33, 34]. Statistical convergence has become one of the most active areas of research due to its wide applicability in various branches of mathematics such as number theory, mathematical analysis, probability theory, etc.

In an attempt to generalize the notion of statistical convergence, in 2012 Ozguc and Yurdakadim [30] introduced the concept of quasi-statistical convergence in terms of quasi-density. They investigated the relationship of the newly introduced notion with statistical convergence. Very recently Ozguc [29] have introduced the notion of quasi statistical limit and cluster points and investigated a few properties. The above two works are the main motivation for us to study the analogus concept in the gradual normed linear spaces.

2. Definitions and Preliminaries

Definition 2.1. [14] *A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}} : (0,1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number is said to be non-negative if for every $\xi \in (0,1]$, $A_{\tilde{r}}(\xi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.*

In [14], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

Definition 2.2. *Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$ respectively. Then $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ given by $A_{\tilde{r}_1 * \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) * A_{\tilde{r}_2}(\xi)\, \forall \xi \in (0,1]$. Then the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $\lambda\tilde{r} (\lambda \in \mathbb{R})$ are defined by*

$$A_{\tilde{r}_1+\tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) + A_{\tilde{r}_2}(\xi) \quad \text{and} \quad A_{\lambda\tilde{r}}(\xi) = \lambda A_{\tilde{r}}(\xi)\ \forall \xi \in (0,1].$$

For any real number $p \in \mathbb{R}$, the constant gradual real number $\tilde{p}$ is defined by the constant assignment function $A_{\tilde{p}}(\xi) = p$ for any $\xi \in (0,1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $A_{\tilde{0}}(\xi) = 0$ and $A_{\tilde{1}}(\xi) = 1$ respectively. One can easily verify that $G(\mathbb{R})$ with the gradual addition and multiplication forms a real vector space [14].

Definition 2.3. [31] *Let X be a real vector space. The function $|| \cdot ||_G : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X if for every $\xi \in (0,1]$, following three conditions are true for any $x, y \in X$:*

(G_1) $A_{||x||_G}(\xi) = A_{\bar{0}}(\xi)$ *iff* $x = 0$;
(G_2) $A_{||\lambda x||_G}(\xi) = |\lambda| A_{||x||_G}(\xi)$ *for any* $\lambda \in \mathbb{R}$;
(G_3) $A_{||x+y||_G}(\xi) \leq A_{||x||_G}(\xi) + A_{||y||_G}(\xi)$.

The pair $(X, ||\cdot||_G)$ is called a gradual normed linear space (GNLS).

Example 2.1. [31] *Let* $X = \mathbb{R}^m$ *and for* $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m, \xi \in (0,1]$, *define* $||\cdot||_G$ *by* $A_{||x||_G}(\xi) = e^{\xi} \sum_{i=1}^{m} |x_i|$. *Then* $||\cdot||_G$ *is a gradual norm on* $\mathbb{R}^m$ *and* $(\mathbb{R}^m, ||\cdot||_G)$ *is a gradual normed linear space.*

Definition 2.4. [31] *Let* (x_k) *be a sequence in the GNLS* $(X, ||\cdot||_G)$. *Then* (x_k) *is said to be gradual convergent to* $x \in X$ *if for every* $\xi \in (0,1]$ *and* $\varepsilon > 0$, *there exists* $N(\xi) \in \mathbb{N}$ *such that* $A_{||x_k - x||_G}(\xi) < \varepsilon$, $\forall k \geq N(\xi)$.

Definition 2.5. [12] *Let* $(X, ||\cdot||_G)$ *be a GNLS. Then a sequence* (x_k) *in* X *is said to be gradual bounded if for every* $\xi \in (0,1]$, *there exists* $M = M(\xi) > 0$ *such that* $A_{||x_k||_G}(\xi) < M$, $\forall k \in \mathbb{N}$.

Definition 2.6. [31] *Let* (x_k) *be a sequence in the GNLS* $(X, ||\cdot||_G)$. *Then* (x_k) *is said to be gradual Cauchy if for every* $\xi \in (0,1]$ *and* $\varepsilon > 0$, *there exists* $N(\xi) \in \mathbb{N}$ *such that* $A_{||x_k - x_j||_G}(\xi) < \varepsilon \, \forall k, j \geq N(\xi)$.

Theorem 2.1. *(*[31]*, Theorem 3.6) Let* $(X, ||\cdot||_G)$ *be a GNLS, then every gradual convergent sequence in* X *is also a gradual Cauchy sequence.*

Definition 2.7. [15] *Let* E *be a subset of the set of natural numbers* $\mathbb{N}$ *and* E_n *denotes the set* $\{k \in E : k \leq n\}$. *The natural density of* E *is denoted and defined by* $\delta(E) = \lim_{n\to\infty} \frac{|E_n|}{n}$. *Here,* $|E_n|$ *denotes the cardinality of the set* E_n.

Definition 2.8. [16] *A sequence* (x_k) *is said to be statistically convergent to* l *if for each* $\varepsilon > 0$, *the set* $\{k \in \mathbb{N} : | x_k - l | \geq \varepsilon\}$ *has natural density zero.* l *is called the statistical limit of the sequence* (x_k) *and symbolically it is expressed as* $x_k \xrightarrow{st} l$.

Definition 2.9. [16] *A sequence* (x_k) *is said to be statistically Cauchy if for every* $\varepsilon > 0$, *there exists a natural number* $N(= N_\varepsilon)$ *such that* $|x_k - x_N| < \varepsilon$ *a.a.k. In other words,* $\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n : |x_k - x_N| \geq \varepsilon\} \mid = 0$.

Definition 2.10. [30] *Let* E *be a subset of* $\mathbb{N}$ *and* E_n *denotes the set* $\{k \in E : k \leq n\}$. *The quasi-density of* E *is given by* $\delta_c(E) = \lim_{n\to\infty} \frac{|E_n|}{c_n}$, *where* $c = (c_n)$ *is a sequence of real numbers satisfying the following properties*

$$c_n > 0 \, \forall n \in \mathbb{N}, \lim_{n\to\infty} c_n = \infty \text{ and } \limsup_{n} \frac{c_n}{n} < \infty. \tag{1}$$

If $c_n = n$, then the above definition turns to the definition of natural density. Throughout the paper $c = (c_n)$ will be used to denote sequences that satisfy (1).

Definition 2.11. [30] *A sequence* (x_k) *is said to be quasi statistical convergent to* l *if for each* $\varepsilon > 0$, *the set* $\{k \in \mathbb{N} : | x_k - l | \geq \varepsilon\}$ *has quasi-density zero.* l *is called the quasi statistical limit of the sequence* (x_k) *and symbolically it is expressed as* $x_k \xrightarrow{st_q} l$.

Definition 2.12. [30] *A real-valued sequence* (x_k) *is said to be strongly quasi summable to* l *in* $\mathbb{R}$ *if*

$$\lim_{n\to\infty} \frac{1}{c_n} \sum_{k=1}^{n} |x_k - l| = 0.$$

Definition 2.13. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradual statistical convergent to $x \in X$ if for every $\xi \in (0,1]$ and $\varepsilon > 0$, the set $\{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}$ has natural density zero. Symbolically, $x_k \xrightarrow{st-||\cdot||_G} x$.*

3. Main Results

Definition 3.1. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradual quasi statistical convergent to $x \in X$ if for every $\xi \in (0,1]$ and $\varepsilon > 0$, the set $\{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}$ has quasi-density zero. Symbolically we write, $x_k \xrightarrow{st_q-||\cdot||_G} x$.*

Theorem 3.1. *Let $(X, ||\cdot||_G)$ be a GNLS. If a sequence (x_k) is gradual convergent to $x \in X$, then (x_k) is gradual quasi statistical convergent to $x \in X$.*

Proof. Since (x_k) is gradual convergent to x, so the set $\{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}$ is a finite set and so has quasi-density zero. $\square$

But the converse of Theorem 3.1 is not true. Example 3.1 illustrates the fact.

Example 3.1. *Let $X = \mathbb{R}^m$ and $||\cdot||_G$ be the norm defined in Example 2.1. Consider the sequence (x_k) in $\mathbb{R}^m$ defined as*

$$x_k = \begin{cases} (0,0,...,0,m) & if\ k = p^2, p \in \mathbb{N} \\ (0,0,....0,0) & otherwise \end{cases}$$

Let $\mathbf{0}$ denotes the element $(0,0,....0,0) \in \mathbb{R}^m$ and (c_n) be the sequence defined as $c_n = n$, $n \in \mathbb{N}$. Then for any $\varepsilon > 0$ and $\xi \in (0,1]$, $\{k \in \mathbb{N} : A_{||x_k - \mathbf{0}||_G}(\xi) \geq \varepsilon\} \subseteq \{1,4,9,..\}$ and eventually $\delta_c\{k \in \mathbb{N} : A_{||x_k - \mathbf{0}||_G}(\xi) \geq \varepsilon\} = 0$. In other words, $x_k \xrightarrow{st_q-||\cdot||_G} \mathbf{0}$ in $\mathbb{R}^m$.

But it is clear from the definition of (x_k) that (x_k) is not gradual convergent to $\mathbf{0}$.

Theorem 3.2. *Let (x_k) be any sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{st_q-||\cdot||_G} x$ in X. Then x is uniquely determined.*

Proof. If possible suppose $x_k \xrightarrow{st_q-||\cdot||_G} x$ and $x_k \xrightarrow{st_q-||\cdot||_G} y$ for some $x \neq y$ in X. Then we have, for any $\varepsilon > 0$ and $\xi \in (0,1]$, $\delta_c(B_1(\xi,\varepsilon)) = \delta_c(B_2(\xi,\varepsilon)) = 1$ where $B_1(\xi,\varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) < \varepsilon\}$ and $B_2(\xi,\varepsilon) = \{k \in \mathbb{N} : A_{||x_k - y||_G}(\xi) < \varepsilon\}$. Choose $m \in B_1(\xi,\varepsilon) \cap B_2(\xi,\varepsilon)$, then $A_{||x_m - x||_G}(\xi) < \varepsilon$ and $A_{||x_m - y||_G}(\xi) < \varepsilon$. Hence $A_{||x-y||_G}(\xi) \leq A_{||x_m - x||_G}(\xi) + A_{||x_m - y||_G}(\xi) < \varepsilon + \varepsilon = 2\varepsilon$. Since ε is arbitrary, so $A_{||x-y||_G}(\xi) = A_{\tilde{0}}(\xi)$ and we must have $x = y$. $\square$

Theorem 3.3. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{st_q-||\cdot||_G} x$ and $y_k \xrightarrow{st_q-||\cdot||_G} y$. Then*
(i) $x_k + y_k \xrightarrow{st_q-||\cdot||_G} x + y$ and (ii) $\lambda x_k \xrightarrow{st_q-||\cdot||_G} \lambda x$, for $\lambda \in \mathbb{R}$.

Proof. (i) Suppose $x_k \xrightarrow{st_q-||\cdot||_G} x$ and $y_k \xrightarrow{st_q-||\cdot||_G} y$. Then for given $\varepsilon > 0$, $\delta_c(C_1) = \delta_c(C_2) = 0$ where $C_1 = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \frac{\varepsilon}{2}\}$ and $C_2 = \{k \in \mathbb{N} : A_{||y_k - y||_G}(\xi) \geq \frac{\varepsilon}{2}\}$. Now as the inclusion $(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \subseteq \{k \in \mathbb{N} : A_{||x_k + y_k - x - y||_G}(\xi) < \varepsilon\}$ holds, so we must have

$$\delta_c(\{k \in \mathbb{N} : A_{||x_k + y_k - x - y||_G}(\xi) \geq \varepsilon\}) \leq \delta_c(C_1 \cup C_2) = 0;$$

and consequently, $x_k + y_k \xrightarrow{st_q-||\cdot||_G} x + y$.

(ii) If $\lambda = 0$, then there is nothing to prove. So let us assume $\lambda \neq 0$. Then since $x_k \xrightarrow{st_q-||\cdot||_G}$

x, we have for given $\varepsilon > 0$, $\delta_c(C_1) = 0$ where $C_1 = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \frac{\varepsilon}{|\lambda|}\}$. Now since $A_{||\lambda x_k - \lambda x||_G}(\xi) = |\lambda| A_{||x_k - x||_G}(\xi)$ holds for any $\lambda \in \mathbb{R}$, we must have $C_2 \subseteq C_1$ where $C_2 = \{k \in \mathbb{N} : A_{||\lambda x_k - \lambda x||_G}(\xi) \geq \varepsilon\}$, which as a consequence implies $\delta_c(C_2) = 0$. This completes the proof. □

Theorem 3.4. *Let (x_k) be any sequence in the GNLS $(X, ||\cdot||_G)$. If every subsequence of (x_k) is gradual quasi statistical convergent to x, then (x_k) is also gradual quasi statistical convergent to x.*

Proof. If possible suppose (x_k) is not gradual quasi statistical convergent to x. Then there exists some $\varepsilon > 0$ and $\xi \in (0, 1]$ such that $\delta_c(C) \neq 0$, where $C = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}$. Since quasi-density of a finite set is zero, so C must be an infinite set. Let $C = \{k_1 < k_2 < ... < k_j < ...\}$. Now define a sequence (y_j) as $y_j = x_{k_j}$ for $j \in \mathbb{N}$. Then (y_j) is a subsequence of (x_k) which is not gradual quasi statistical convergent to x, a contradiction. □

Remark 3.1. *The converse of the above theorem is not true. One can easily verify this from Example 3.1.*

Theorem 3.5. *Let $(X, ||\cdot||_G)$ be a GNLS. A subsequence (x_{k_j}) of a gradual quasi statistical convergent sequence (x_k) in X is gradual quasi statistical convergent if and only if $\delta_c(\{k_j : j \in \mathbb{N}\}) = 1$.*

Proof. The proof is easy and so omitted. □

Theorem 3.6. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, ||\cdot||_G)$ such that (y_k) is gradual convergent and $\delta_c(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$. Then (x_k) is gradual quasi statistical convergent.*

Proof. Suppose $\delta_c(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$ holds and $y_k \xrightarrow{||\cdot||_G} y$. Then by definition for every $\varepsilon > 0$ and $\xi \in (0, 1]$, $\{k \in \mathbb{N} : A_{||y_k - y||_G}(\xi) \geq \varepsilon\}$ is a finite set and therefore

$$\delta_c(\{k \in \mathbb{N} : A_{||y_k - y||_G}(\xi) \geq \varepsilon\}) = 0. \tag{2}$$

Now since the inclusion

$$\{k \in \mathbb{N} : A_{||x_k - y||_G}(\xi) \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : A_{||y_k - y||_G}(\xi) \geq \varepsilon\} \cap \{k \in \mathbb{N} : x_k \neq y_k\}$$

holds, so using Equation (2) and the hypothesis we get

$$\delta_c(\{k \in \mathbb{N} : A_{||x_k - y||_G}(\xi) \geq \varepsilon\}) = 0.$$

Hence $x_k \xrightarrow{st_q - ||\cdot||_G} y$ and the proof is complete. □

In ([12], Theorem 3.5), Ettefagh et.al. proved that in a GNLS every gradual convergent sequence is gradual bounded. But this result is not true in the case of gradual quasi statistical convergence. Put $c_n = n, n \in \mathbb{N}$ and consider the gradual normed space $(\mathbb{R}^2, ||\cdot||_G)$ with the gradual norm defined in Example 2.1. Consider the sequence (x_k) in $\mathbb{R}^2$ defined as

$$x_k = \begin{cases} (0, k), & if\ k = p^2, p \in \mathbb{N}; \\ (0, 0), & otherwise. \end{cases}$$

Then it is clear that (x_k) is not gradual bounded but gradual quasi statistical convergent to $(0, 0) \in \mathbb{R}^2$.

Definition 3.2. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradual quasi statistical bounded if for every $\xi \in (0, 1]$, there exists $M(= M(\xi)) > 0$ such that $\delta_c(\{k \in \mathbb{N} : A_{||x_k||_G}(\xi) > M\}) = 0$.*

Theorem 3.7. *Let $(X, ||\cdot||_G)$ be a GNLS and suppose (x_k) be a gradual quasi statistical convergent sequence. Then (x_k) is gradual quasi statistical bounded.*

Proof. The proof is straightforward so omitted. □

Theorem 3.8. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{st_q - ||\cdot||_G} x$. Then, $x_k \xrightarrow{st-||\cdot||_G} x$.*

Proof. Let $x_k \xrightarrow{st_q - ||\cdot||_G} x$. Then, by definition, $\delta_c(\{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}) = 0$. Suppose $H = \sup\limits_n \frac{c_n}{n}$. Then, for any $\varepsilon > 0$ the inclusion

$$\frac{1}{n} \mid \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \mid \leq \frac{H}{c_n} \mid \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \mid$$

holds. Now by the assumption, the right-hand side of the inequation tends to zero, so the left-hand side of the above inequation also tends to 0. Hence $x_k \xrightarrow{st-||\cdot||_G} x$. □

The converse of the above theorem is not necessarily true. The following example illustrates the fact.

Example 3.2. *Let $X = \mathbb{R}^m$ and $||\cdot||_G$ be the norm defined in Example 2.1. Let (c_n) be a sequence satisfying $\lim\limits_{n\to\infty} c_n = \infty$ and $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n}}{c_n} = \infty$. We can choose a subsequence (c_{n_p}) such that $c_{n_p} > 1$ for all $p \in \mathbb{N}$. Consider the sequence (x_k) in $\mathbb{R}^m$ defined by*

$$x_k = \begin{cases} (0, 0, ..., 0, c_k), & if\, k\, is\, a\, perfect\, cube\, and\, c_k \in \{c_{n_p} : p \in \mathbb{N}\} \\ (0, 0, ..., 0, 3), & if\, k\, is\, a\, perfect\, cube\, and\, c_k \notin \{c_{n_p} : p \in \mathbb{N}\} \\ (0, 0, ...0, 0), & otherwise. \end{cases}$$

Then, it is easy to verify that $x_k \xrightarrow{st-||\cdot||_G} \mathbf{0}$ in $\mathbb{R}^m$. Now we will show that (x_k) is not gradual quasi statistical convergent to $\mathbf{0}$. Take $\varepsilon = e^\xi, \xi \in (0, 1]$. Then,

$$\begin{aligned} \frac{1}{c_n} \mid \{k \in \mathbb{N} : A_{||x_k - \mathbf{0}||_G}(\xi) \geq e^\xi\} \mid &= \frac{1}{c_n}[|\sqrt[3]{n}|] \\ &= \frac{1}{c_n}(\sqrt[3]{n} - t_n), \end{aligned}$$

where $0 \leq t_n < 1$ for each $n \in \mathbb{N}$. Taking $n \to \infty$ on both sides of the above relation, we conclude that $\delta_c(\{k \in \mathbb{N} : A_{||x_k - \mathbf{0}||_G}(\xi) \geq \varepsilon\}) \neq 0$ for $\varepsilon = e^\xi$. Hence, (x_k) is not gradual quasi statistical convergent to $\mathbf{0}$.

Theorem 3.9. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{st-||\cdot||_G} x$. Then, $x_k \xrightarrow{st_q - ||\cdot||_G} x$ holds if $\inf\limits_n \frac{c_n}{n} > 0$.*

Proof. The proof easily follows from the following inequation

$$\frac{1}{n} \mid \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq e^\xi\} \mid \geq (\inf_n \frac{c_n}{n}) \cdot \frac{1}{c_n} \mid \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq e^\xi\} \mid .$$

□

Corollary 3.1. *A necessary and sufficient condition for a gradual statistical convergent sequence (x_k) in the GNLS $(X, ||\cdot||_G)$ to be gradual quasi statistical convergent is that $\inf\limits_n \frac{c_n}{n} > 0$.*

Definition 3.3. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be strongly gradual quasi summable to x in X if*

$$\lim_{n\to\infty} \frac{1}{c_n} \sum_{k=1}^{n} A_{||x_k - x||_G}(\xi) = 0.$$

Theorem 3.10. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. If (x_k) is strongly gradual quasi summable to x then $x_k \xrightarrow{st_q - ||\cdot||_G} x$.*

Proof. Let (x_k) be strongly gradual quasi summable to x in X. Then, by definition

$$\lim_{n\to\infty} \frac{1}{c_n} \sum_{k=1}^{n} A_{||x_k - x||_G}(\xi) = 0 \tag{3}$$

holds and eventually, we have

$$\begin{aligned} \frac{1}{c_n} \sum_{k=1}^{n} A_{||x_k-x||_G}(\xi) &= \frac{1}{c_n} \sum_{\substack{k=1 \\ A_{||x_k-x||_G}(\xi) \ge \varepsilon}}^{n} A_{||x_k-x||_G}(\xi) + \frac{1}{c_n} \sum_{\substack{k=1 \\ A_{||x_k-x||_G}(\xi) < \varepsilon}}^{n} A_{||x_k-x||_G}(\xi) \\ &\ge \frac{1}{c_n} \sum_{\substack{k=1 \\ A_{||x_k-x||_G}(\xi) \ge \varepsilon}}^{n} A_{||x_k-x||_G}(\xi) \\ &\ge \frac{\varepsilon}{c_n} \mid \left\{ k \le n : A_{||x_k-x||_G}(\xi) \ge \varepsilon \right\} \mid . \end{aligned}$$

Letting $n \to \infty$ and using Eq.(3), we obtain

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \left\{ k \le n : A_{||x_k-x||_G}(\xi) \ge \varepsilon \right\} \mid = 0.$$

Hence, $x_k \xrightarrow{st_q - ||\cdot||_G} x$. □

Theorem 3.11. *Let (x_k) be a gradually bounded sequence in the GNLS $(X, ||\cdot||_G)$. If $x_k \xrightarrow{st_q - ||\cdot||_G} x$ then (x_k) is strongly gradual quasi summable to x provided that $\inf_n \frac{c_n}{n} > 0$.*

Proof. Since $x_k \xrightarrow{st_q - ||\cdot||_G} x$, so for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \left\{ k \le n : A_{||x_k-x||_G}(\xi) \ge \varepsilon \right\} \mid = 0. \tag{4}$$

Now since (x_k) is gradually bounded, so there exists some $M > 0$ such that $\forall k \in \mathbb{N}$ and for any $\xi \in (0, 1]$, $A_{||x_k-x||_G}(\xi) \le M$. Then for any $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{c_n} \sum_{k=1}^{n} A_{||x_k-x||_G}(\xi) &= \frac{1}{c_n} \sum_{\substack{k=1 \\ A_{||x_k-x||_G}(\xi) \ge \varepsilon}}^{n} A_{||x_k-x||_G}(\xi) + \frac{1}{c_n} \sum_{\substack{k=1 \\ A_{||x_k-x||_G}(\xi) < \varepsilon}}^{n} A_{||x_k-x||_G}(\xi) \\ &\le \frac{M}{c_n} \sum_{\substack{k=1 \\ A_{||x_k-x||_G}(\xi) \ge \varepsilon}}^{n} 1 + \frac{1}{c_n} \sum_{\substack{k=1 \\ A_{||x_k-x||_G}(\xi) < \varepsilon}}^{n} A_{||x_k-x||_G}(\xi) \\ &\le \frac{M}{c_n} \mid \left\{ k \le n : A_{||x_k-x||_G}(\xi) \ge \varepsilon \right\} \mid + \varepsilon \cdot \frac{n}{c_n}. \end{aligned}$$

From Eq.(4) and the above inequation we have

$$\lim_{n\to\infty}\frac{1}{c_n}\sum_{k=1}^{n}A_{||x_k-x||_G}(\xi)=0.$$

Hence, (x_k) is strongly gradual quasi summable to x. This completes the proof. □

Definition 3.4. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradual quasi statistical Cauchy if for every $\varepsilon > 0$ and $\xi \in (0,1]$, there exists a natural number $N(= N(\xi,\varepsilon))$ such that $\delta_c(\{k\in\mathbb{N} : A_{||x_k-x_N||_G}(\xi)\geq\varepsilon\}) = 0$.*

Theorem 3.12. *Let $(X, ||\cdot||_G)$ be a GNLS. Then every gradual quasi statistical convergent sequence in X is a gradual quasi statistical Cauchy sequence.*

Proof. Let (x_k) be a sequence in X such that $x_k \xrightarrow{st_q-||\cdot||_G} x$. Then for every $\varepsilon > 0$ and $\xi \in (0,1]$,

$$\delta_c(B_1(\xi,\varepsilon)) = 0 \text{ where } B_1(\xi,\varepsilon) = \{k\in\mathbb{N} : A_{||x_k-x||_G}(\xi)\geq\varepsilon\}. \tag{5}$$

Choose $N \in \mathbb{N}\setminus B_1(\xi,\varepsilon)$. Then we have $A_{||x_N-x||_G}(\xi) < \varepsilon$.

Let $B_2(\xi,\varepsilon) = \{k\in\mathbb{N} : A_{||x_k-x_N||_G}(\xi)\geq 2\varepsilon\}$. Now we prove that the following inclusion is true

$$B_2(\xi,\varepsilon)\subseteq B_1(\xi,\varepsilon). \tag{6}$$

For if $p\in B_2(\xi,\varepsilon)$ we have

$$2\varepsilon\leq A_{||x_p-x_N||_G}(\xi)\leq A_{||x_p-x||_G}(\xi)+A_{||x-x_N||_G}(\xi) < A_{||x_p-x||_G}(\xi)+\varepsilon,$$

which implies $p\in B_1(\xi,\varepsilon)$ and so Eq.(6) is true. From Eq.(5) and Eq.(6) we conclude that $\delta_c(B_2(\xi,\varepsilon)) = 0$ which means that (x_k) is gradual quasi statistical Cauchy sequence. □

4. Conclusions

In this paper, we have investigated a few fundamental properties of quasi statistical convergence in the gradual normed linear spaces. Further, we introduced strong quasi statistical summability in the gradual normed linear spaces and proved that every strongly gradual quasi summable sequence is gradual quasi statistical convergent. Finally, we have introduced the concept of quasi statistical Cauchy sequences in the gradual normed space and established the interrelationship between gradual quasi statistical convergent and gradual quasi statistical Cauchy sequences.

Summability theory and the convergence of sequences have wide applications in various branches of mathematics particularly, in mathematical analysis. Research in this direction based on gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. The obtained results may be useful for future researchers to explore various notions of convergences in the gradual normed linear spaces in more detail.

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