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MODIFIED BISECTION ALGORITHM IN ESTIMATING THE EXTREME VALUE INDEX UNDER RANDOM CENSORING

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Abstract. The Generalized Pareto Distribution (GPD) has long been employed in the theories of extreme values. In this paper, we are interested by estimating the extreme value index under censoring. Using a maximum likelihood estimator (MLE) and a numerical method algorithm, a new approach is proposed to estimate the extreme value index by maximizing the adaptive log-likelihood of GPD given censored data. We also show how to construct the maximum likelihood estimate of the GPD parameters (shape and scale) using censored data. Lastly, numerical examples are provided at the end of the paper to show the method's reliability and to better illustrate the findings of this research.

Keywords: extreme value index, random censoring, generalized pareto distributions, maximum likelihood, the modified bisection algorithm.

AMS Subject Classification: 62G05, 62G20.

1. Introduction

Recent research has concentrated on extreme values on both a theoretical and practical level. These theories include a broad range of topics, intending to study rare events with a small possibility of occurring, such as natural disasters, economic crises, and other occurrences. The studies of extreme values are related to the investigation of the distribution function's tail index. Since the distribution of generalized Pareto (GPD) is based on the tail index. It is practically regarded, as useful distribution of extreme values, it has several applications especially in the fields of hydrology, finance, biology, social sciences due to its heavy tail properties. Let's remember that the GPD's cumulative distribution function (CDF) is known as:

$$P(X \leq x) = F_{\gamma,\sigma}(x) := \left\{ \begin{array}{c} 1 - \left(1 + \frac{\gamma}{\sigma}x\right)^{-1/\gamma}, \text{ for } \gamma \neq 0 \\ 1 - \exp\left(-\frac{\gamma}{\sigma}x\right), \text{ for } \gamma = 0 \end{array} \right. , \tag{1}$$

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where γ is called the shape parameter and σ the scale parameter, which is positive. Then, x is positive when $\gamma \geq 0$ and $0 < x < |\gamma|/\sigma$, when $\gamma < 0$.

Tail behavior is the assessment of the extreme value index, it has attracted a lot of attention recently and there has been a great deal of interest in their application for incomplete data. Specifies the issue of censored data, which was first noted as a problem by Reiss and Thomas, (1997) [14]. Furthermore, in the context of censored data, Einmahl et al. (2008), [6] adapted multiple conventional extreme value index estimators and proposed a unified way to establish their asymptotic normality. In this context we can also cite the article of Talha et al (2020) [20] which estimates location and scale paramters of the moyal distribution model. Recall that there are a variety of numerical approaches for computing the ML estimator for tail behavior utilizing maximum likelihood estimates (MLE) of the GPD parameters based on full data (without censoring) (γ, σ). The important classical MLE of the GPD parameters (γ, σ) for CDF given in (1) are the algorithms of Hosking and Wallis, 1987 [9] when $-1/2 < \gamma < 1/2$ Grimshaw, (1993) [7] and Kouider, (2019) [10] when $\gamma \geq -1$.

To investigate and improve this theory's contribution. We propose an approach for estimating the index of extreme values using the ML method and the shape and scale parameters of the GPD distribution under random censoring. We use a numerical method to estimate the GPD parameters under censored data with fewer conditions and without needing additional calculations. For numerous roots, we apply the Modified Bisection Algorithm (MBA), which is provided in section (3). In particular, we provide a detailed description of our algorithm. The GPD parameters under censored data are also presented, together with confidence intervals for the estimated parameters. In section(4), we illustrate our achievements by a simulation study in two numerical examples. The first one is generated by a sample that follows GPD parameters (the shape and the scale) under censored data and the latter is given with real data.

Let X_i, Y_i, $i = 1 \ldots, n$ two random variables (rv's) independent and identically distribution (i.i.d) with continuous CDF F and G respectively. We said that X_i is right-censored by $Y_i, i = 1 \ldots, n$ if $X_i > Y_i$., then we have $\delta_i = 0$. As a result, the variable $Z_i = \min(X_i, Y_i)$ is only observed when $X_i \leq Y_i$ for $i = 1 \ldots, n$, then $\delta_i = 1$ equal to one which represents the indicator function of censored data. Assuming now that H the cumulative distribution function of the observed variable Z_i, for $i = 1 \ldots, n$ and $\tau_H = \sup\{x : H(x) < 1\}$ the supremum of $H's$support. Let's call the attraction domains of an extreme value distribution F and G, so the extreme value index for distribution function (df)of (Z, δ) exists and we noted by γ where $\gamma = \gamma_1\gamma_2/(\gamma_1 + \gamma_2)$, such that γ_1 is the extreme value index for df of X and γ_2 is the extreme value index for df of Y. The adapted ML estimator with censoring noted $\widehat{\gamma}_1^{(c,ML)}$ which defined by split $\widehat{\gamma}^{(ML)}$ the extreme value index estimator of Z_i for $i = 1 \ldots, n$ (without censoring) into $\widehat{p}$ the proportion of non-censored observation in k , the number of Z's over threshold t (see, Einmahl et al (2008) [6]),

$$\widehat{\gamma}_1^{(c,ML)} := \frac{\widehat{\gamma}^{(ML)}}{\widehat{p}}, \tag{2}$$

where $\widehat{p} = \frac{r}{k}$ and $r := \sum_{i=1}^{n} \delta_{[i,n]} 1_{Z_{[i,n]} \geq t}$ (r is the number of censored observation in the k largest Z's), $\delta_{[1,n]}, \delta_{[2,n]}, \ldots, \delta_{[n,n]}$ the δ corresponding to $Z_{[1,n]}, Z_{[2,n]}, \ldots, Z_{[n,n]}$ respectively and $Z_{[1,n]} < Z_{[2,n]} < \cdots < Z_{[n,n]}$ the order statistics of Z_i for $i = 1 \ldots, n$. While γ divided by $\gamma_2/(\gamma_1 + \gamma_2)$ is equal to γ_1, then under (2), we conclude that $\widehat{p}$ estimate $\gamma_2/(\gamma_1 + \gamma_2)$. As well, Pham et al [13] discussed the estimation of the GPD parameters

(γ_1, σ_1) via MLE for CDF given in (1) under censoring data, that based on Grimshaw's estimation [7], for $\gamma_1 \geq -1$ with many mathematical operations. The sensitivity and specificity of their estimation method were also studied in detail and wonderfully by Pham, Tsokos and Choi, (2019) [13]. The current study aims is to reduce count compression. Based on [10], a new algorithm for MLE of the GPD parameters (γ_1, σ_1) is proposed to estimate the extreme value index under censored data. Furthermore, using our MLE approach, we introduce a new tail behavior estimator under random censoring based on the ML method, indicated $\widehat{\gamma}_1^{(c,KIB)}$.

The remainder of the paper is organized as follows. The adapt likelihood equation of GPD under random censoring was shown in section 2. In section 3, we propose a new algorithm for MLE of GPD parameters (γ_1, σ_1) based on Kouider's estimation (2019) (see [10]) under censored data when $\gamma_1 \geq -1$. It will be used, to reduce account compression for MLE for the parameters (γ_1, σ_1) of GPD under censoring, and a new MLE-based tail behavior estimator $\widehat{\gamma}_1^{(c,KIB)}$ will also be introduced. In section 4, a simulation study is carried out on the adaptive estimator of the GPD parameters (γ_1, σ_1) using two examples, the first of which is based on censored data and the second of which is based on real data, to better illustrate our outcomes.

2. The Adapt Likelihood Equation of GPD Under Random Censoring

Let (Z_j, δ_j) be a sample from a couple of rv's (Z, δ), and noted the concomitant of the i-th order statistic by $\delta_{[i,n]} = \delta_j$ where $Z_{[i,n]} = Z_j$, $1 \leq j \leq n$. In the case of censoring, the ML estimator is an adaptation estimator for the extreme value index where the shape parameter of the GPD as we defined in (2). In this paper, we use the likelihood approach under censoring. This approach methodology is one of the most concepts in the statistics of extremes value.

The likelihood approach, which is based on two works by Balkema [2] and Pickands [12], uses GPD parameters to approximate the distribution of excesses $C_j = Z_j - t$, given $Z_j > t$ over a threshold $t > 0$ (sufficiently high) when $t \to \tau_F$, where τ_F is the supremum of the support of F,. Noted the number of absolute excesses over t by k , wherein the asymptotic setting $k = k_n$ is an intermediate sequence, that is, $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. Under censoring, the maximum likelihood estimate strategy uses the maximization of an adapt likelihood function (see [1]),

$$\ell\left(C_j, \delta_j\right) = \prod_{j=1}^{k} f_{\gamma_1,\sigma_1}\left(C_j\right)^{\delta_j} \left(1 - F_{\gamma_1,\sigma_1}\left(C_j\right)\right)^{1-\delta_j}, \tag{3}$$

where $f_{\gamma_1,\sigma_1}(x) = (1/\sigma_1)\left(1 + (\gamma_1/\sigma_1)\, x\right)^{-(1/\gamma_1)-1}$ is the associated GPD density. As a consequence, we estimate the unknown parameters γ_1 and σ_1 by the ML estimator if we maximize (3). From equation (3) the log- likelihood given by:

$$\log\left(\ell\left(C_j, \delta_j\right)\right) = \sum_{j=1}^{k} \left(\delta_j \log\left(\frac{1}{\sigma_1}\right) - \left(\delta_j + \frac{1}{\gamma_1}\right) \log\left(1 + \frac{\gamma_1}{\sigma_1} C_j\right)\right). \tag{4}$$

Then for any $\gamma_1 < -1$ with $C_{k,k} = \max\left(C_j\right)$ for $1 \leq j \leq k$,

$$\lim_{\frac{\sigma_1}{\gamma_1} \to C_{k,k}^{+}} \log\left(\ell\left(C_j, \delta_j\right)\right) = \begin{cases} +\infty \text{ if } \delta_j = 1, \\ -\infty \text{ if } \delta_j = 0. \end{cases}$$

From (4), the likelihood equations are given in terms of the partial:

$$\begin{cases} \frac{\partial \log(\ell(C_j,\delta_j))}{\partial \gamma_1} = \sum\limits_{j=1}^{k} \left(\left(\frac{1}{\gamma_1}\right)^2 \log\left(1+\frac{\gamma_1}{\sigma_1}C_j\right) - \left(\delta_j+\frac{1}{\gamma_1}\right)\left(\frac{1}{\sigma_1}C_j\right)\left(1+\frac{\gamma_1}{\sigma_1}C_j\right)^{-1}\right) \\ \frac{\partial \log(\ell(C_j,\delta_j))}{\partial \sigma_1} = \sum\limits_{j=1}^{k} \frac{1}{\sigma_1}\left(-\delta_j + \left(\delta_j+\frac{1}{\gamma_1}\right)\left(\frac{\gamma_1}{\sigma_1}C_j\right)\left(1+\frac{\gamma_1}{\sigma_1}C_j\right)^{-1}\right) \end{cases}$$

For $\gamma_1 = 0$, these concepts should be evaluated in terms of Taylor's development, therefore we have:

$$\begin{cases} \left.\frac{\partial \log(\ell(C_j,\delta_j))}{\partial \gamma_1}\right|_{\gamma_1=0} = \sum\limits_{j=1}^{k} \left(-\delta_j\left(\frac{C_j}{\sigma_1}\right) + \frac{1}{2}\left(\frac{C_j}{\sigma_1}\right)^2\right) \\ \left.\frac{\partial \log(\ell(C_j,\delta_j))}{\partial \sigma_1}\right|_{\gamma_1=0} = \sum\limits_{j=1}^{k} \left(-\delta_j\left(\frac{1}{\sigma_1}\right) + \left(\frac{C_j}{(\sigma_1)^2}\right)\right) \end{cases} \tag{5}$$

From formulas (5), we find that $\sigma_1 = (k/r)\,\overline{C}$ where $\overline{C} := (1/k)\sum\limits_{j=1}^{k} C_j$, and $r := \sum\limits_{j=1}^{k} \delta_j$.

For $\gamma_1 \neq 0$, the likelihood equations will given respectively by :

$$\sum_{j=1}^{k}\left(\left(\frac{1}{\gamma_1}\right)\log\left(1+\frac{\gamma_1}{\sigma_1}C_j\right) - \left(\delta_j+\frac{1}{\gamma_1}\right)\left(\frac{\gamma_1}{\sigma_1}C_j\right)\left(1+\frac{\gamma_1}{\sigma_1}C_j\right)^{-1}\right) = 0 \tag{6}$$

and

$$\sum_{j=1}^{k}\left(\left(\delta_j+\frac{1}{\gamma_1}\right)\left(\frac{\gamma_1}{\sigma_1}C_j\right)\left(1+\frac{\gamma_1}{\sigma_1}C_j\right)^{-1}\right) = r \tag{7}$$

Then, we can simplified (6) by :

$$\frac{1}{r}\sum_{j=1}^{k}\log\left(1+\frac{\gamma_1}{\sigma_1}C_j\right) = \gamma_1, \tag{8}$$

and (7) by:

$$\frac{1}{k}\sum_{i=1}^{n}(\gamma_1\delta_j+1)\left(1+\frac{\gamma_1}{\sigma_1}C_j\right)^{-1} = 1 \tag{9}$$

Noted r-value for the case $\delta_j = 1$ and m-value for the case $\delta_j = 0$ equivalent to $k = r+m$ value for $C_1, C_2, \ldots, C_k$ to be a sequence of i.i.d. random variables. Therefore, the equation (4) will be given by:

$$\log\left(\ell\left(C_j,\delta_j\right)\right) = r\log\left(\frac{1}{\sigma_1}\right) - \sum_{j=1}^{r}\log\left(1+\frac{\gamma_1}{\sigma_1}C_{j,r}\right) - \frac{1}{\gamma_1}\sum_{j=1}^{k}\log\left(1+\frac{\gamma_1}{\sigma_1}C_{j,k}\right).$$

where $\delta_{1,k}, \delta_{2,k}, \ldots, \delta_{k,k}$ the δ' corresponding to $C_{1,k}, C_{2,k}, \ldots, C_{k,k}$, respectively and $C_{1,k} < C_{2,k} < \cdots < C_{k,k}$ the order statistics correspondence, such that $\sigma_1 > 0$ for $\gamma_1 > 0$ and $\sigma_1 > -\gamma_1 C_{k,k}$ for $\gamma_1 < 0$. Hence, if $\gamma_1 < -1$ so there is no MLE. And, if we take $\gamma_1 = -1$ with $\sigma_1 > C_{k,k}$ we obtain :

$$M = \left.\log\left(\ell\left(C_j,\delta_j\right)\right)\right|_{\gamma_1=-1} = r\log\left(\frac{1}{\sigma_1}\right) + \sum_{j=1}^{m}\log\left(1-\frac{C_{j:m}}{\sigma_1}\right) \tag{a}$$

For $m = k - r$, in this case we are going to take $C_{k,k}$ estimator of σ_1 with $\gamma_1 = -1$. Since $C_{k,k} = C_{m,m}$ is censored (not observed), M not exist. According to Jensen's inequality, if we consider $\overline{C'} := (1/m) \sum_{j=1}^{m} C_{j:m}$ then we can rewrite the formula (a) as

$$M_1 \simeq r \log\left(\frac{1}{\sigma_1}\right) + m \log\left(1 - \frac{\overline{C'}}{\sigma_1}\right) \tag{b}$$

However, if $C_{k,k} = C_{r,r}$ is observed so M exist. Then, with $\widehat{\gamma}_1^{(c)} = -1$ we can find $\widehat{\sigma}_1^{(c)} = C_{k,k}$. We are already aware that the estimator the statistical analysis if one exists, is lonely. In this situation, the problem is complicated because we use the optimization in an open set under censored data. But there is no problem with the close-set. Hence, the MLE of GPD parameters under random censoring, denoted by $\left(\widehat{\gamma}_1^{(c)}, \widehat{\sigma}_1^{(c)}\right)$. We recognize it in three cases as follows :

Case1: $C_{k,k}$ is observed, $\left(\widehat{\gamma}_1^{(c)}, \widehat{\sigma}_1^{(c)}\right)$ is given by the local maximum if

$$\log\left(\ell\left(C_j, \delta_j\right)\right) > M|_{\widehat{\sigma}_1^{(c)} = C_{k,k}},$$

and is given by the boundary maximum if $\log\left(\ell\left(C_j, \delta_j\right)\right) < M|_{\widehat{\sigma}_1^{(c)} = C_{k,k}}$,

Case2: $C_{k,k}$ is censored, $\left(\widehat{\gamma}_1^{(c)}, \widehat{\sigma}_1^{(c)}\right)$ is given by the local maximum if $\log\left(\ell\left(C_j, \delta_j\right)\right) > M_1|_{\widehat{\sigma}_1^{(c)} = C_{k,k}}$, and is given by the boundary maximum if

$$\log\left(\ell\left(C_j, \delta_j\right)\right) < M_1|_{\widehat{\sigma}_1^{(c)} = C_{k,k}},$$

Case3: if no local maximum is found, then there is no MLE of parameters of GPD under random censoring. In this situation, use an adaptive estimator via the use of the alternative estimators given by Hosking and Wallis, (1987) [9] without censoring, to obtain a finite maximum of the GPD log-likelihood under censored data as we defined in (2).

Consequently, the constraint $\gamma_1 \geq -1$ must be imposed for the existence of an ML-estimator. We have to count the MLE of GPD parameters with random censoring is an optimization on the constrained space $A = \{\gamma_1 > 0, \sigma_1 > 0\} \cup \{-1 \leq \gamma_1 < 0, \sigma_1/\gamma_1 < -C_{k,k}\}$. Furthermore, in order to compute the MLE of GPD with censored data. We have to compute two values of (γ_1, σ_1) that have to count together. The first is the local maximum of the log-likelihood on the space A. And the second is at the boundary of A , where $\gamma_1 = -1$. Therefore from Eq. (8) and (9), the system likelihood equation can be simplified as well:

$$\psi\left(\theta_1\right) = \frac{1}{k} \sum_{j=1}^{k} \left[\left(\delta_j \left(\frac{1}{r} \sum_{j=1}^{k} \log\left(1 - \theta_1 C_j\right)\right) + 1\right) \times \left(1 - \theta_1 C_j\right)^{-1}\right] - 1 = 0, \tag{10}$$

for $\theta_1 = -\left(\gamma_1/\sigma_1\right)$ and $1 - \theta_1 C_{j,k} > 0$, $j = 1, 2, \ldots, k$, so it must be computed numerically on space $B = \{\theta_1 < 1/C_{k,k}, \theta_1 \neq 0\}$. For $\gamma_1 \in \mathbb{R}$ and in terms of (C_j, δ_j), the Eq (10) with the likelihood equations under censored data will be given by:

$$\begin{aligned} \psi\left(\theta_1\right) &= \mathcal{CD} + \frac{1}{k} \sum_{j=1}^{k} \left(1 - \theta_1 C_{j,k}\right)^{-1} - 1 \quad when \quad \gamma_1 \neq 0 \\ \widehat{\sigma}_1^{(c)} &= (k/r)\,\overline{C}, \qquad\qquad when \;\; \gamma_1 = 0 \end{aligned}$$

where $\mathcal{C} =\frac{1}{r}\sum\limits_{j=1}^{k}\log\left(1-\theta_1 C_{j,k}\right)$ and $\mathcal{D} =\frac{1}{k}\sum\limits_{j=1}^{r}\left(1-\theta_1 C_{j,r}\right)^{-1}$. Then, the ML estimator for the GPD parameters under censored data can be approximated in the following procedure :

- Find the root $\widehat{\theta}_1^{(c)}$ of $\psi\left(\theta_1\right)=0$ where:

$$\psi\left(\theta_1\right)=\left(\left(r/k\right)\mathcal{C}\right)\left(\left(k/r\right)\mathcal{D}\right)+\frac{1}{k}\sum_{j=1}^{k}\left(1-\theta_1 C_{j,k}\right)^{-1}-1, \tag{11}$$

- Compute $\widehat{\gamma}_1^{(c)}$ by

$$\frac{1}{r}\sum_{j=1}^{k}\log\left(1-\widehat{\theta}_1^{(c)}C_j\right)=\widehat{\gamma}_1^{(c)}. \tag{12}$$

- $\widehat{\theta}_1^{(c)}=-\left(\widehat{\gamma}_1^{(c)}/\widehat{\sigma}_1^{(c)}\right)$ then $\widehat{\sigma}_1^{(c)}=-\left(\widehat{\gamma}_1^{(c)}/\widehat{\theta}_1^{(c)}\right)$.

Otherwise, we give a new algorithm for MLE of GPD parameters (γ_1,σ_1) based on [10]. If we consider the case of uncensored data, where $C_j=Z_j-t$ and $k=r$ which has already been studied in many articles including the important works [7] and [10], the ML estimator for the GPD parameters can be approximated in the following procedure:

- Find the root $\widehat{\theta}$ of $\psi\left(\theta\right)=0$ where:

$$\psi\left(\theta\right)=\left(\frac{1}{k}\sum_{j=1}^{k}\log\left(1-\theta C_{j,k}\right)\right)\times\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\theta_1 C_{j,k}\right)^{-1}\right)+\frac{1}{k}\sum_{j=1}^{k}\left(1-\theta_1 C_{j,k}\right)^{-1}-1, \tag{13}$$

- Compute $\widehat{\gamma}$ by

$$\frac{1}{k}\sum_{j=1}^{k}\log\left(1-\widehat{\theta}C_j\right)=\widehat{\gamma}. \tag{14}$$

- $\widehat{\theta}=-\left(\widehat{\gamma}/\widehat{\sigma}\right)$ then $\widehat{\sigma}=-\left(\widehat{\gamma}/\widehat{\theta}\right)$.

We can find that $\widehat{\theta}_1^{(c)}\neq\widehat{\theta}$ because the zero of $\psi\left(\theta_1\right)$ in (11) is different from $\psi\left(\theta\right)$ in (13). As consequence, we cannot define $\widehat{\gamma}_1^{(c)}$ that is given in (12) as adapted ML estimator in censoring which is defined previously in (2). Noted $\left(\widehat{\gamma}_1^{(c)},\widehat{\sigma}_1^{(c)},\widehat{\theta}_1^{(c)}\right)$ by

$$\left(\widehat{\gamma}_1^{(c,KIB)},\widehat{\sigma}_1^{(c,KIB)},\widehat{\theta}_1^{(c,KIB)}\right)$$

respectively. Refer to the previous procedure of the ML estimator for the GPD parameters without censoring as $C_j=X_j-t$, for $X_j>t$ and $\theta_1=-\left(\gamma_1/\sigma_1\right)$ it gives $\widehat{\theta}_1^{(c,KIB)}\neq\widehat{\theta}_1$.

If $\widehat{\theta}_1^{(c,KIB)}\leq\widehat{\theta}_1$ equivalent $\widehat{\gamma}_1^{(c,KIB)}\leq\widehat{\gamma}_1$ and $\widehat{\sigma}_1\leq\widehat{\sigma}_1^{(c,KIB)}$. Then we find the following asymptotic behavior :

$$\sqrt{k}\left(\widehat{\gamma}_1^{(c,KIB)}-\gamma_1\right)\leq\sqrt{k}\left(\widehat{\gamma}_1-\gamma_1\right),$$

where $\sqrt{k}\left(\frac{\widehat{\sigma}_1^{(c,KIB)}}{\sigma_1}-1\right)\geq\sqrt{k}\left(\frac{\widehat{\sigma}_1}{\sigma_1}-1\right)$, such that, the asymptotic $\sqrt{k}\left(\widehat{\gamma}_1-\gamma_1\right)$ and $\sqrt{k}\left(\frac{\widehat{\sigma}_1}{\sigma_1}-1\right)$ is based on the X-sample (uncensored situation) for $k=r$ (see [4]).

When $\widehat{\theta}_1^{(c,KIB)} \geq \widehat{\theta}_1$ we also have $\sqrt{k}\left(\widehat{\gamma}_1^{(c,KIB)} - \gamma_1\right) \geq \sqrt{k}\left(\widehat{\gamma}_1 - \gamma_1\right)$ with $\sqrt{k}\left(\frac{\widehat{\sigma}_1^{(c,KIB)}}{\sigma_1} - 1\right) \leq \sqrt{k}\left(\frac{\widehat{\sigma}_1}{\sigma_1} - 1\right)$. Now if $\widehat{\theta}_1^{(c,KIB)} = \widehat{\theta}_1$ the estimator $\widehat{\gamma}_1^{(c,KIB)}$ becomes an adaptive estimator to censoring as we define in (2). The following theorem provides some useful properties for obtaining the zeros of the function $\psi(\theta_1)$ provided in section (3) using the numerical method discussed in (11).

Theorem 2.1. *Let's the function $\psi(\theta_1)$ given in (11) and defined on the space B. We have:*

$$\text{(1)} \lim_{\theta_1 \to 1/C_{k,k}^-} \psi(\theta_1) = -\infty.$$

$$\text{(2)} \lim_{\theta_1 \to 0} \psi(\theta_1) = 0 \tag{15}$$

$$\text{(3)}\ \psi(\theta_1) < 0 \text{ for all } \theta_1 < (\theta_1)_L = \frac{2\left(C_{1,k} - \overline{C}\right)}{\left(C_{1,k}\right)^2}. \tag{16}$$

where $\overline{C} = \frac{1}{k}\sum_{j=1}^{k} C_j$, and $C_{1,k} = \min(C_j)$ for $j = 1, 2, \ldots, k$.

In formula (5) if $\gamma_1 \to 0$ then $\theta_1 \to 0$ where $\sigma_1 = \frac{k}{r}\overline{C}$; as a similar result, we obtain $\gamma_1 = 10^{-s}$ (Noted that s is a natural number). Consequently, if we take $\varepsilon = 10^{-s}/\frac{k}{r}\overline{C}$ around them, while the upper bound $1/C_{k,k} \notin B$, we can use $1/C_{k,k} - \varepsilon$ as the upper bound and $(\theta_1)_L - \varepsilon$ for $\varepsilon > 0$ as lower bound. Therefore the space B will be define by: $\mathcal{B} = \{\theta_1 \in [(\theta_1)_L ; -\varepsilon] \cup [\varepsilon ; (\theta_1)_U], \theta_1 \neq 0\}$, where $(\theta_1)_U = 1/C_{k,k} - \varepsilon$, with $\varepsilon = 10^{-s}/\frac{k}{r}\overline{C}$ for s is a natural number.

Proof. Results (1) and (2) are accurate.

The proof of Result (3). We can rewrite the function (11) as

$$\psi(\theta_1) = \left(\frac{1}{r}\sum_{j=1}^{k}\log\left(1-\theta_1 C_{j,k}\right) + 1\right)\left(\frac{1}{k}\sum_{j=1}^{r}\left(\frac{1}{1-\theta_1 C_{j,r}}\right)\right) + \left(\frac{1}{k}\sum_{j=1}^{m}\left(\frac{1}{1-\theta_1 C_{j,m}}\right)\right) - 1 \tag{17}$$

Follows by using Jensen inequality $\frac{1}{r}\sum_{j=1}^{k}\log\left(1-\theta_1 C_{j,k}\right) \leq \frac{k}{r}\log\left(1-\theta_1\overline{C}\right)$ for $\theta_1 \in \mathcal{B}$ and since $C_{1,r} < C_{j,r}$ and $C_{1,m} < C_{j,m}$ for all $j = 1, \ldots, k$ respectively , and with $\theta_1 < 0$ we have

$$\frac{1}{k}\sum_{j=1}^{r}\left(\frac{1}{1-\theta_1 C_{j,r}}\right) < \frac{r}{k}\left(\frac{1}{1-\theta_1 C_{1,r}}\right) \text{ and } \frac{1}{k}\sum_{j=1}^{m}\left(\frac{1}{1-\theta_1 C_{j,m}}\right) < \frac{m}{k}\left(\frac{1}{1-\theta_1 C_{1,m}}\right)$$

For $\theta_1 < 0$ it follows that

$$\psi(\theta_1) \leq \left(\frac{k}{r}\log\left(1-\theta_1\overline{C}\right) + 1\right)\left(\frac{r}{k}\left(\frac{1}{1-\theta_1 C_{1,r}}\right)\right) + \frac{m}{k}\left(\frac{1}{1-\theta_1 C_{1,m}}\right) - 1,$$

then

$$\psi(\theta_1) \leq \log\left(1-\theta_1\overline{C}\right)\left(\frac{1}{1-\theta_1 C_{1,r}}\right) + \frac{r}{k}\left(\frac{1}{1-\theta_1 C_{1,r}}\right) + \frac{m}{k}\left(\frac{1}{1-\theta_1 C_{1,m}}\right) - 1 < 0$$

Since $\theta_1 < 0$ with $C_{1,r} < C_{1,m}$ ($\min(C_{1,m}; C_{1,r})$) we have $\left(\frac{1}{1-\theta_1 C_{1,m}}\right) < \left(\frac{1}{1-\theta_1 C_{1,r}}\right)$, we get

$$\psi(\theta_1) \leq \log\left(1-\theta_1\overline{C}\right)\left(\frac{1}{1-\theta_1 C_{1,r}}\right) + \left(\frac{1}{1-\theta_1 C_{1,r}}\right) - 1 < 0,$$

then $\log\left(1-\theta_1\overline{C}\right) + 1 < (1-\theta_1 C_{1,r})$, and $1-\theta_1\overline{C} < \exp(-\theta_1 C_{1,r})$, hence $1-\theta_1\overline{C} < 1 + (-\theta_1 C_{1,r}) + \frac{1}{2}(-\theta_1 C_{1,r})^2$.

Assuming that $\theta_1 < (\theta_1)_L$, then,
$(\theta_1)_L = \frac{2\left(C_{1,r}-\overline{C}\right)}{(C_{1,r})^2}$. By the way, if $C_{1,m} < C_{1,r}$ ($\min(C_{1,m}; C_{1,r})$) we have $\left(\frac{1}{1-\theta_1 C_{1,r}}\right) < \left(\frac{1}{1-\theta_1 C_{1,m}}\right)$ and we get $(\theta_1)_L = \frac{2\left(C_{1,m}-\overline{C}\right)}{(C_{1,m})^2}$.

Knowing that $C_{1,k} := \min(C_j; \delta_j)$ for all $j = 1, \ldots, k$. this implies that:
1. if $C_{1,r} := C_{1,k}$ that is mean is observed then:

$$(\theta_1)_L = \frac{2\left(C_{1,k}-\overline{C}\right)}{(C_{1,k})^2}$$

2. if $C_{1,m} := C_{1,k}$ that is mean is censored then:

$$(\theta_1)_L = \frac{2\left(C_{1,k}-\overline{C}\right)}{(C_{1,k})^2}$$

This achieves the proof of (16).

□

3. Algorithm for the GPD maximum likelihood estimates under censored data

Many numerical strategies for computing the MLE of GPD parameters (γ_1, σ_1) by solving the ML-equations based on the full data have been proposed in the literature. Among them, the reader is invited to see [10]. In this work, we use MBA to search the multi-roots of $\psi(\theta_1)$ given in (11) which proposed in [11], as the following steps:

Step 1: Separate the interval $[\alpha; \beta]$ into several intervals $[\alpha_i; \beta_i]$ by $h = (\beta - \alpha)/d$ where $d \in \mathbb{N}$ is the number of intervals (for example $d = 10$), and $\beta > \alpha$.

Step 2: For $i = 1, \ldots d$, where $\alpha_1 = \alpha_i$, $\beta_1 = \beta_i$ and $f(\alpha_i).f(\beta_i) < 0$.

Step 3: For $n \geq 1$, calculate the middle of $[\alpha_n; \beta_n]$ by $(\theta_1)^*_n = (\alpha_n + \beta_n)/2$.

Step 4: Compute for a sub-interval $[\alpha^*_n, \beta^*_n]$ by:

$$[\alpha^*_n; \beta^*_n] = \begin{cases} [\alpha_n; (\theta_1)^*_n], & \text{if } f(\alpha_n).f((\theta_1)^*_n) < 0 \\ [(\theta_1)^*_n; \beta_n], & \text{if } f((\theta_1)^*_n).f(\beta_n) < 0 \end{cases}$$

Step 5: Compute the value of $(\theta_1)_{n+1}$ by:

$$(\theta_1)_{n+1} = \beta^*_n - f(\beta^*_n)\frac{\beta^*_n - \alpha^*_n}{f(\beta^*_n) - f(\alpha^*_n)} \text{ or } (\theta_1)_{n+1} = \alpha^*_n - f(\alpha^*_n)\frac{\beta^*_n - \alpha^*_n}{f(\beta^*_n) - f(\alpha^*_n)}$$

Step 6: For tolerance 1×10^{-7}, if $\left|(\theta_1)_{n+1} - (\theta_1)^*_n\right| < 1 \times 10^{-7}$, then $(\theta_1)_{n+1} = (\theta_1)^*_n$ and stop the algorithm. Consequently, the zero is $(\theta_1)_{n+1}$. Else

$$\left[\alpha^*_{n+1}; \beta^*_{n+1}\right] = \begin{cases} \left[\alpha^*_n; (\theta_1)_{n+1}\right], & \text{if } f(\alpha^*_n).f\left((\theta_1)_{n+1}\right) < 0 \\ \left[(\theta_1)_{n+1}; \beta^*_n\right], & \text{if } f\left((\theta_1)_{n+1}\right).f(\beta^*_n) < 0 \end{cases}$$

and set $n = n + 1$, then come back to Step3. Inside some regularity conditions, the MLE has a consistent estimator with the lowest variance. The inverse of the Fisher information matrix is used to estimate the estimators' asymptotic variance covariance matrix of the estimators $\left(\widehat{\gamma}_1^{(c,KIB)}, \widehat{\sigma}_1^{(c,KIB)}\right)$. Which is given by:

$$\begin{pmatrix} Var\left(\widehat{\gamma}_1^{(c,KIB)}\right) & Cov\left(\widehat{\gamma}_1^{c,KIB}, \widehat{\sigma}_1^{c,KIB}\right) \\ Cov\left(\widehat{\gamma}^{c,KIB}, \widehat{\sigma}^{c,KIB}\right) & Var\left(\widehat{\sigma}_1^{c,KIB}\right) \end{pmatrix} =$$

$$\begin{pmatrix} \frac{\partial^2 \log(\ell(C_j;\delta_j))}{\partial^2 \widehat{\gamma}_1^{(c,KIB)}} & \frac{\partial^2 \log(\ell(C_j;\delta_j))}{\partial \widehat{\gamma}_1^{(c,KIB)} \partial \widehat{\sigma}_1^{(c,KIB)}} \\ \frac{\partial^2 \log(\ell(C_j;\delta_j))}{\partial \widehat{\gamma}_1^{(c,KIB)} \partial \widehat{\sigma}_1^{(c,KIB)}} & \frac{\partial^2 \log(\ell(C_j;\delta_j))}{\partial^2 \widehat{\sigma}_1^{(c,KIB)}} \end{pmatrix}^{-1}$$

As a result, given censored data, the second derivatives of the log-likelihood of the GPD are computed for $\theta_1 = -(\gamma_1/\sigma_1)$ by:

$$\begin{cases} \frac{\partial^2 \log(\ell(C_j;\delta_j))}{\partial^2 \gamma_1} := \left(\frac{1}{\gamma_1}\right)^3 \sum\limits_{j=1}^{k} \left[(1+\gamma_1\delta_j)(\mathcal{A})^2 - 2\log(\mathcal{R}) + 2(\mathcal{A})\right], \\ \frac{\partial^2 \log(\ell(C_j;\delta_j))}{\partial^2 \sigma_1} := \left(\frac{\theta_1}{\gamma_1}\right)^2 \sum\limits_{j=1}^{k} \left[\delta_j - \left(\delta_j + \frac{1}{\gamma_1}\right)(2\mathcal{A}) - (\mathcal{A})^2\right)], \\ \frac{\partial^2 \log(\ell(C_j;\delta_j))}{\partial \gamma_1 \partial \sigma_1} := \frac{\theta_1}{(\gamma_1)^2} \sum\limits_{j=1}^{k} \left[\left(\delta_j + \frac{1}{\gamma_1}\right)(\mathcal{A})^2 - \delta_j(\mathcal{A})\right]. \end{cases}$$

When $\mathcal{A} = 1 - \frac{1}{1-\theta_1 C_j}$ and $\mathcal{R} = 1 - \theta_1 C_j$. In the data sets, we used investigating the MLE of GPD that agree with the finding of [9], [7] and [10]. The results show that on each interval, there are either no zeros, two zeros, or one zero. This is what their numerical study indicates. We will use this advantage in order to present an algorithm for computing the MLE for the parameters (γ_1, σ_1) of GPD under random censoring. First, we look for multi-roots with the properties stated in Theoreme (2.1) in the roots of the function given on 11) by the MBA. Next, for the function given on (11), we compute the shape and scale parameters of each root finding. Then, for GPD parameter estimations, we choose a maximum of local maximum likelihood. Lastly, using confidence intervals for each estimator, we estimate their asymptotic variance-covariance matrix. The steps below describe the procedure for computing the MLE of GPD with CDF given in (1) for the parameters (γ_1, σ_1) under censoring:

A. Choose a value for ε close to zero enough, for example, let $\varepsilon = 10^{-8}/\overline{C}$.

B. Compute$(\theta_1)_L$ and $(\theta_1)_U$ the lower and upper bounds of $\psi(\theta_1)$ respectively which given by :

$$(\theta_1)_L = \frac{2\left(C_{1,k} - \overline{C}\right)}{\left(C_{1,k}\right)^2}; (\theta_1)_U = \frac{1}{C_{k:k}} - \varepsilon.$$

C. To find the zero of $\psi(\theta_1) = 0$ given on (11) on the tow intervals $[(\theta_1)_L; -\varepsilon]$ and $[\varepsilon; (\theta_1)_U]$, it has to be used the MBA for multi-roots algorithm given in section(3) with each interval.

D. For each value of $(\theta_1)_s^{(0)}$ obtained from the previous step C , compute the value of $(\gamma_1)_s$ given by $(\gamma_1)_s = \frac{1}{r}\sum\limits_{j=1}^{k} \log\left(1 - (\theta_1)_s^{(0)} C_j\right)$ for $r := \sum\limits_{j=1}^{k} \delta_j$. And compute the value of $(\sigma_1)_s = -\left((\gamma_1)_s / (\theta_1)_s^{(0)}\right)$.

E. Let $(\gamma_1)_s$ and $(\sigma_1)_s$ denote the results of the previous step D which belongs on space A, and if find a local maximum exists, the ML estimator of the GPD parameters is

$\left(\widehat{\gamma}_1^{(c,KIB)},\widehat{\sigma}_1^{(c,KIB)}\right)$ which have a maximum $\log\left(\ell\left(C_j,\delta_j;\left(\widehat{\gamma}_1^{(c,KIB)},\widehat{\sigma}_1^{(c,KIB)}\right)\right)\right)$.

F. For $\left(\widehat{\gamma}_1^{(c,KIB)},\widehat{\sigma}_1^{(c,KIB)}\right)$ compute all the second-order partial derivatives estimators:

$\frac{\partial^2 \log(\ell(C_j,\delta_j))}{\partial^2\widehat{\gamma}_1^{(c,KIB)}}$, $\frac{\partial^2 \log(\ell(C_j,\delta_j))}{\partial^2\widehat{\sigma}_1^{(c,KIB)}}$ and $\frac{\partial^2 \log(\ell(C_j,\delta_j))}{\partial\widehat{\gamma}_1^{(c,KIB)}\partial\widehat{\sigma}_1^{(c,KIB)}}$ and the Jacobian determinant of the second-order derivatives :

$$\frac{\partial^2 \log(\ell(C_j,\delta_j))}{\partial^2\widehat{\gamma}_1^{(c,KIB)}}\times\frac{\partial^2 \log(\ell(C_j,\delta_j))}{\partial^2\widehat{\sigma}_1^{(c,KIB)}}-\left(\frac{\partial^2 \log(\ell(C_j,\delta_j))}{\partial\widehat{\gamma}_1^{(c,KIB)}\partial\widehat{\sigma}_1^{(c,KIB)}}\right)^2.$$

G. Calculate $100(1-\kappa)\%$ confidence intervals of (γ_1,σ_1) constructing by using

$\widehat{\gamma}_1^{(c,KIB)}\underset{+}{-}t_{\kappa/2}\sqrt{Var\left(\widehat{\gamma}_1^{(c,KIB)}\right)}$ and $\widehat{\sigma}_1^{(c,KIB)}\underset{+}{-}t_{\kappa/2}\sqrt{Var\left(\widehat{\sigma}_1^{(c,KIB)}\right)}$ respectively where $t_{\kappa/2}$ denotes the $1-\kappa/2$ quantile of the standard normal distribution is symmetric about 0.

H. After calculating the confidence intervals of γ_1 and σ_1 respectively is the final of our algorithm as we presented in the following figure:

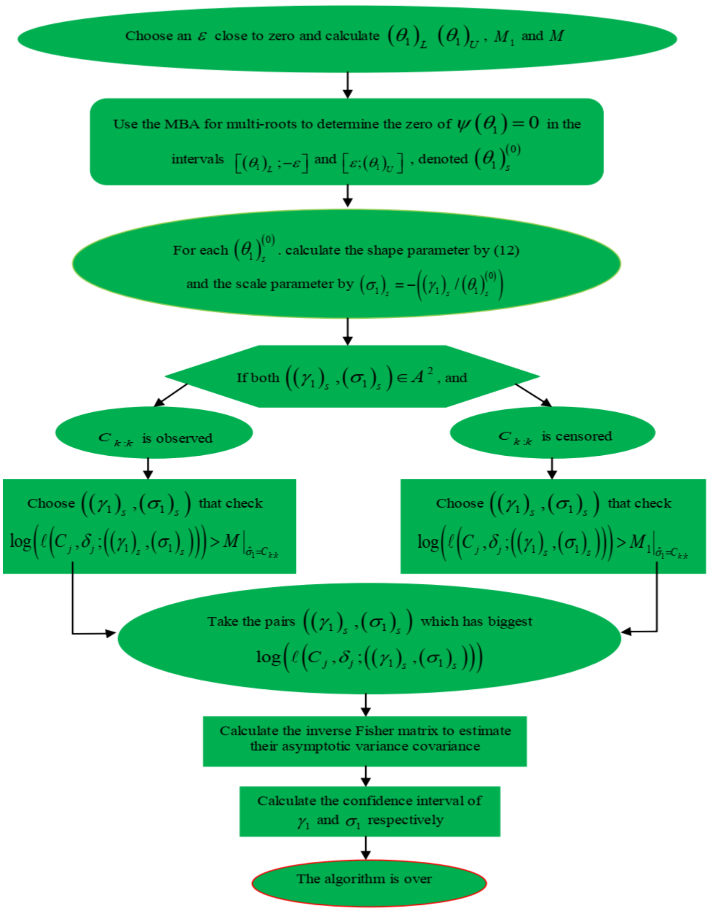

FIGURE 1. Algorithm for the GPD maximum likelihood estimates under censored data Process.

4. Simulation and Illustrative Example

4.1. **Simulation . Example 1**

A simulation study is carried out, using the generalized Pareto distribution under censored data. We generate X_i sample of size 15 which follows the GPD distribution with parameters $\gamma_1 = 0.5$ and $\sigma_1 = 25$. A second sample is indeed generated Y_i with the same size as the first and with parameters $\gamma_2 = 0.8$, $\sigma_2 = 20$. The 15 value are ranked in the following table:

36.330	4.903	0.078	70.547	5.377
13.065	16.349	4.321	11.689	6.322
0.226	6.552	22.814	6.004	58.550

Table 1. Z-sample of 15 size generated with the GPD parameters under censoring

The observed data $r = 9$ (the data were uncensored at different times 36.330, 4.903, 5.377, 13.065, 4.321, 6.322, 0.226, 22.814 and 58.550 respectively), and the censored data $m = 6$ with $Z_{15,15}$ is censored.

When the data is uncensored, our algorithm produces gives GPD maximum likelihood estimates (based on full data), such that $r = n = 15$ and $m = 0$, (In this situation, the algorithm is similar to the classic MLE of GPD given in [10]). Then, the extreme value index which based on $(Z_n; n \geq 1)$ is $\gamma := \left(\frac{\gamma_1\gamma_2}{\gamma_2+\gamma_1}\right) = 0.30769$ with $\theta = -(\gamma/\sigma)$.

The boundaries are calculated $\varepsilon = 3.42 \times 10^{-9}$ and $\theta_L = -6.81 \times 10^2$, $\theta_U = 1.42 \times 10^{-2}$, to apply MBA for multi-roots on the intervals $[\varepsilon; \theta_U]$and $[\theta_L; -\varepsilon]$, where $\psi(\theta)$ given in (13). Then, we find $\theta_1^{(0)} = -0.024207$ and $\theta_2^{(0)} = 0.013466$.

Noted $Z_j = C_j$ We determine which values correspond to $\theta_1^{(0)}$ and $\theta_2^{(0)}$. The following table illustrates these results:

$\theta_1^{(0)} = -0.024207$	$\gamma_1^* = 0.3049$	$\sigma_1^* = 12.5956$	$\log(\ell(C_j; 1; (\gamma_1^*, \sigma_1^*))) = -57.5737$
$\theta_2^{(0)} = 0.013466$	$\gamma_2^* = -0.44494$	$\sigma_2^* = 33.04180$	$\log(\ell(C_j; 1; (\gamma_2^*, \sigma_2^*))) = -60.79249$

Table 2. An account of the values $(\gamma_1^*; \sigma_1^*)$ and $(\gamma_2^*; \sigma_2^*)$.

Now, we have $\gamma_1^* > 0$, $\sigma_1^* > 0$ and $\sigma_2^*/\gamma_2^* < -70.457$, where $Z_{15,15} := 70.457$ the local maximum of the GPD log-likelihood into A. The boundary maximum for $\widehat{\gamma} = -1$ and $\widehat{\sigma} = 70.457$ given by $\log(\ell(C_j; 1; (-1, 70.457))) := -63.84419$, in (a) or (b), where $m = 0$.

So if we consider $\log(\ell(C_j; 1; (\gamma_1^*, \sigma_1^*))) > \log(\ell(C_j; 1; (\gamma_2^*, \sigma_2^*)))$, then the GPD maximum likelihood estimates obtained for uncensored data are: $\widehat{\gamma}^{ML} = 0.3049$, $\widehat{\sigma}^{ML} = 12.5956$ and $\widehat{\theta}^{ML} = -0.024207$.

The algorithm given in section(3) uses $\varepsilon = 3.42 \times 10^{-9}$ and the bounds $(\theta_1)_L = -6.81 \times 10^2$ where $(\theta_1)_U = 1.42 \times 10^{-2}$. It will search for either two root of $\psi(\theta_1)$ given in (11) on the intervals $[\varepsilon; (\theta_1)_U]$ and $[(\theta_1)_L; -\varepsilon]$ respectively.

On the interval $[(\theta_1)_L; -\varepsilon]$ using the MBA for multi-roots, as explained in section (3) converges to $\widehat{\theta}_1^{(c,KIB)} = -0.0241$. Then, the root $\widehat{\theta}_1^{(c,KIB)}$ correspond to the estimated

$\widehat{\gamma}_1^{(c,KIB)} = 0.506$ and $\widehat{\sigma}_1^{(c,KIB)} = 21.014$ with $\log\left(\ell\left(C_j;\left(\widehat{\gamma}_1^{(c,KIB)},\widehat{\sigma}_1^{(c,KIB)}\right)\right)\right) = -39.108$. Then, we find $\widehat{\gamma}_1^{(c,KIB)} > 0$ and $\widehat{\sigma}_1^{(c,KIB)} > 0$ the local maximum of the GPD log-likelihood under censoring into A.

As $Z_{n,n}$ is failure time we get the boundary maximum $\widehat{\gamma}_1^{(c,KIB)} = -1$ and $\widehat{\sigma}_1^{(c,KIB)} = 70.457$ given in (b) by: $\log\left(\ell\left(C_j;(-1,70.457)\right)\right) := -40.136$, but on the interval $[\varepsilon;(\theta_1)_U]$, so no zero of $\psi(\theta_1)$ exists. By the way, for this particular censored data, the MLE for GPD is: $\widehat{\gamma}_1^{(c,KIB)} = 0.506$, $\widehat{\sigma}_1^{(c,KIB)} = 21.014$ and $\widehat{\theta}_1^{(c,KIB)} = -0.0241$. And their corresponding 95% confidence intervals of γ_1 and σ_1 exist when $-1.08 < \gamma_1 < 2.09$ with $-2.08 < \sigma_1 < 44.09$.

In other side, we have $\widehat{\gamma}_1^{(c,KIB)} = 0.506$ and the adaptive ML estimator:

$$\widehat{\gamma}_1^{(c,ML)} = \frac{\widehat{\gamma}^{(ML)}}{\widehat{p}} = 0.50817 \quad where \quad \widehat{\gamma}^{ML} = 0.3049 \quad and \quad \widehat{p} = \frac{9}{15},$$

then, we obtain $\frac{\widehat{\sigma}^{ML}}{\widehat{\sigma}_1^{(c,KIB)}} = 8.99909 \simeq 9$ which is the number of observed data given in this simulation study. To compare and evaluate the performance of two distinct estimation methods for determining the extreme value index, a simulation study is carried out based on bias and RMSE of γ_1 where $n = \left\{15^{th}, 50^{th}, 100^{th}, 200^{th}, 500^{th}\right\}$. The outcomes of utilizing the GPD method with censored data are reported in table (3).

				$\sigma_1 = 25, \gamma_2 = 0.8, \sigma_2 = 25$			
γ_1	n	$\widehat{r}$	Methods	$\widehat{\gamma}_1^{(c,.)}$	Bias (γ_1)	MSE (γ_1)	RMSE (γ_1)
0.5	15	9	Adaptive ML	0.508	0.008	0.000128	0.01131371
			KIB-Estimator	0.506	0.006	7.2×10^{-5}	0.008485281
	50	30	Adaptive ML	0.97	0.47	0.4418	0.6646804
			KIB-Estimator	0.499	-0.001	2×10^{-6}	0.001414214
	100	53	Adaptive ML	0.696	0.196	0.076832	0.2771859
			KIB-Estimator	0.488	-0.012	0.000288	0.01697056
	200	98	Adaptive ML	0.757	0.257	0.132098	0.3634529
			KIB-Estimator	0.508	0.008	0.000128	0.01131371
	500	251	Adaptive ML	0.641	0.114	0.025992	0.1612203
			KIB-Estimator	0.51	0.01	2×10^{-4}	0.01414214

TABLE 3. Bias and RMSE of extreme value index estimates using the GPD method.

The focus of this part of simulation is to compare $\widehat{\gamma}_1^{(c,KIB)}$ with $\widehat{\gamma}_1^{(c,ML)}$. We note that the bias and RMSE of $\widehat{\gamma}_1^{(c,KIB)}$ are lower than $\widehat{\gamma}_1^{(c,ML)}$. Because the amount of data points observed is also estimated , $\widehat{\gamma}_1^{(c,KIB)}$ is a virtual estimator that is both efficient and robust. These results show that our method provides a robust tailed behavior estimator under random censoring.

4.2. **Example 2.** We apply the extreme value model, with Peak-Over-Threshold (POT). We take the tensile-strength fiber data which is given in [10]. The data is gathered from tensile-strength testing for a random value that exceeds the threshold $C_1, C_1, \ldots, C_{15}$ of 15 nylon carpet fibers, listed in an increasing order in table (4), (Only the fact that we have not yet given us the test's threshold value is known due to proprietary considerations).

0.011	0.030	0.051	0.056	0.092
0.100	0.140	0.184	0.200	0.286
0.338	0.365	0.518	0.561	0.876

Table 4. A Random Sample of $n = 15$ Nylon Carpet Fibers.

In his study, he found that the GPD (MLE) were $\widehat{\gamma}^{ML} = -0.1176979$ and also $\widehat{\sigma}^{ML} = 0.283040$ for no censoring data $(r = k = 15)$.

Considering now a censored individual sample (a right censored data with test cessation after the 13^{th} test unit failure). We employ our algorithm, which produces GPD maximum likelihood estimates from censored data $\widehat{\gamma}_1^{(c,KIB)} = -0.214426$ and $\widehat{\sigma}_1^{(c,KIB)} = 0.350239$.

Their corresponding 95% confidence intervals are $-0.933661 < \gamma_1 < 0.504809$ and $0.0537193 < \sigma_1 < 0.6467594$. So, if we compare them with those were reported in [10], we find that our interval estimation still covering as [10]'s results. The P-P interaction graph of C-sample with GPD$\left(\widehat{\gamma}_1^{(c,KIB)}, \widehat{\sigma}_1^{(c,KIB)}\right)$, and their histogram are shown in the following figures respectively.

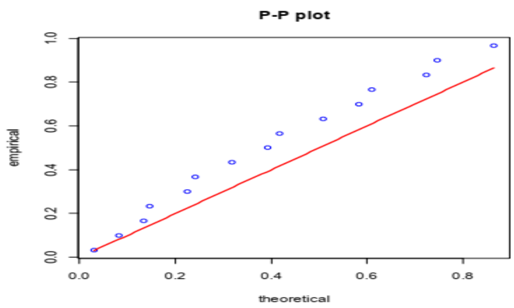

Figure 2. P-P plot of Exceedance (in kg/mm2) of the Testing Threshold in Tensile-Strength Tests for a Random Sample of $n = 15$ Nylon with GPD $\left(\widehat{\gamma}_1^{(c,KIB)}, \widehat{\sigma}_1^{(c,KIB)}\right)$.

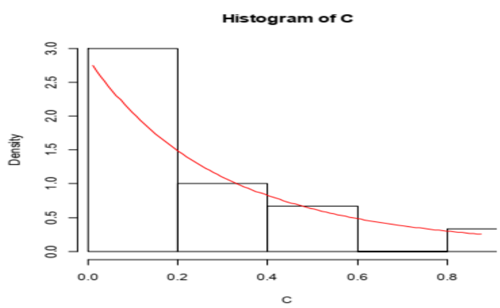

Figure 3. Histogram of Exceedance (in kg/mm2) of the Testing Threshold in Tensile-Strength Tests for a Random Sample of $n = 15$ Nylon with GPD $\left(\widehat{\gamma}_1^{(c,KIB)}, \widehat{\sigma}_1^{(c,KIB)}\right)$.

5. Conclusion

In this study, we propose a new algorithm for MLE of the GPD parameters (γ_1, σ_1) with CDF given in (1) and shape parameter $\gamma_1 \geq -1$ to estimate the extreme value index under censored data, based on [10]. Hence, we can use our MLE of the GPD under censored data with $\gamma_1 \leq 1$ because the aim of this present study is to reduce the compressing of counts. Therefore, our algorithm is satisfied without a sign of γ_1 (i.e $\gamma_1 \leq 1$ or $\gamma_1 \geq -1$). Furthermore, we define a new tail behavior estimator under random censoring based on ML method by our MLE algorithm denoted $\widehat{\gamma}_1^{(c,KIB)}$.

Since the proportion of non-censored observation in the k largest Z's is estimated in (2) (denoted $\widehat{p}$), we find that $\widehat{\gamma}_1^{(c,KIB)}$ is not adapted estimator (see table(3)). However, is adapted if $\widehat{\theta}_1^{(c)} := \widehat{\theta}$, in this case we get the adaptive estimator of the two GPD parameters (γ_1, σ_1) under censored data (Section(4)).

As an outcome, in future research, we investigate the properties of this adaptive estimator when the data is censored. This is the following research subject, and the idea has indeed been established in a new paper which is in the final stages.

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Fatah Benatia for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.13, N.3.