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# ON THE COEFFICIENT BOUNDS FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS BY USING SĂLĂGEAN DIFFERENTIAL OPERATORS

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Abstract. Making use of Sălăgean differential operator, in this paper, we introduce and investigate an interesting subclass $S_{\Sigma}^{h,p}(k,\lambda)$ of bi-univalent functions in the open unit disk $\mathbb{U}$. Furthermore, we find estimates on the $|a_2|$ and $|a_3|$ coefficients for functions in this subclass. The results presented in this paper would generalize and improve some recent works.

Keywords: bi-univalent functions, coefficient estimates, univalent functions, Sălăgean differential operator.

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## 1. Introduction

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also we let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.
Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*[\alpha]$ of starlike functions of order $\alpha$ in $\mathbb{U}$ and the class $\mathcal{K}[\alpha]$ of convex functions of order $\alpha$ in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*[\alpha] = \left\{ f : f \in \mathcal{S},\ \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{U},\ 0 \le \alpha < 1 \right\}$$

and

$$\mathcal{K}[\alpha] = \left\{ f : f \in \mathcal{S},\ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathbb{U},\ 0 \le \alpha < 1 \right\}$$

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The Koebe one-quarter Theorem [6] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdot. \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$ (see[13]).

Let $\Sigma$ denote the class of bi-univalent functions defined in $\mathbb{U}$ given by (1).

Examples of functions in the class $\Sigma$ are

$$\frac{z}{1-z},\ -\log(1-z),\ \frac{1}{2}\log(\frac{1+z}{1-z}),$$

and so on. However, the familiar Koebe function $(\frac{z}{(1-z)^2})$ is not a member of $\Sigma$. Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of $\Sigma$.

Lewin [8] investigated the class $\Sigma$ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to $\Sigma$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [10], on the other hand, showed that $\max_{f\in\Sigma}|a_2| = \frac{4}{3}$.

Various subclasses of the bi-univalent functions class $\Sigma$ were introduced and non-sharp estimates on the first two coefficient $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) were found in several recent investigations (see, for example, [1, 3, 12, 13, 17, 19]).

In 1983, Sălăgean [11] introduced differential operator $D^k : \mathcal{A} \to \mathcal{A}$ defined by

$$D^0f(z) = f(z),$$

$$D^1f(z) = Df(z) = zf'(z),$$

$$D^kf(z) = D(D^{k-1})f(z) = z(D^{k-1}f(z))',\ k = 1, 2, 3, ....$$

We note that

$$D^kf(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n,\ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Jothibasu [7] introduced the following two subclasses of the bi-univalent function class $\Sigma$ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

**Definition 1.1.** *(see [7]) Let $0 \leq \alpha < 1, 0 \leq \lambda < 1$ and $k \in \mathbb{N}_0$. A function $f(z)$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma}^{k,\lambda}(\alpha)$ if the following conditions are satisfied:*

$$f \in \Sigma \text{ and } \left|arg\left(\frac{D^{k+1}f(z)}{(1-\lambda)D^kf(z) + \lambda D^{k+1}f(z)}\right)\right| < \frac{\alpha\pi}{2} \ (z \in \mathbb{U}),$$

*and*

$$\left|arg\left(\frac{D^{k+1}g(w)}{(1-\lambda)D^kg(w)+\lambda D^{k+1}g(w)}\right)\right| < \frac{\alpha\pi}{2}\ (w\in\mathbb{U}),$$

*where the function $g$ is given by* (2).

**Remark 1.1.** *Taking $\lambda = 0$ in the class $\mathcal{S}_{\Sigma}^{k,\lambda}(\alpha)$, we have $\mathcal{S}_{\Sigma}^{k,0}(\alpha) = \mathcal{S}_{\Sigma}^{k}(\alpha)$ and $f \in \mathcal{S}_{\Sigma}^{k}(\alpha)$ if the following conditions are satisfied:*

$$f\in\Sigma \text{ and } \left|arg\left(\frac{D^{k+1}f(z)}{D^kf(z)}\right)\right| < \frac{\alpha\pi}{2}\ (0\le\alpha<1, z\in\mathbb{U}),$$

*and*

$$\left|arg\left(\frac{D^{k+1}g(w)}{D^kg(w)}\right)\right| < \frac{\alpha\pi}{2}\ (0\le\alpha<1, w\in\mathbb{U}),$$

*where the function $g$ is given by* (2).

We note that for $k = 0$ and $\lambda = 0$ the class $\mathcal{S}_{\Sigma}^{0,0}(\alpha) = \mathcal{S}_{\Sigma}^{*}[\alpha]$ is class of strongly bi-starlike functions of order $\alpha(0 \le \alpha < 1)$ which defined as following.

**Definition 1.2.** *( see* [13]*) Let $0 \le \alpha < 1$. A function $f(z)$ given by* (1) *is said to be in the class $\mathcal{S}_{\Sigma}^{*}[\alpha]$ if the following conditions are satisfied:*

$$f\in\Sigma \text{ and } \left|arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}\ (z\in\mathbb{U}),$$

*and*

$$\left|arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}\ (w\in\mathbb{U}),$$

*where the function $g$ is given by* (2).

When $k = 1$ and $\lambda = 0$ the class $\mathcal{S}_{\Sigma}^{1,0}(\alpha) = \mathcal{K}_{\Sigma}[\alpha]$ is class of strongly bi-convex functions of order $\alpha(0 \le \alpha < 1)$ which defined as following.

**Definition 1.3.** *( see* [13]*) Let $0 \le \alpha < 1$. A function $f(z)$ given by* (1) *is said to be in the class $\mathcal{K}_{\Sigma}[\alpha]$ if the following conditions are satisfied:*

$$f\in\Sigma \text{ and } \left|arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}\ (z\in\mathbb{U}),$$

*and*

$$\left|arg\left(1+\frac{wg''(w)}{g'(w)}\right)\right| < \frac{\alpha\pi}{2}\ (w\in\mathbb{U}),$$

*where the function $g$ is given by* (2).

**Theorem 1.1.** *( see* [7]*) Let $f(z)$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{k,\lambda}(\alpha)$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha(1-\lambda)3^k+[2\alpha(\lambda^2-1)-(\alpha-1)(1-\lambda)^2]2^{2k}}},$$

*and*

$$|a_3| \le \frac{\alpha}{3^k(1-\lambda)} + \frac{4\alpha^2}{2^{2k}(1-\lambda)^2}.$$

**Definition 1.4.** *( see [7]) Let $0 \leq \beta < 1, 0 \leq \lambda < 1$ and $k \in \mathbb{N}_0$. A function $f(z)$ given by (1) is said to be in the class $\mathcal{M}_{\Sigma}^{k,\lambda}(\alpha)$ if the following conditions are satisfied:*

$$f \in \Sigma \text{ and } \Re\left(\frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z) + \lambda D^{k+1}f(z)}\right) > \beta \ (z \in \mathbb{U}),$$

*and*

$$\Re\left(\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w) + \lambda D^{k+1}g(w)}\right) > \beta \ (w \in \mathbb{U}),$$

*where the function $g$ is given by (2).*

**Remark 1.2.** *Taking $\lambda = 0$ in the class $\mathcal{M}_{\Sigma}^{k,\lambda}(\beta)$, we have $\mathcal{M}_{\Sigma}^{k,0}(\beta) = \mathcal{M}_{\Sigma}^{k}(\beta)$ and $f \in \mathcal{M}_{\Sigma}^{k}(\beta)$ if the following conditions are satisfied:*

$$f \in \Sigma \text{ and } \Re\left(\frac{D^{k+1}f(z)}{D^k f(z)}\right) > \beta \ (z \in \mathbb{U}),$$

*and*

$$\Re\left(\frac{D^{k+1}g(w)}{D^k g(w)}\right) > \beta \ (w \in \mathbb{U}),$$

*where the function $g$ is given by (2).*

We note that for $k = 0$ and $\lambda = 0$ the class $\mathcal{M}_{\Sigma}^{0,0}(\beta) = \mathcal{S}_{\Sigma}^{*}(\beta)$ is class of strongly bi-starlike functions of order $\beta(0 \leq \beta < 1)$. When $k = 1$ and $\lambda = 0$ the class $\mathcal{M}_{\Sigma}^{1,0}(\beta) = \mathcal{K}_{\Sigma}(\beta)$ is class of strongly bi-convex functions of order $\beta(0 \leq \beta < 1)$.

**Theorem 1.2.** *( see [7]) Let $f(z)$ given by (1) be in the class $\mathcal{M}_{\Sigma}^{k,\lambda}(\beta)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2^{2k}(\lambda^2-1) + 2(1-\lambda)3^k}},$$

*and*

$$|a_3| \leq \frac{(1-\beta)}{3^k(1-\lambda)} + \frac{4(1-\beta)^2}{2^{2k}(1-\lambda)^2}.$$

The purpose of this paper is to investigate the bi-univalent function class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$ introduced in Definition 2.1 and derive coefficient estimates on the first two Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$. Our results for the bi-univalent function class $f \in \mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$ would generalize and improve some recent works of Jothibasu [7] and Brannan and Taha[3].

## 2. Coefficient bounds for the function class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$

In this section, we introduce and investigate the general subclass $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$.

**Definition 2.1.** *Let the analytic functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that*

$$\min\{\Re(h(z)), \Re(p(z))\} > 0 \ (z \in \mathbb{U}) \text{ and } h(0) = p(0) = 1.$$

*Let $0 \leq \lambda < 1$ and $k \in \mathbb{N}_0$. A function $f \in \mathcal{A}$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma \text{ and } \frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z) + \lambda D^{k+1}f(z)} \in h(\mathbb{U}) \ (z \in \mathbb{U}), \tag{3}$$

*and*

$$\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w)+\lambda D^{k+1}g(w)} \in p(\mathbb{U}) \ (w \in \mathbb{U}), \tag{4}$$

*where the function $g$ is defined by* (2).

**Remark 2.1.** *There are many choices of $h$ and $p$ which would provide interesting subclasses of class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$. For example, If we take*

$$h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\alpha} \ (0 \le \alpha < 1, 0 \le \lambda < 1, z \in \mathbb{U}),$$

*it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f \in \mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$, then*

$$f \in \Sigma \text{ and } \left| arg\left(\frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z)+\lambda D^{k+1}f(z)}\right)\right| < \frac{\alpha\pi}{2} \ (z \in \mathbb{U}),$$

*and*

$$\left| arg\left(\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w)+\lambda D^{k+1}g(w)}\right)\right| < \frac{\alpha\pi}{2} \ (w \in \mathbb{U}),$$

*where the function $g$ is given by* (2).
*If we take*

$$h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z} \ (0 \le \beta < 1, 0 \le \lambda < 1, z \in \mathbb{U}),$$

*then the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f \in \mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$, then*

$$f \in \Sigma \text{ and } \Re\left(\frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z)+\lambda D^{k+1}f(z)}\right) > \beta \ (z \in \mathbb{U}),$$

*and*

$$\Re\left(\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w)+\lambda D^{k+1}g(w)}\right) > \beta \ (w \in \mathbb{U}),$$

*where the function $g$ is given by* (2).

## 3. Coefficient Estimates

Now, we obtain by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$.

**Theorem 3.1.** *Let $f(z)$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$. Then*

$$|a_2| \le \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2^{2k+1}(1-\lambda)^2}}, \sqrt{\frac{|h''(0)|+|p''(0)|}{2|4.3^k(1-\lambda)+2^{2k+1}(\lambda^2-1)|}}\right\}, \tag{5}$$

*and*

$$\begin{aligned} |a_3| \le \min\{&\frac{|h''(0)|+|p''(0)|}{8.3^k(1-\lambda)}+\frac{|h'(0)|^2+|p'(0)|^2}{2^{k+1}(1-\lambda)^2}, \\ &\frac{|h''(0)|+|p''(0)|}{8.3^k(1-\lambda)}+\frac{|h''(0)|+|p''(0)|}{2|4.3^k(1-\lambda)+2^{k+1}(\lambda^2-1)|}\}. \end{aligned} \tag{6}$$

*Proof.* First of all, we write the argument inequalities in (3) and (4) in their equivalent forms as follows:

$$\frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z)+\lambda D^{k+1}f(z)} = h(z)\ (z\in\mathbb{U}), \tag{7}$$

and

$$\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w)+\lambda D^{k+1}g(w)} = p(w)\ (w\in\mathbb{U}), \tag{8}$$

respectively, where functions $h$ and $p$ satisfy the conditions of Definition 2.1. Also, the functions $h$ and $p$ have the following Taylor-Maclaurin series expensions:

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdot, \tag{9}$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + p_3 w^3 + \cdot. \tag{10}$$

Now, upon substituting from (9) and (10) into (7) and (8), respectively, and equating the coefficients, we get

$$2^k(1-\lambda)a_2 = h_1, \tag{11}$$

$$2^{2k}(\lambda^2-1)a_2^2 + 3^k(2-2\lambda)a_3 = h_2, \tag{12}$$

$$-2^k(1-\lambda)a_2 = p_1, \tag{13}$$

and

$$2(1-\lambda)(2a_2^2-a_3)3^k + 2^{2k}(\lambda^2-1)a_2^2 = p_2. \tag{14}$$

From (11) and (13), we get

$$h_1 = -p_1, \tag{15}$$

and

$$2^{2k+1}(1-\lambda)^2 a_2^2 = h_1^2 + p_1^2. \tag{16}$$

Adding (12) and (14), we get

$$[4.3^k(1-\lambda) + 2^{k+1}(\lambda^2-1)]a_2^2 = p_2 + h_2. \tag{17}$$

Therefore, from (16) and (17), we have

$$a_2^2 = \frac{h_1^2+p_1^2}{2^{k+1}(1-\lambda)^2}, \tag{18}$$

and

$$a_2^2 = \frac{p_2+h_2}{4.3^k(1-\lambda)+2^{k+1}(\lambda^2-1)}, \tag{19}$$

respectively. Therefore, we find from the equations (18) and (19), that

$$|a_2|^2 \le \frac{|h'(0)|^2+|p'(0)|^2}{2^{2k+1}(1-\lambda)^2},$$

and

$$|a_2|^2 \le \frac{|h''(0)|+|p''(0)|}{2|4.3^k(1-\lambda)+2^{k+1}(\lambda^2-1)|},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (5).
Next, in order to find the bound on the coefficient $|a_3|$, by subtracting (14) from (12), we get

$$4.3^k(1-\lambda)a_3 - 4.3^k(1-\lambda)a_2^2 = h_2 - p_2. \tag{20}$$

Upon substituting the value of $a_2^2$ from (18) into (20), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{2^{2k+1}(1-\lambda)^2} + \frac{h_2 - p_2}{4.3^k(1-\lambda)},$$

Therefore, we get

$$|a_3| \le \frac{|h'(0)|^2 + |p'(0)|^2}{2^{2k+1}(1-\lambda)^2} + \frac{|h''(0)| + |p''(0)|}{8.3^k(1-\lambda)}, \tag{21}$$

On the other hand, upon substituting the value of $a_2^2$ from (19) into (20), it follows that

$$a_3 = \frac{(p_2 + h_2)}{4.3^k(1-\lambda) + 2^{2k+1}(\lambda^2 - 1)} + \frac{(h_2 - p_2)}{4.3^k(1-\lambda)},$$

Therefore, we get

$$|a_3| \le \frac{|h''(0)| + |p''(0)|}{8.3^k(1-\lambda)} + \frac{|h''(0)| + |p''(0)|}{2|4.3^k(1-\lambda) + 2^{2k+1}(\lambda^2 - 1)|}. \tag{22}$$

So we obtain from (21) and (22) the desired estimate on the coefficient $|a_3|$ as asserted in (6). This completes the proof. □

## 4. Conclusions

By choosing

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (0 \le \alpha < 1,\ z \in \mathbb{U}),$$

in Theorem 3.1, we conclude the following corollary.

**Corollary 4.1.** *Let the function $f$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$. Then*

$$|a_2| \le \min\left\{\frac{\alpha}{2^{k-1}(1-\lambda)}, \frac{\alpha}{\sqrt{|3^k(1-\lambda) + 2^{2k-1}(\lambda^2 - 1)|}}\right\},$$

*and*

$$|a_3| \le \min\left\{\frac{\alpha^2}{3^k(1-\lambda)} + \frac{4\alpha^2}{2^{2k}(1-\lambda)^2}, \frac{\alpha^2}{3^k(1-\lambda)} + \frac{\alpha^2}{|3^k(1-\lambda) + 2^{2k-1}(\lambda^2 - 1)|}\right\}.$$

**Remark 4.1.** *It is easy to see, for the coefficient $|a_3|$, that*

$$\frac{\alpha^2}{3^k(1-\lambda)} + \frac{4\alpha^2}{2^{2k}(1-\lambda)^2} \le \frac{\alpha}{3^k(1-\lambda)} + \frac{4\alpha^2}{2^{2k}(1-\lambda)^2}.$$

*Thus, clearly, Corollary 4.1 is an improvement of Theorem 1.1.*

Taking $\lambda = 0$ and $k = 0$ in Corollary 4.1, we obtain the following corollary.

**Corollary 4.2.** *Let the function $f$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$. Then*

$$|a_2| \le \sqrt{2}\alpha \ (0 \le \alpha < 1),$$

*and*

$$|a_3| \le 3\alpha^2 \ (0 \le \alpha < 1).$$

**Remark 4.2.** *Corollary 4.2 provides an improvement of estimates which obtained by Brannan* [3].

Taking $\lambda = 0$ and $k = 1$ in Corollary 4.1, we have

**Corollary 4.3.** *Let the function $f$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$. Then*

$$|a_2| \leq \alpha \ (0 \leq \alpha < 1),$$

*and*

$$|a_3| \leq \frac{4\alpha^2}{3} \ (0 \leq \alpha < 1).$$

**Remark 4.3.** *Corollary 4.3 provides an refinement of estimates which obtained by Brannan* [3].

By letting

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} \ (0 \leq \beta < 1, \ z \in \mathbb{U}),$$

in Theorem 3.1, we deduce the following corollary.

**Corollary 4.4.** *Let the function $f$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$. Then*

$$|a_2| \leq \min\left\{\frac{(1-\beta)}{2^{k-1}(1-\lambda)}, \sqrt{\frac{2(1-\beta)}{|2.3^k(1-\lambda)+2^{2k}(\lambda^2-1)|}}\right\},$$

*and*

$$|a_3| \leq \min\left\{\frac{(1-\beta)}{3^k(1-\lambda)} + \frac{4(1-\beta)^2}{2^{2k}(1-\lambda)^2}, \frac{(1-\beta)}{3^k(1-\lambda)} + \frac{(1-\beta)}{|3^k(1-\lambda)+2^{2k-1}(\lambda^2-1)|}\right\}.$$

Taking $\lambda = 0$ and $k = 0$ in Corollary 4.4, we get

**Corollary 4.5.** *Let the function $f$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$. Then*

$$|a_2 - \rho a_{m+1}^2| \leq \begin{cases} \sqrt{2(1-\beta)}, \ 0 \leq \beta \leq \frac{1}{2} \\ 2(1-\beta); \ \frac{1}{2} \leq \beta < 1. \end{cases}$$

*and*

$$|a_3 - \rho a_{m+1}^2| \leq \begin{cases} 3(1-\beta); \ 0 \leq \beta \leq \frac{1}{2} \\ (1-\beta) + 4(1-\beta)^2; \ \frac{1}{2} \leq \beta < 1. \end{cases}$$

**Remark 4.4.** *Corollary 4.5 provides an improvement of estimates which obtained by Brannan* [3].

Taking $\lambda = 0$ and $k = 1$ in Corollary 4.4, we have

**Corollary 4.6.** *Let the function $f$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$. Then*

$$|a_2| \leq (1-\beta) \ (0 \leq \beta < 1).$$

*and*

$$|a_3| \leq \frac{(1-\beta)}{3} + (1-\beta)^2 \ (0 \leq \beta < 1).$$

**Remark 4.5.** *Corollary 4.6 provides an refinement of estimates which obtained by Brannan* [3].

**Suggestions for future study**: The subordination property is interesting and recently studied by authors. I think the class which defined by subordination will be considered for research.

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