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FIXED POINT RESULTS IN A_b-METRIC SPACE

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Abstract. We derive some fixed point results in A_b- metric space in which map need not be continuous. In addition, we find fixed point and common fixed theorems having rational expressions in the contractive condition. Our results extend and improve various results from the current existing literature. Also, we provide examples.

Keywords: Fixed point, A_b-metric space, discontinuity, S-metric space.

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1. Introduction and Preliminaries

Many authors generalized Banach contraction principle via using different forms of contractive conditions in various generalized metric spaces ([1], [2], [3], [4], [9], [10], [11], [12], [15], [20]). Such generalizations are established via contractive conditions formulated by rational terms (see, [17], [18], [19]).
In 2012, A three dimensional metric space was introduced by Sedghi et al. [5], and it is called S-metric space, which is defined by modifying D-metric and G-metric spaces.

Definition 1.1. [5] *Let X be a nonempty set, An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(1) $S(x, y, z) \geq 0$;
(2) $S(x, y, z) = 0$ *if and only if* $x = y = z$;
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then (X, S) is called S-metric space.

After that, Kim et al. [8] derived common fixed point theorem for two single-valued mappings in S-metric spaces. In the same year (2016), Nizar and Nabil [6] introduced the concept of S_b- metric space which is a combination of b-metric space and S-metric space.

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Definition 1.2. *Let X be a nonempty set and let $s \geq 1$, An S_b-metric on X is a function $S_b : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(1) $S_b(x, y, z) = 0$ *if and only if* $x = y = z$;
(2) $S_b(x, x, y) = S_b(y, y, x)$;
(3) $S_b(x, y, z) \leq s[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$.

Then (X, S) is called S_b-metric space.

Remark 1.1. *Note that, the class of S_b -metric space is larger than the class of S-metric spaces. In fact, every S-metric space is an S_b-metric space with $s = 1$. However, the converse is not always true.*

In definition of S_b-metric space, condition(2) is not true in general. In order to make a general one, Y. Rohen et al. [7] modified as follows:

Definition 1.3. [7] *Let X be a nonempty set and let $s \geq 1$, An S_b-metric on X is a function $S_b : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(1) $S_b(x, y, z) = 0$ *if and only if* $x = y = z$;
(2) $S_b(x, y, z) \leq s[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$.

Then (X, S) is called S_b-metric space.

Recently Mustafa et al. [20] introduced S_p-metric space, at which he replace constant s by one variable continuous increasing function in the definition (1.3) and derived some results in that space. In 2015, Abbas et al. [13] introduced the notion of A-metric space which is a generalization of S-metric space and defined it as follows:

Definition 1.4. *Let X be a nonempty set and the function $A : X^n \to [0, \infty)$ that satisfies the following conditions, for all $x_1, x_2, ..., x_n, a \in X$,*

(1) $A(x_1, x_2, ..., x_n) \geq 0$;
(2) $A(x_1, x_2, ..., x_n) = 0$ *if and only if* $x_1 = x_2 = ... = x_n$;
(3) $A(x_1, x_2, ..., x_n) \leq$
$A(x_1, x_1, .., x_{1_{(n-1)}}, a) + A(x_2, x_2, .., x_{2_{(n-1)}}, a) + \ldots + A(x_n, x_n, ...x_{n_{(n-1)}}, a)$.

Then (X, A) is called an A-metric space.

Example 1.1. *Let $X = R$ and $A(x_1, x_2, x_3, ..., x_n) = |x_1 - x_n| + |x_2 - x_n| + ... + |x_{n-1} - x_n|$. Then, (X, A) is an A-metric space.*

Manoj et al. [14] introduced the A_b-metric space, which is a combination of A-metric space and S_b-metric space and defined as below:

Definition 1.5. *Let X be a nonempty set and let $s \geq 1$, the function $A_b : X^n \to [0, \infty)$ that satisfies, for all $x_1, x_2, ..., x_n, a \in X$,*

(1) $A_b(x_1, x_2, ..., x_n) \geq 0$;
(2) $A_b(x_1, x_2, ..., x_n) = 0$ *if and only if* $x_1 = x_2 = ... = x_n$;
(3) $A_b(x_1, x_2, ..., x_n) \leq$
$s[A_b(x_1, x_1, .., x_{1_{(n-1)}}, a) + A_b(x_2, x_2, .., x_{2_{(n-1)}}, a) + \ldots + A_b(x_n, ..., x_{n_{(n-1)}}, a)]$.

Then (X, A_b) is called an A_b-metric space.

Remark 1.2. (i) *S_b-metric space is the particular case of A_b-metric space with $n = 3$.*
(ii) *Every A-metric space will be A_b-metric space with $s = 1$. However the converse need not be true.*

Example 1.2. [14] *Let* $X = [1, \infty)$ *and* $A_b : X^n \to [0, \infty)$ *defined by*

$$A_b(x_1, x_2, ..., x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2 \ ; \forall x_i \in X, \ i = 1, 2, ..., n$$

Then, (X, A_b) *is an* A_b*-metric space with* $s = 2$.

Example 1.3. [14] *Let* $X = R$ *and* $A_b : X^n \to [0, \infty)$ *defined by*

$$\begin{aligned} A_b(x_1, x_2, ..., x_{n-1}, x_n) &= |\sum_{i=n}^{2} x_i - (n-1)x_1|^2 + |\sum_{i=n}^{3} x_i - (n-2)x_2|^2 + \ldots \\ &\quad + |\sum_{i=n}^{n-3} x_i - 3x_{n-3}|^2 + |\sum_{i=n}^{n-2} x_i - 2x_{n-2}|^2 + |x_n - x_{n-1}|^2 \end{aligned}$$

for all $x_i \in X, \ i = 1, 2, ..., n$. *Then,* (X, A_b) *is an* A_b*-metric space with* $s = 2$.

Example 1.4. *Let* $X = R$ *and* $A_b(x_1, x_2, ..., x_n) = |x_1 - x_n|^2 + |x_2 - x_n|^2 + \ldots + |x_{n-1} - x_n|^2$.

Proof.

$$\begin{aligned} A_b(x_1, x_2, ..., x_n) &= |x_1 - x_n|^2 + |x_2 - x_n|^2 + \ldots + |x_{n-1} - x_n|^2 \\ &\leq 2\{|x_1 - a|^2 + |x_n - a|^2\} + \ldots + 2\{|x_{n-1} - a|^2 + |x_n - a|^2\} \\ &\leq 2(n-1)\{|x_1 - a|^2 + |x_2 - a|^2 + \ldots + |x_n - a|^2\} \\ &= 2[A_b(x_1, x_1, ..., a) + A_b(x_2, x_2, ..., a) + \ldots + A_b(x_n, x_n, ..., a)] \end{aligned}$$

Hence, (X, A_b) is an A_b-metric space with $s = 2$. □

Lemma 1.1. [14] *Let* (X, A_b) *be an* A_b*-metric space with* $s \geq 1$. *Then for all* $x, y \in X$,

$$A_b(x, x, ..., x, y) \leq sA_b(y, y, ..., y, x).$$

The concepts of convergence, Cauchy sequence and completeness in an A_b-metric space are defined in similar manner.

Definition 1.6. [14] *Let* (X, A_b) *be an* A_b*-metric space and* $\{x_n\}$ *be a sequence in* X. *Then*

(i) *A sequence* $\{x_n\}$ *is called convergent if and only if there exists* $u \in X$ *such that* $A_b(x_n, x_n, ..., x_n, u) \to 0$ *as* $n \to \infty$. *So, we can write* $\lim_{n\to\infty} x_n = u$.
(ii) *A sequence* $\{x_n\}$ *is called a Cauchy sequence if and only if* $\lim_{n\to\infty} A_b(x_n, x_n, ..., x_n, x_m) \to 0$.
(iii) (X, A_b) *is said to be a complete* A_b*- metric space if every Cauchy sequence* $\{x_n\}$ *is convergent to a point* $u \in X$.

In our last result, The following lemma will be helpful to mange discontinuity of the A_b-metric space.

Lemma 1.2. *Let* (X, A_b) *be an* A_b*-metric space with* $s \geq 1$, *then we have the following:*

(i) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $x_n \to x, y_n \to y$ *and the elements of* $\{x, y, x_n, y_n : n \in \mathbb{N}\}$ *are totally distinct. Then, we have*

$$\begin{aligned} s^{-2}A_b(x, x, ..., x, y) \leq \lim_{n\to\infty} \inf A_b(x_n, x_n, ..., x_n, y_n) &\leq \lim_{n\to\infty} \sup A_b(x_n, x_n, ..., x_n, y_n) \\ &\leq s^2 A_b(x, x, ..., x, y). \end{aligned}$$

(ii) *Let $\{x_n\}$ be a Cauchy sequence in X converging to x. If x_n has infinitely many distinct terms, then*

$$\begin{aligned} s^{-2}A_b(x,x,...,x,y)) \leq \lim_{n\to\infty}\inf A_b(x_n,x_n,...,x_n,y) &\leq \lim_{n\to\infty}\sup A_b(x_n,x_n,...,x_n,y) \\ &\leq s^2A_b(x,x,...,x,y), \end{aligned}$$

for all $y \in X$ with $x \neq y$.

Proof. (i) Using the second condition from the definition of A_b-metric space, we have

$$\begin{aligned} A_b(\underbrace{x,x,...,x}_{\text{(N-1) terms}},y) &\leq s[(N-1)A_b(x,x,...,x,x_n)+A_b(y,y,...,y,x_n)] \\ &\leq s(N-1)A_b(x,x,...,x,x_n) \\ &\quad +s^2[(N-1)A_b(y,y,...,y,y_n)+A_b(x_n,x_n,...,x_n,y_n)], \end{aligned}$$

And

$$\begin{aligned} A_b(x_n,x_n,...,x_n,y_n) &\leq s[(N-1)A_b(x_n,x_n,...,x_n,x))+A_b(y_n,y_n,...,y_n,x)] \\ &\leq s(N-1)A_b(x_n,x_n,...,x_n,x) \\ &\quad +s^2[(N-1)A_b(y_n,...,y_n,y)+A_b(x,...,x,y)]. \end{aligned}$$

Taking the lower limit as $n \to \infty$ in the first inequality and upper limit in the second inequality, we obtain our required result.

(ii) Using the second condition from the definition of A_b-metric space, we get

$$\begin{aligned} A_b(x,x,...,x,y) &\leq s[(N-1)A_b(x,x,...,x,x_n)+A_b(y,y,...,y,x_n)] \\ &\leq s(N-1)A_b(x,x,...,x,x_n) \\ &\quad +s^2[(N-1)A_b(y,y,...,y,y)+A_b(x_n,x_n,...,x_n,y)]. \end{aligned}$$

That is

$$A_b(x,x,...,x,y) \leq s[(N-1)A_b(x,x,...,x,x_n)+sA_b(x_n,x_n,...,x_n,y)], \quad (1)$$

And

$$A_b(x_n,x_n,...,x_n,y) \leq s[(N-1)A_b(x_n,x_n,...,x_n,x))+A_b(y,y,...,y,x)]. \quad (2)$$

From lemma 1.1, one obtain

$$A_b(x_n,x_n,...,x_n,y) \leq s[(N-1)A_b(x_n,x_n,...,x_n,x))+sA_b(x,x,...,x,y)]$$

. Taking the lower limit as $n \to \infty$ in the equation (1) and upper limit in the equation (2), we obtain desired result.

□

Manoj ughade et al. [14] proved the following theorem for continuous map.

Theorem 1.1. *Let (X, A_b) be a complete A_b-metric space. Let f be a continuous self map satisfying the following:*

$$A_b(fx^1, fx^2, ..., fx^n) \leq \psi(A_b(x^1, x^2, ..., x^n)),$$

for all $x^1, x^2, ..., x^n \in X$, where $\psi : [0, +\infty) \to [0, +\infty)$ is an increasing function such that $\lim_{k\to\infty} \psi^k(t) = 0$, for each fixed $t > 0$. Then f has a unique fixed point in X.

Motivated by [14], in this paper we present an important results of A_b-metric space and then we obtain Banach type contraction principle and Kannan type fixed point theorem as corollaries. At last, we derive fixed point theorem having rational terms, which is the answer to the open problem given in [16]. Moreover, we find common fixed point theorems for four maps which involving rational terms. Our results extend and generalize several results from the existing literature especially the results of Manoj et al. [14]. In addition, we provide examples for the justification of our results.

2. Main results

Let us recall the following definitions.

Definition 2.1. [23] *Let X be a non-empty set and $T_1, T_2 : X \to X$. If $w = T_1x = T_2x$ for some $x \in X$, then x is called a coincidence point of T_1 and T_2, and w is called a point of coincidence of T_1 and T_2.*

Definition 2.2. [21] *Let X be a non-empty set and $T_1, T_2 : X \to X$. The pair $\{T_1, T_2\}$ is said to be weakly compatible if $T_1T_2t = T_2T_1t$, whenever $T_1t = T_2t$ for some t in X.*

Now, we start with the main result of our paper in which map need not be continuous. Also note that throughout the paper, $N \geq 2$.

Theorem 2.1. *Let (X, A_b) be a complete A_b-metric space with $s \geq 1$. Let T be a self map satisfying the following:*

$$\begin{aligned}&A_b(Tu_1, Tu_2, ..., Tu_N)\\&\leq \lambda_1[A_b(u_1, u_1, ..., Tu_1) + A_b(u_2, u_2, ..., Tu_2) + ... + A_b(u_N, u_N, ..., Tu_N)]\\&+\lambda_2 A_b(u_1, u_2, ..., u_N)\\&+\lambda_3[A_b(u_1, u_1, ..., Tu_2) + A_b(u_2, u_2, ..., Tu_3) + ... + A_b(u_N, u_N, ..., Tu_1)],\end{aligned} \tag{3}$$

for all $u_1, u_2, ..., u_N \in X$, where λ_1, λ_2 and λ_3 are non negative real numbers such that $0 < \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < 1$; and, $\alpha = s(N-1)^2 + 1, \beta = s(N-1), \gamma = s(N-1)(N-2) + s^2(N-1)^2 + s$. Then T has a unique fixed point in X.

Note: Here $\alpha \geq 2, \beta \geq 1, \gamma \geq 2$; for any value of s and N. So, according to the value of α, β, γ ; one can choose λ_1, λ_2 and λ_3 such that $0 < \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < 1$.

Proof. Let us define sequence $\{y_n\}$ as $Ty_n = y_{n+1}$. From definition of A_b-metric space and for $n > m$, we have

$$\begin{aligned}A_b(\underbrace{y_n, y_n, ..., y_n}_{\text{(N-1) terms}}, y_m) \leq\; & s^2 A_b(\underbrace{y_m, y_m, ..., y_m}_{\text{(N-1) terms}}, y_{m+1})\\&+s^3(N-1)A_b(\underbrace{y_{m+1}, y_{m+1}, ..., y_{m+1}}_{\text{(N-1) terms}}, y_{m+2})\\&+s^4(N-1)^2A_b(\underbrace{y_{m+2}, y_{m+2}, ..., y_{m+2}}_{\text{(N-1) terms}}, y_{m+3})\\&+s^5(N-1)^3A_b(\underbrace{y_{m+3}, y_{m+3}, ..., y_{m+3}}_{\text{(N-1) terms}}, y_{m+4})\\&+s^5(N-1)^4A_b(\underbrace{y_n, y_n, ..., y_n}_{\text{(N-1) terms}}, y_{m+4}).\end{aligned}$$

Continuing in similar way, we obtain

$$\begin{aligned} A_b(y_n, y_n, ..., y_m) \leq\ & s^2 A_b(y_m, y_m, ..., y_{m+1}) \\ & + s^3(N-1)A_b(y_{m+1}, y_{m+1}, ..., y_{m+2}) \\ & + s^4(N-1)^2 A_b(y_{m+2}, y_{m+2}, ..., y_{m+3}) \\ & + s^5(N-1)^3 A_b(y_{m+3}, y_{m+3}, ..., y_{m+4}) \\ & + s^6(N-1)^4[A_b(y_{m+4}, ..., y_{m+5}) + (N-1)A_b(y_n, y_n, ..., y_{m+4})]. \end{aligned}$$

We arrive at

$$A_b(\underbrace{y_n, y_n, ..., y_n}_{\text{(N-1) terms}}, y_m) \leq \sum_{i=1}^{n-m} S^{i+1}(N-1)^{i-1} A_b(\underbrace{y_{m+i-1}, y_{m+i-1}, ..., y_{m+i-1}}_{\text{(N-1) terms}}, y_{m+i}). \tag{4}$$

Again using definition of A_b-metric space and the contractive condition, one gets

$$\begin{aligned} & A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty_{n-1}, Ty_n) \\ & \leq \lambda_1[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_n, y_n, ..., y_n, y_{n+1})] \\ & + \lambda_2 A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) \\ & + \lambda_3[\{(N-2) + s(N-1)\}A_b(y_{n-1}, ..., y_{n-1}, y_n) + sA_b(y_n, ..., y_n, y_{n+1})]. \end{aligned}$$

That implies

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \frac{\lambda_1(N-1) + \lambda_2 + \lambda_3[N-2+s(N-1)]}{1-\lambda_1-\lambda_3 s} A_b(y_{n-1}, ..., y_{n-1}, y_n).$$

That is

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \mu\ A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n), \tag{5}$$

where, $\mu = \frac{\lambda_1(N-1)+\lambda_2+\lambda_3[N-2+s(N-1)]}{1-\lambda_1-\lambda_3 s}$.
Then, the equation (5) results in

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \mu^n A_b(y_0, y_0, ..., y_0, y_1).$$

This yields the follows by the use of equation(4),

$$A_b(y_n, y_n, ..., y_m) \leq \sum_{i=1}^{n-m} s^{i+1}(N-1)^{i-1}\mu^{m+i-1} A_b(y_0, y_0, ..., y_1). \tag{6}$$

Let , $a_i = s^{i+1}(N-1)^{i-1}\mu^{m+i-1}$. Then,

$$\lim_{i\to\infty} \frac{a_i}{a_{i+1}} = \frac{1}{s(N-1)\mu}$$

Since, $\alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < 1$; where, $\alpha = s(N-1)^2 + 1$, $\beta = s(N-1)$,
$\gamma = s(N-1)(N-2) + s^2(N-1)^2 + s$.
We have

$$\begin{aligned} & \{s(N-1)^2+1\}\lambda_1 + s(N-1)\lambda_2 + \{s(N-1)(N-2) + s^2(N-1)^2 + s\}\lambda_3 < 1 \\ \implies & s(N-1)^2\lambda_1 + s(N-1)\lambda_2 + \{s(N-1)(N-2) + s^2(N-1)^2\}\lambda_3 < 1 - \lambda_1 - \lambda_3 s \\ \implies & (N-1)\lambda_1 + \lambda_2 + \{(N-2) + s(N-1)\}\lambda_3 < \frac{1-\lambda_1-\lambda_3 s}{s(N-1)} \\ \implies & \frac{(N-1)\lambda_1 + \lambda_2 + \{(N-2) + s(N-1)\}\lambda_3}{1-\lambda_1-\lambda_3 s} < \frac{1}{s(N-1)} \\ \implies & \mu < \frac{1}{s(N-1)} \end{aligned}$$

This yields, $\lim_{i\to\infty} \frac{a_i}{a_{i+1}} > 1$. Therefore utilizing the ratio test, $\sum a_i$ is convergent.
So, from the equation(6), we conclude that

$$\lim_{n,m\to\infty} A_b(y_n, y_n, ..., y_m) \to 0.$$

Hence, $\{y_n\}$ is a Cauchy sequence. With the use of completeness, one gets

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0.$$

Again by contractive condition, one finds that

$$\begin{aligned} A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty) \leq\ & \lambda_1[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y, y, ..., Ty)] \\ & +\lambda_2[A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y)] \\ & +\lambda_3[(N-2)A_b(y_{n-1}, y_{n-1}, ..., y_n) + A_b(y_{n-1}, y_{n-1}, ..., Ty)] \\ & +\lambda_3 A_b(y, y, ..., y_n) \end{aligned}$$

Taking limit both sides, we arrive at $(1 - \lambda_1 - \lambda_3)A_b(y, y, ..., Ty) \leq 0$.
Here, $1 - \lambda_1 - \lambda_3 > 0$ as $0 < \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < 1$. Hence, $y = Ty$.
Suppose $y*$ is another fixed point of T, then one can come across

$$A_b(Ty*, Ty*, ..., Ty) \leq (\lambda_2 + \lambda_3)A_b(y*, y*, ..., y*, y) + \lambda_3 A_b(y, y, ..., y*).$$

With the use of lemma (1.1), we have

$$(1 - \lambda_2 - (1+s)\lambda_3)A_b(y*, y*, ..., y) \leq 0. \quad (7)$$

If $(1 - \lambda_2 - (1+s)\lambda_3) < 0$, then $1 < \lambda_2 + (1+s)\lambda_3$. That implies

$$\alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 < \lambda_2 + (1+s)\lambda_3,$$

which is not possible for any value of s and N. So, $(1 - \lambda_2 - (1+s)\lambda_3) > 0$.
Therefore from (7), we have

$$A_b(y*, y*, ..., y) = 0.$$

Hence, y is the unique fixed point of T in X. □

If we put $\lambda_2 = 0$ and $\lambda_3 = 0$ in the previous theorem (2.1), then we have Kannan theorem as corollary.

Corollary 2.1. *Let (X, A_b) be a complete A_b-metric space. Let T be a self map satisfying the following:*

$$A_b(Tu_1, Tu_2, ..., Tu_N)]) \leq \lambda[A_b(u_1, u_1, ..., Tu_1) + A_b(u_2, ..., Tu_2) + ... + A_b(u_N, u_N, ..., Tu_N)] \quad (8)$$

$\forall u_1, u_2, ..., u_n \in X$ and $0 < \lambda < \frac{1}{1+s(N-1)^2}$. Then T has a unique fixed point in X.

Proof. Here $max\ \{\frac{1}{1+s(N-1)^2}\} = \frac{1}{2}$.
Let us define sequence $\{y_n\}$ as $Ty_n = y_{n+1}$. With the same process adopted in previous theorems, one observes that $\{y_n\}$ is a Cauchy sequence and hence convergent to y(say), So, finally one can write

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0.$$

Now our claim is to prove that y is the fixed point of T. So, by contractive condition, one gets

$$A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty) \leq \lambda(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_n) + \lambda A_b(y, y, ..., y, Ty).$$

Letting limit $n \to \infty$, we have

$$(1 - \lambda)A_b(y, y, ..., y, Ty) \leq 0.$$

Since, $\lambda < \frac{1}{2}$. Therefore, $A_b(y, y, ..., y, Ty) = 0$.
That is $y = Ty$. It is easy to prove that, T has a unique fixed point in X. □

Example 2.1. *Let $X = R - (-1, 1) \cup \{0\} \cup \{\frac{1}{6}\}$ and $A_b(x_1, x_2, x_3, x_4) = |x_1 - x_4|^2 + |x_2 - x_4|^2 + |x_3 - x_4|^2$. Then, it is clear from example (1.4) that (X, A_b) is an A_b-metric space with $s = 2$. Define $T : X \to X$ by*

$$T(x) = \begin{cases} \frac{1}{6} & ; if\ x = 0, \frac{1}{6} \\ 0, & ; otherwise \end{cases}$$

Here, T is discontinuous at $\{\frac{1}{6}\}$ and 0. Now, we have following possibilities:

Case 1: $x_i = \frac{1}{6}$ *or* 0 *; $i = 1, 2, 3, 4$*
we have $A_b(Tx_1, Tx_2, Tx_3, Tx_4) = 0$. It is trivially true.
Case 2: $x_i \neq \frac{1}{6}$ *and* 0 *; $i = 1, 2, 3, 4$. It is trivially true.*
Case 3: $x_i \neq \frac{1}{6}, 0$ *; $i = 1, 2, 3$ and $x_4 = 0$*
$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(0, 0, 0, \frac{1}{6}) = \frac{3}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 3|x_1|^2$,
$A_b(x_2, x_2, x_2, Tx_2) = 3|x_2|^2, A_b(x_3, x_3, x_3, Tx_3) = 3|x_3|^2$ $A_b(x_4, x_4, x_4, Tx_4) = \frac{3}{36}$
From contractive condition, one observe that

$$\frac{3}{36} \leq 3\lambda\{|x_1|^2 + |x_2|^2 + |x_3|^2 + \frac{1}{36}\}.$$

Case 4: $x_i \neq \frac{1}{6}, 0$ *; $i = 1, 2$ and $x_3 = 0, x_4 = 0$*
$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(0, 0, \frac{1}{6}, \frac{1}{6}) = \frac{2}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 3|x_1|^2$,
$A_b(x_2, x_2, x_2, Tx_2) = 3|x_2|^2$, $A_b(x_3, x_3, x_3, Tx_3) = \frac{3}{36}, A_b(x_4, x_4, x_4, Tx_4) = \frac{3}{36}$
From contractive condition, one gets

$$\frac{3}{36} \leq 3\lambda\{|x_1|^2 + |x_2|^2 + \frac{1}{36} + \frac{1}{36}\}.$$

Case 5: $x_1 \neq \frac{1}{6}, 0$ *and $x_i = 0$; $i = 2, 3, 4$*
$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) = \frac{1}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 3|x_1|^2$,
$A_b(x_2, x_2, x_2, Tx_2) = \frac{3}{36}$ $A_b(x_3, x_3, x_3, Tx_3) = \frac{3}{36}, A_b(x_4, x_4, x_4, Tx_4) = \frac{3}{36}$
This implies

$$\frac{3}{36} \leq 3\lambda\{|x_1|^2 + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}\}.$$

Case 6: $x_1 = \frac{1}{6}$ *and $x_i \neq 0, \{\frac{1}{6}\}$; $i = 2, 3, 4$*
$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(\frac{1}{6}, 0, 0, 0) = \frac{1}{36}$, $A_b(x_1, x_1, x_1, Tx_1) = 0$,
$A_b(x_2, x_2, x_2, Tx_2) = 3|x_2|^2$ $A_b(x_3, x_3, x_3, Tx_3) = 3|x_3|^2, A_b(x_4, x_4, x_4, Tx_4) = 3|x_4|^2$
We arrive at

$$\frac{1}{36} \leq 3\lambda\{0 + |x_2|^2 + |x_3|^2 + |x_4|^2\}.$$

Case 7: $x_1 = \frac{1}{6}, x_2 = \frac{1}{6}$ *and $x_i \neq 0, \{\frac{1}{6}\}$; $i = 3, 4$*
$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(\frac{1}{6}, \frac{1}{6}, 0, 0) = \frac{2}{36}$ $A_b(x_1, x_1, x_1, Tx_1) = 0$,
$A_b(x_2, x_2, x_2, Tx_2) = 0$ $A_b(x_3, x_3, x_3, Tx_3) = 3|x_3|^2, A_b(x_4, x_4, x_4, Tx_4) = 3|x_4|^2$
We have

$$\frac{2}{36} \leq 3\lambda\{|x_3|^2 + |x_4|^2\}.$$

Case 8: $x_1, x_2, x_3 = \frac{1}{6}$ *and $x_4 \neq 0, \{\frac{1}{6}\}$.*
$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = A_b(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0) = \frac{3}{36}$ $A_b(x_1, x_1, x_1, Tx_1) = 0$,

$A_b(x_2, x_2, x_2, Tx_2) = 0 \ A_b(x_3, x_3, x_3, Tx_3) = 0, A_b(x_4, x_4, x_4, Tx_4) = 3|x_4|^2$
So,

$$\frac{2}{36} \leq 3\lambda\{|x_4|^2\}.$$

So, for $0 < \lambda < \frac{1}{19}$, all the above cases are satisfied. Hence, all the condition required in Theorem (2.1) are satisfied. Thus, $\frac{1}{6}$ is the unique fixed point of T in X at which map is discontinuous.

Remark 2.1. *In next results, Continuity of the A_b-metric space is not necessary.*

If we put $\lambda_1 = 0$ and $\lambda_2 = 0$ in the theorem (2.1), we have following result.

Corollary 2.2. *Let (X, A_b) be a complete A_b-metric space with $s \geq 1$. Let T be a self map satisfying the following:*

$$A_b(Tu_1, Tu_2, ..., Tu_N) \leq \lambda[A_b(u_1, u_1, ..., Tu_2)+A_b(u_2, u_2, ..., Tu_3)+...+A_b(u_N, u_N, ..., Tu_1)], \tag{9}$$

for all $u_1, u_2, ..., u_N \in X$, where, $0 < \lambda < \frac{1}{s[1+(N-1)\{(N-2)+(N-1)s\}]}$. Then T has a unique fixed point in X.

Proof. With the same process adopted in previous theorem, one observes that $\{y_n\}$ is a Cauchy sequence and hence convergent to y(say), So, one can write

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0.$$

From contractive condition, one finds that

$$A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty) \leq \lambda(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_n) + \lambda A_b(y, y, ..., y, y_n).$$

Which together with (1.1), we have

$$A_b(y_n, y_n, ..., Ty) \leq \lambda(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_n) + \lambda s A_b(y_n, y_n, ..., y_n, y).$$

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., Ty) = 0.$$

Therefore, $\{y_n\}$ converges to both y and Ty. It must be the case $y = Ty$. Suppose $y*$ is another fixed point of T, then

$$A_b(Ty*, Ty*, ..., Ty) \leq \lambda[A_b(y*, y*, ..., y*, y) + sA_b(y*, y*, ..., y*, y)].$$

$$[1-(1+s)\lambda]A_b(y*, y*, ..., y*, y) \leq 0. \tag{10}$$

If $[1-(1+s)\lambda] < 0$, then

$$\frac{1}{1+s} < \lambda < \frac{1}{s[1+(N-1)\{(N-2)+(N-1)s\}]},$$

which is not possible for any $N \geq 2$. That means $[1-(1+s)\lambda] > 0$. Hence, from equation (10), It easy to say that, y is the unique fixed point of T in X. □

If we put $\lambda_1 = 0$ and $\lambda_3 = 0$ in the theorem 2.1, then the theorem turns into Banach type contractive condition. In a similar way, one can easily prove the following.

Corollary 2.3. *Let (X, A_b) be a complete A_b-metric space. Let T be a self map satisfying the following:*

$$A_b(Tu_1, Tu_2, ..., Tu_N) \leq \lambda A_b(u_1, u_2, ..., u_N) \ ; \forall u_1, u_2, ..., u_N \in X. \tag{11}$$

where, $0 < \lambda < \frac{1}{s(N-1)}$. Then T has a unique fixed point.

Proof. Here *max* $\{\frac{1}{s(N-1)}\} = 1$. □

Example 2.2. *Let* $X = [0, \infty)$ *and* $A_b(x_1, x_2, x_3, x_4) = (max\{x_1, x_2, x_3\} - x_4)^2$. *It is clear that* (X, A_b) *is an* A_b*-metric space with* $s = 2$. *Also define* $T : X \to X$ *by* $T(x) = \frac{x}{4}$. *We have*

$$A_b(Tx_1, Tx_2, Tx_3, Tx_4) = (max\{\frac{x_1}{4}, \frac{x_2}{4}, \frac{x_3}{4}\} - \frac{x_4}{4})^2$$

and

$$A_b(x_1, x_2, x_3, x_4) = (max\{x_1, x_2, x_3\} - x_4)^2$$

So, for $\frac{1}{16} \leq \lambda < \frac{1}{6}$ *, the equation(11) is satisfied. Hence,* 0 *is the unique fixed point of* T *in* X.

In the sequel, we have following result which gives the answer to the open problem given in paper by G. S. Saluja [16].

Theorem 2.2. *Let* (X, A_b) *be a complete* A_b*-metric space. Let* T *be a self map satisfying the following:*

$$A_b(Tu_1, Tu_2, ..., Tu_N) \leq \lambda \cdot \frac{M^*}{N^*} \quad ; \forall u_1, u_2, ..., u_N \in X,$$

where,

$$M^* = [A_b(u_1, u_1, ..., Tu_1) + A_b(u_2, u_2, ..., Tu_2) + ... + A_b(u_N, u_N, ..., Tu_N)]A_b(u_1, u_2, ..., u_N),$$

$$N^* = [A_b(u_1, u_1, ..., Tu_2) + ... + A_b(u_{N-2}, u_{N-2}, ..., Tu_{N-1})] + A_b(u_1, u_2, ..., u_N) + A_b(Tu_1, Tu_2, ..., Tu_N),$$

$0 < \lambda < \frac{1}{s(N-1)}$ *and* $N^* \neq 0$. *Then* T *has a unique fixed point in* X.

Proof. Let us define sequence $\{y_n\}$ as $Ty_n = y_{n+1}$. Using definition of A_b-metric space and the contractive condition, one can see that

$$A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty_{n-1}, Ty_n) \leq \lambda \cdot \{\frac{[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_n, y_n, ..., y_n, y_{n+1})]A_b(y_{n-1}, ..., y_{n-1}, y_n)}{(N-2)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_n, ..., y_n, y_{n+1})}\}$$

yielding there by

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \lambda^n A_b(y_0, y_0, ..., y_0, y_1).$$

With the same process adopted in previous theorems, one observes that $\{y_n\}$ is a Cauchy sequence and hence convergent to y(say). That is $\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0$
Again using contractive condition, we have

$$A_b(Ty_{n-1}, Ty_{n-1}, ..., Ty_{n-1}, Ty) \leq \lambda\{\frac{[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y, y, ..., y, Ty)]A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y)}{(N-2)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y) + A_b(y_n, y_n, ..., y_n, Ty)}\}.$$

That is

$$A_b(y_n, y_n, ..., y_n, Ty) \leq \lambda\{\frac{[(N-1)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y, y, ..., y, Ty)]A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y)}{(N-2)A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n) + A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y) + A_b(y_n, y_n, ..., y_n, Ty)}\}.$$

Taking limit both sides , we conclude that $A_b(y, y, ..., Ty) = 0$. Hence $y = Ty$. It is easy to check that, T has a unique fixed point in X. □

At last, we find common fixed point theorem for the setting of four maps.

Theorem 2.3. *Let (X, A_b) be a complete A_b-metric space with coefficient $s \geq 1$. Let S, T, A and B be self mappings of X such that $TX \subseteq AX, SX \subseteq BX$ and*

$$A_b(Su_1, Su_2, .., Su_{N-1}, Tu_N) \leq \frac{\lambda A_b(Au_1, Au_2, .., Au_{N-1}, Bu_N) + \mu A_b(Au_1, Au_2, ..., Au_{N-1}, Su_1)A_b(Bu_N, ..., Bu_N, Tu_N)}{1 + A_b(Au_1, Au_2, ..., Au_{N-1}, Bu_N)}$$

;for all $u_1, u_2, ..., u_N \in X$ where, $0 < s(N-1)\lambda + \mu < 1$. If one of the ranges AX or BX is a closed subset of (X, A_b), then

(i) *A and S have a coincidence point.*
(ii) *B and T have a coincidence point.*

Moreover, if the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then A, B, T and S have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $TX \subseteq AX$, there exists $x_1 \in X$ such that $Ax_1 = Tx_0$, and $SX \subseteq BX$, there exists $x_2 \in X$ such that $Bx_2 = Sx_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Ax_{2n+1} = Tx_{2n},\ y_{2n+1} = Bx_{2n+2} = Sx_{2n+1}\ \forall n \in N$$

Using contractive condition, one can see that

$$A_b(Sx_{2n+1}, Sx_{2n+1}, ..., Sx_{2n+1}, Tx_{2n+2}) \leq \frac{\lambda A_b(Ax_{2n+1}, ..., Ax_{2n+1}, Bx_{2n+2}) + \mu \cdot l^*}{1 + A_b(Ax_{2n+1}, Ax_{2n+1}, ..., Ax_{2n+1}, Bx_{2n+2})}.$$

where,

$$l^* = A_b(Ax_{2n+1}, ..., Ax_{2n+1}, Sx_{2n+1})A_b(Bx_{2n+2}, Bx_{2n+2}, ..., Tx_{2n+2})$$

Which yields

$$A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}) \leq \frac{\lambda A_b(y_{2n}, y_{2n}, ..., y_{2n}, y_{2n+1}) + \mu \cdot m^*}{1 + A_b(y_{2n}, y_{2n}, ..., y_{2n}, y_{2n+1})},$$

where,

$$m^* = A_b(y_{2n}, y_{2n}, ..., y_{2n}, y_{2n+1})A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}),$$

Which results in

$$A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}) \leq \lambda A_b(y_{2n}, ..., y_{2n}, y_{2n+1}) + \mu A_b(y_{2n+1}, ..., y_{2n+1}, y_{2n+2}).$$

We arrive at

$$A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}) \leq \frac{\lambda}{1-\mu} A_b(y_{2n}, y_{2n}, ..., y_{2n}, y_{2n+1}). \tag{12}$$

Similarly, one gets

$$A_b(y_{2n+2}, y_{2n+2}, ..., y_{2n+2}, y_{2n+3}) \leq \frac{\lambda}{1-\mu} A_b(y_{2n+1}, y_{2n+1}, ..., y_{2n+1}, y_{2n+2}). \tag{13}$$

Therefore, from (12) and (13),

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq \frac{\lambda}{1-\mu} A_b(y_{n-1}, y_{n-1}, ..., y_{n-1}, y_n).$$

Likewise,

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq [\frac{\lambda}{1-\mu}]^2 A_b(y_{n-2}, y_{n-2}, ..., y_{n-2}, y_{n-1}).$$

Continuing this process, one arrives at

$$A_b(y_n, y_n, ..., y_n, y_{n+1}) \leq [\frac{\lambda}{1-\mu}]^n A_b(y_0, y_0, ..., y_0, y_1).$$

Say, $h = \frac{\lambda}{1-\mu}$ and with the use of the equation (4),

$$A_b(y_n, y_n, ..., y_m) \leq \sum_{i=1}^{n-m} s^{i+1}(N-1)^{i-1}h^{m+i-1}A_b(y_0, y_0, ..., y_1). \tag{14}$$

Let, $a_i = s^{i+1}(N-1)^{i-1}h^{m+i-1}$. Then

$$\lim_{i\to\infty} \frac{a_i}{a_{i+1}} = \frac{1}{s(N-1)h}$$

Since, $s(N-1)\lambda + \mu < 1$. We have $\lim_{i\to\infty} \frac{a_i}{a_{i+1}} > 1$. By Ratio test, $\{y_n\}$ is a Cauchy sequence and hence convergent to y(say). That means

$$\lim_{n\to\infty} A_b(y_n, y_n, ..., y) = 0.$$

Thus, one finds that

$$\lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Bx_{2n+2} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Ax_{2n+1} = y. \tag{15}$$

Now without loss of generality, one can suppose that AX is a closed subset of (X, A_b). From the equation (15), there exists $z \in X$ such that $y = Az$. Employing the definition of A_b-metric space, we have

$$A_b(Sz, Sz, ..., Sz, y) \leq s[(N-1)A_b(Sz, Sz, ..., Sz, Tx_{2n}) + A_b(y, y, ..., y, Tx_{2n})]$$

From the contractive condition,

$$\begin{aligned} &A_b(Sz, Sz, ..., Sz, y) \\ &\leq s(N-1)\{\frac{\lambda A_b(Az, ..., Az, Bx_{2n}) + \mu A_b(Az, ..., Az, Sz)A_b(Bx_{2n}, ..., Bx_{2n}, Tx_{2n})}{1 + A_b(Az, ..., Az, Bx_{2n})}\} \\ &+sA_b(y, y, ..., y, Tx_{2n}). \end{aligned}$$

From lemma 1.2 (ii) and letting limit supremum both sides, one obtains

$$A_b(Sz, Sz, ..., Sz, y) = 0.$$

That is, $y = Sz = Az$. Since, $SX \subseteq BX$, there exists $w \in X$ such that $Bw = y$.
Again using contractive condition, one arrives at

$$A_b(y, y, ..., y, Tw) = A_b(Sz, Sz, ...Sz, Tw) = 0.$$

Thus, $y = Tw = Bw = Sz = Az$. That is, A and S have coincidence point z and B and T have coincidence point w.
Let, A and S are weakly compatible, So we have

$$Ay = ASz = SAz = Sy.$$

Now our claim is to prove $Sy = y$.

$$\begin{aligned} A_b(Sy, Sy, ..., Sy, y) &= A_b(Sy, Sy, ..., Sy, Tw) \\ &\leq \frac{\lambda A_b(Ay, ..., Ay, Bw) + \mu A_b(Ay, ..., Ay, Sy)A_b(Bw, ..., Bw, Tw)}{1 + A_b(Ay, ..., Ay, Bw)}, \end{aligned}$$

which results in

$$A_b(Sy, Sy, ..., Sy, y) = 0.$$

Therefore, $Sy = y = Ay$. Similarly, B and T are weakly compatible, one concludes that $Ty = y = By$. Finally, we have $Sy = Ay = Ty = By = y$. So, y is the common fixed point of S, T, A and B. It is easy to check that y is the unique common fixed point. □

3. Future work

It is observed that, fixed point and common fixed point results in A_b-metric space can be derive for Wardowski F-contraction for discontinuous map and even without taking continuity of the metric space.

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