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CRITICAL POINT RESULTS AND THEIR APPLICATIONS

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ABSTRACT. The purpose of this paper is to study the concept of a critical point in complete cone metric spaces and its application in Ekeland type variational principle. For this, we consider the concept of a λ-space, which is weaker than a cone metric space in general. Actually, we rectify some critical point results in λ-spaces, and complete cone metric spaces. Indeed, we try to correct some gaps in some definitions and main results of the previous works and apply them in the set valued case. Moreover, we give an improved version of Ekeland type variational principle in complete cone metric spaces.

Keywords: Cone metric space, λ-space, Critical point, Ekeland Type Variational Principle.

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1. INTRODUCTION

The Ekeland variational principle (EVP) [1], as one of the most important results in nonlinear analysis, has many applications in the geometry of Banach spaces, optimization, variational inequalities, game theory, optimal control theory and other related issues; see, for example, [2]-[8], and references therein. The relation between EVP and the famous Bishop-Phelps theorem (BPT) is studied in [9], which plays an essential role in optimization problems. Dancs et al. [10] presented a critical point theorem in complete metric spaces and proved EVP. Alleche and Radulescu [4] presented the equilibrium version of EVP ($EEVP$) in complete metric spaces. Ansari [2] extended $EEVP$ for vector valued functions in the setting of complete quasimetric spaces with a W-distance. He established some equivalent results to $EEVP$ for vector valued functions and also established Caristi-Kirk fixed point theorem for multivalued maps [11] in a more general setting. Al-Homidan et al. [12] established $EEVP$ in the setting of quasimetric spaces with a Q-function which generalizes the notions of τ-function and a w-distance [8]. They proved some equivalences of $EEVP$ with a fixed point theorem of Caristi-Kirk type for multivalued maps [11] and some other related results. Sabetghadam and Masiha [13]

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defined the concept of generalized φ-pair mappings and proved some common fixed point theorems for this type of mappings. Asadi and Soleimani [14] presented some fixed point theorems for generalized contractions by altering distance functions in a complete cone metric spaces endowed with a partial order. Tavakoli et al. [15] presented some examples in order to show that the imagination of many authors that the behavior of ordering induced by a strongly minihedral cone is just as the behavior of usual ordering on the real line, that has caused an error in their proofs, is not correct. They established a relationship between strong minihedrality and total orderness and, then a fixed point theorem for a contractive mapping is investigated. Bae and Kim [16] used the critical point theorem to establish $EEVP$ for multi-valued bifunctions. They also investigated critical point theorems for the continuously Gateaux differentiable functionals. Khanh and Quy [17] apply their results to establish $EEVP$ for the vector-valued mappings. In this paper, we consider the concepts of λ-function and λ-space, which are respectively weaker than those of cone metric and cone metric space. We rectify some critical point results in λ-spaces, and complete cone metric spaces, improve by the results of Ekeland type variational principles. This article intends not only to rectify some gaps in [18] but also improve the Ekeland type variational principle.

2. Preliminaries

Let E be a topological vector space, a non-empty subset P of E is called a convex cone if $P + P \subseteq P$ and $\lambda P \subseteq P$ for $\lambda \geq 0$. A convex cone P is said to be pointed if $P \cap (-P) = \{0\}$.

Remark 2.1. *The definition of partial order "$\leq$" with respect to P in* [18] *is not logically correct and, which is rectified as follows. Also, to correct the conditions mentioned for real topological vector space E in* [18], *it is necessary to add the condition $intP \neq 0$.*

For a given nontrivial, pointed, closed and convex cone P in E, we define on E a partial order $\leq$ with respect to P by $x \leq y$ iff $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$ (i.e. $y - x \in P \setminus \{0\}$), while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. In the case that E is a normed linear space, the nontrivial, pointed, closed and convex cone P is called normal if there exists a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The least positive number K satisfying the above inequality is called the normal constant of P.
Throughout this note, unless otherwise specified, we always assume that X is a non-empty set, E is a real topological vector space with its zero 0, ordered by a nontrivial, pointed, closed and convex cone P with $intP \neq \emptyset$ and $\lambda : X \times X \to E$ is a vector-valued function. Denote by 2^X the family of all subsets of X.

Definition 2.1. [19] *Let X be a non-empty set. A vector-valued function $d : X \times X \to E$ is said to be a cone metric, if the following conditions hold:*

(d1) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$,
(d2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

The set X with a cone metric d is called a cone metric space and denoted by (X, d).

Definition 2.2. [18] *A vector-valued function $\lambda : X \times X \to E$ is said to be a λ-function if for all $x, y \in X$, $\lambda(x, y) \geq 0$. If $x \neq y$ then $\lambda(x, y) \neq 0$.*
A nonempty set X with a λ-function is called a λ-space, and denoted by (X, λ).

Remark 2.2. *Obviously every cone metric space is a λ-space. The following example shows that the converse is not true in general.*

Example 2.1. *Let $X = E = \mathbb{R}$ and $P = \mathbb{R}_+$. Define $\lambda : X \times X \to E$ by*

$$\lambda(x, y) = \max\{| x |, | y |\}.$$

Then (X, λ) is a λ-space but (X, λ) is not a cone metric space. Because $x = y$ doesn't imply $\lambda(x, y) = 0$ in general.

Remark 2.3. *There are some gaps in the Definition (2.3)* [18]*, which must be corrected as follows in order to be applied in th proof of Theorem 3.1, 3.14, and Remark 4 in* [18]*.*

(i) *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a λ-space (X, λ) is said to be:*
  (a) *λ-Cauchy sequence (resp., quasi-λ-Cauchy sequence) if for every $c \in E$ with $c \gg 0$ (resp., $c > 0$), there exists a positive integer N such that $\lambda(x_n, x_m) \ll c$ (resp., $\lambda(x_n, x_m) < c$) for all $n, m \geq N$,*
  (b) *λ-convergent (resp., quasi-λ-convergent) if there exists $x \in X$ such that for every $c \in E$ with $c \gg 0$ (resp., $c > 0$), there exists a positive integer N such that $\lambda(x_n, x) \ll c$ (resp., $\lambda(x_n, x) < c$) for all $n \geq N$. In this case, we say that $\{x_n\}$ λ-converges (resp., quasi-λ-converges) to x in (X, λ), and we denote it by $x_n \xrightarrow{\lambda} x$ (resp., $x_n \xrightarrow{q-\lambda} x$). The point $x \in X$ is called a λ-limit point (resp., quasi-λ-limit point) of the sequence $\{x_n\}$,*

(ii) *A λ-space (X, λ) is said to be λ-complete (resp., quasi-λ-complete) if every λ-Cauchy sequence (resp., quasi-λ-Cauchy sequence) is a λ-convergent (resp., quasi-λ-convergent) sequence.*

(iii) *A subset D of a λ-space (X, λ) is said to be the following:*
  (a) *λ-closed (resp., quasi-λ-closed) in (X, λ) if for every $x \in X$ with a sequence $\{x_n\} \subset D$ such that $\{x_n\}$ λ-converges (resp., quasi-λ-converges) to x in (X, λ), then $x \in D$; the λ-closure of a set D in(X, λ) is the intersection of all λ-closed sets containing D.*
  (b) *λ-open (resp., quasi-λ-open) in (X, λ) if $D^c = X \setminus D$ (the complement of D in X) is λ-closed (resp., quasi-λ-closed).*

The following example shows that λ-limit (quasi-λ-limit) in the setting of λ-spaces is not unique.

Example 2.2. *Let $E = \mathbb{R}$, $P = \mathbb{R}_+$ and $X = [0, 1]$. Define $\lambda : X \times X \to \mathbb{R}_+$ by*

$$\lambda(x, y) = \begin{cases} y - x & y \geq x, \\ 1 + y - x & y < x \text{ but } (x, y) \neq (1, 0), \\ 1 & (x, y) = (1, 0). \end{cases}$$

Then (X, λ) is a λ-space but not a cone metric space. Consider sequence $\{\frac{n}{n+1}\} \subseteq [0, 1)$ increasing to 1. Thus, both 1 and 0 are λ-limit (quasi-λ-limit) points of the sequence $\{\frac{n}{n+1}\}$. Indeed,

$$\forall c \in \mathbb{R} \text{ with } c \in (0, \infty)\ \exists N \geq 0 \text{ s.t } \forall n \geq N\ ,\ c - \lambda(\frac{n}{n+1}, 0) \in (0, \infty)$$

and

$$\forall c \in \mathbb{R} \text{ with } c \in (0, \infty)\ \exists N \geq 0 \text{ s.t } \forall n \geq N\ ,\ c - \lambda(\frac{n}{n+1}, 1) \in (0, \infty).$$

Therefore, λ-limit (quasi-λ-limit) in a λ-space (X, λ) is not always unique.

We provide an example to show that, unlike metric spaces and cone metric spaces, in λ-spaces if $x \in \overline{A}^{\lambda}$, then there is not necessarily $\{x_n\} \subset A$ such that $x_n \xrightarrow{\lambda} x$.

Example 2.3. *Let $X = E = \mathbb{R}$ and $P = \mathbb{R}_+$. Consider λ-space (X, λ), which*

$$\lambda(x, y) = \max\{| x |, | y |\}.$$

Then, $\frac{1}{2} \in \overline{(0,1)}^{\lambda} = [0, 1)$ but there is not exist a sequence $\{x_n\} \subset (0, 1)$ such that $x_n \xrightarrow{\lambda} \frac{1}{2}$. Since, if $x_n \xrightarrow{\lambda} \frac{1}{2}$ then

$$\forall c \in \mathbb{R} \textit{ with } c \in (0, \infty)\ \exists N \geq 0 \textit{ s.t } \forall n \geq N, c - \max\{|x_n|, \frac{1}{2}\} \in (0, \infty).$$

This leads to a contradiction.

Remark 2.4. *(i) The statement "We replace λ by d in Definition 2.3 (i) and (ii), we obtain the definition of d-Cauchy Sequence and d-convergent sequence in a cone metric space" in Remark 2.6 of* [18], *is not correct. So the corrected definitions of d-Cauchy sequence and d-convergent sequence are following.*

(a) *If (X, d) is a cone metric space, and if we replace λ by d in the definition of λ-Cauchy sequence and λ- convergent sequence, we obtain the definitions of d-Cauchy sequence and d-convergent sequence in a cone metric space and the definition of a d-complete cone metric space in* [19]*, respectively. If there is no danger of confusion, then we will not use d before these definitions.*

(b) *As it is proved in* [19]*, every convergent sequence is a Cauchy sequence in a cone metric space. However, this assertion is not true for a λ-convergent sequence in (X, λ). For example, let $X = E = \mathbb{R}$ and $P = \mathbb{R}_+$. Define $\lambda : X \times X \to E$ by*

$$\lambda(x, y) = \begin{cases} | x - y | & \textit{if } x{=}0 \textit{ or } y{=}0, \\ 1 & \textit{otherwise.} \end{cases}$$

Then, $\{1/n\}$ is a λ-convergent sequence with λ-limit 0, but it is not a λ-Cauchy sequence.

(c) *The family of λ-open (resp., quasi-λ-open) and λ-closed (resp., quasi-λ-closed) sets, makes a topology on X which is weaker than the topology generated by quasi-metric spaces and cone metric spaces.*

(ii) Definition 2.7 in [18] *has some fundamental logical defects. The mentioned definition has been modified as follows. Let A be a non-empty subset of a λ-space (X, λ), $\{A_n\}$ be a sequence of non-empty subsets in (X, λ) and let $\delta : 2^X \to E$ be a map. We adopt the following notations:*

(i) *$\delta(A) < c$ for some $c \in E$ with $c \geq 0$ if and only if $\lambda(x, y) < c$ for all $x, y \in A$.*
(ii) *$\delta(A) \ll c$ for some $c \in E$ with $c \geq 0$ if and only if $\lambda(x, y) \ll c$ for all $x, y \in A$.*
(iii) *$\rho(A) = \sup\{\| \lambda(x, y) \| : x, y \in A\}$ if E is a normed vector space with an ordered cone P.*
(iv) *$\delta(A_n) \xrightarrow{\lambda} 0$ of the first type if for every $c \in E$ with $c > 0$, there exists a positive integer N such that $\delta(A_n) < c$ for all $n \geq N$.*
(v) *$\delta(A_n) \xrightarrow{\lambda} 0$ of the second type if for every $c \in E$ with $c \gg 0$, there exists a positive integer N such that $\delta(A_n) \ll c$ for all $n \geq N$.*
(vi) *$\delta(A_n) \xrightarrow{\lambda} 0$ of the first type w.r.t. $\{y_n\} \subseteq X$ if for every $c \in E$ with $c > 0$, there exists a positive integer N such that for each $n \geq N$, we have $\lambda(y_n, u) < c$ for all $u \in A_n$.*
(vii) *$\delta(A_n) \xrightarrow{\lambda} 0$ of the second type w.r.t. $\{y_n\} \subseteq X$ if for every $c \in E$ with $c \gg 0$, there exists a positive integer N such that for each $n \geq N$, we have $\lambda(y_n, u) \ll c$ for all $u \in A_n$.*

In order to continue, we recall the definitions of τ-functions and weak τ-functions.

Definition 2.3. [21] *Let (X, d) be a metric space. A function $p : X \times X \to \mathbb{R}_+$ is said to be a τ-function if the following conditions hold:*

($\tau1$) *For all $x, y, z \in X$, $p(x, z) \leq p(x, y) + p(y, z)$;*
($\tau2$) *$p(x, .)$ is $\mathbb{R}_+$-lower semicontinuous, for each $x \in X$ (if $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ and $p(x, y_n) \leq M$ for some $M > 0$ then $p(x, y) \leq M$);*
($\tau3$) *For any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, and if there exists a $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;*
($\tau4$) *For $x, y, z \in X$, $p(x, y) = 0$, and $p(x, z) = 0$ imply $y = z$.*

Lemma 2.1. [21] *Let p be a τ-function on $X \times X$. If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

Definition 2.4. [18] *Let (X, d) be a quasi metric space (i.e., symmetricity is not required). A function $p : X \times X \to \mathbb{R}_+$ is said to be a weak τ-function if the conditions ($\tau1$), ($\tau3$), and ($\tau4$) hold.*

Remark 2.5. *The definition of weak τ-functions on a metric space is given in* [17].

Definition 2.5. [10] *Let $F : X \to 2^X$ be a multi-valued map. A point $x \in X$ is said to be a critical point of F if and only if $F(x) = \{x\}$.*

Theorem 2.1. [18] *Let (X, d) be a complete metric space and $F : X \to 2^X$ satisfy the following conditions*

(i) *$F(x)$ is closed and $x \in F(x)$ for all $x \in X$,*
(ii) *for all $x, y \in X$, $y \in F(x)$ implies $F(y) \subseteq F(x)$,*
(iii) *$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ if $x_{n+1} \in F(x_n)$ for all n.*

Then, there exists $\bar{x} \in X$ such that $F(\bar{x}) = \{\bar{x}\}$.

3. Critical Point Theorems

Remark 3.1. *Without correcting the definition of quasi-λ-Cauchy sequence and part(i) of Defintion(2.7) in* [18]*, it wouldn't be possible to prove the following theorem.*

Theorem 3.1. [18] *Let (X, λ) be a quasi-λ-complete space, and let $F : X \to 2^X$ be a multi-valued map with non-empty quasi-λ-closed values. Assume that*

(i) *for all $x, y \in X$, $y \in F(x)$ implies $F(y) \subseteq F(x)$,*
(ii) *for every sequence $\{x_n\}$ with $x_{n+1} \in F(x_n)$, one has $\delta(F(x_n)) \xrightarrow{\lambda} 0$ of the first type.*

Then, for each $\widehat{x} \in X$, there exists $x^* \in F(\widehat{x})$ such that $F(x^*) = \{x^*\}$ and $\lambda(x^*, x^*) = 0$.

Remark 3.2. *When $E = \mathbb{R}$, $P = \mathbb{R}_+$ and (X, λ) is a complete metric space, Theorem 3.1 reduces to Theorem 3.1 in* [10].

Definition 3.1. [18] *For a transitive relation $\Re$ (i.e., $x\Re y$ and $y\Re z$ imply $x\Re z$) in a topological space Y, we say that*

(i) *$\Re$ is lower closed if for any $\Re$-monotone (i.e., $\cdots \Re x_n \Re \cdots \Re x_2 \Re x_1$) convergent sequence $x_n \longrightarrow \overline{x}$ one has $\overline{x} \Re x_n$ for all $n \in \mathbb{N}$,*
(ii) *a subset $A \subseteq Y$ is $\Re$-complete if any Cauchy sequence in A (if the definition of Cauchy sequence is given) which is $\Re$-monotone, converges to a point of A.*

Remark 3.3. *In Theorem 3.1, if (X, λ) is λ-complete then F has λ-closed values. But, if (X, λ) is quasi-λ-complete, F has not necessarily quasi-λ-closed values.*

Lemma 3.1. [18] *Let (X, λ) be a λ-space, and let E be a normed vector space with an ordering cone P. Let $\{A_n\}$ be a sequence of subsets of X such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} \rho(A_n) = 0$ implies $\delta(A_n) \xrightarrow{\lambda} 0$ of the second type in (X, λ). Moreover, the converse holds if P is a normal cone.*

Lemma 3.2. [18] *Let E be a normed vector space ordered by a normal cone P, then the following statements are equivalent.*

(i) *$\{x_n\}$ is a λ-Cauchy sequence in (X, λ).*

(ii) *For every $\varepsilon > 0$, there exists a positive N such that $\| \lambda(x_i, x_j) \| < \varepsilon$ for all $i, j \geq N$.*

Remark 3.4. *Obviously, if E is assumed to be a normed vector space with a normal ordering cone P, (X, λ) is λ-complete (not necessarily quasi-λ-complete) and the values of mapping F is λ-closed (not necessarily quasi-λ-closed). Then, condition (ii) of Theorem 3.1 can be replaced by the following.*

$(ii)''$ *For every sequence $\{x_n\}$ with $x_{n+1} \in F(x_n)$,we have $\delta(F(x_n)) \xrightarrow{\lambda} 0$ of the second type.*

Now, we are ready to establish the corrected version of the critical point theorem in the setting of cone metric spaces. First we present the theorem as it is stated and proved in [18].

Theorem 3.2. [18] *Let (X, d) be a complete cone metric space, E a normed vector space, and $F : X \to 2^X$ a multi-valued map with non-empty closed values. Assume that*

(i) *for all $x, y \in X$, $y \in F(x)$ implies $F(y) \subseteq F(x)$,*

(ii) *for every sequence $\{x_n\}$ with $x_{n+1} \in F(x_n)$, one has $\lim_{n\longrightarrow\infty} \|d(x_n, x_{n+1})\| = 0$.*

Then, for each $\widehat{x} \in X$, there exists $x^* \in F(\widehat{x})$ such that $F(x^*) = \{x^*\}$.

Proof. Without loss of generality, we may assume that for each $x \in X$, $F(x)$ is bounded; that is, $\rho(F(x))$ exists. For any given $\varepsilon > 0$ and any fixed element $\widehat{x} \in X$, let $x_1 = \widehat{x}$ and choose $x_2 \in F(x_1)$ such that

$$\| d(x_1, x_2) \| > \frac{\rho(F(x_1))}{2} - \frac{\varepsilon}{2}. \tag{1}$$

Continuing in this way, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n+1} \in F(x_n)$ and

$$\| d(x_n, x_{n+1}) \| > \frac{\rho(F(x_n))}{2} - \frac{\varepsilon}{2^n} \text{ , } \forall n \in \mathbb{N}. \tag{2}$$

Since $\lim_{n\longrightarrow\infty} \|d(x_n, x_{n+1})\| = 0$ and ε is arbitrary positive number, we have

$$\lim_{n\longrightarrow\infty} \rho(F(x_n)) = 0. \tag{3}$$

By Lemma 3.2, sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, $\{x_n\}$ converges to some $x^* \in X$. □

Remark 3.5. *Since the main space is supposed to be partially ordered and also the notion of boundedness is not given in [18], so it can not be applied to prove the results. Also, for all $n \in \mathbb{N}$, $x_n \notin F(x_n)$, inequalities (1) and (2) are not satisfied; therefore, (3) is not valid. On the other hand, since P is not a normal cone, Lemma 3.2 can not be applied. Also, the limit of sequences in the cone metric spaces is unique, therefore, the above proof is incorrect. Thus, we first define the notion of boundedness from above in a cone metric space. Then, to ensure existence of $\rho(F(x))$, for all $x \in X$, assuming P is a normal cone with K as normal constant and for all $x \in X$, $F(x)$ is bounded above, we have the following proof.*

Definition 3.2. [22] *Let (X, d) be a cone metric space. Then $A \subset X$ is called bounded above if there exists $c \in E$, $c \gg 0$ such that $d(x, y) \leq c$, for all $x, y \in A$.*

What follows is the corrected version of the above theorem.

Proof. Without loss of generality, we may assume that for each $x \in X$, $\rho(F(x))$ exists. For any given $\varepsilon > 0$ and any fixed element $\widehat{x} \in X$, let $x_1 = \widehat{x}$ and choose for all $n \in \mathbb{N}$, $x_{n+1} \in F(x_n)$ such that

$$\begin{aligned} \| d(x_2, x_3) \| &\geq \frac{\rho(F(x_1))}{2} - \frac{\varepsilon}{2} \\ \| d(x_3, x_4) \| &\geq \frac{\rho(F(x_2))}{2} - \frac{\varepsilon}{2^2} \\ &\vdots \\ \| d(x_n, x_{n+1}) \| &\geq \frac{\rho(F(x_{n-1}))}{2} - \frac{\varepsilon}{2^{n-1}}\ , \ \forall n \in \mathbb{N}. \end{aligned} \tag{4}$$

Since $\lim_{n\longrightarrow\infty} \|d(x_n, x_{n+1})\| = 0$ and ε is arbitrary positive number, by taking the limit of both sides (4), we have

$$0 \geq \lim_{n\longrightarrow\infty} \rho(F(x_{n-1})) \Longrightarrow \lim_{n\longrightarrow\infty} \rho(F(x_{n-1})) = 0$$

and since

$$\rho(F(x_n)) = \sup\{\|d(x, y)\| : x, y \in F(x_n)\} \leq \sup\{\|d(x, y)\| : x, y \in F(x_{n-1})\} = \rho(F(x_{n-1})).$$

Limiting from both sides, we have

$$\lim_{n\longrightarrow\infty} \rho(F(x_n)) \leq 0 \Longrightarrow \lim_{n\longrightarrow\infty} \rho(F(x_n)) = 0.$$

By Lemma 3.2, sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, $\{x_n\}$ converges to $x^* \in X$. By hypothesis, $F(x_n)$ is closed and $F(x_{n+1}) \subseteq F(x_n)$ for all $n \in \mathbb{N}$. Then, $x^* \in F(x_n)$ for all $n \in \mathbb{N}$. Since $\lim_{n\longrightarrow\infty} \rho(F(x_n)) = 0$, we have

$$\cap_{n\in\mathbb{N}} F(x_n) = \lim_{n\longrightarrow\infty} F(x_n) = \{x^*\}.$$

Indeed, if there exists $y \in \cap_{n\in\mathbb{N}} F(x_n)$ with $y \neq x^*$. Then,

$$0 <\| d(x^*, y) \|\leq \rho(F(x_n)),\ \forall n \in \mathbb{N} \tag{5}$$

Since $\rho(F(x_n)) \longrightarrow 0$, thus by taking the limit from both sides in (5), we have $\| d(x^*, y) \|= 0$ which this leads to a contradiction. Therefore,

$$\emptyset \neq F(x^*) \subseteq \cap_{n\in\mathbb{N}} F(x_n) = \{x^*\},$$

and hence $F(x^*) = \{x^*\}$. □

Remark 3.6. *If for each $u_n \in \overline{\{x_n\}}$ with $u_{n+1} \in F(u_n)$ there exists $n \in \mathbb{N}$ such that $u_n \in \overline{\{x_n\}}_{n\in\mathbb{N}} \setminus \{x_n\}_{n\in\mathbb{N}}$, it can not be concluded that $\delta(F(u_n)) \xrightarrow{\lambda} 0$ of the second type. In addition, if for all $n \in \mathbb{N}$ the critical point of the map F $(x^* \in X) \neq x_n$ then it is possible $x^* \notin \overline{\{x_n\}}$; hence it is not possible to have $x^* \in \bigcap_{n\in\mathbb{N}} A_n$; so the proof of second part of Theorem (3.12)* [18] *is not true. Moreover, the following example shows that in Theorem (3.12)* [18], *if the multi-valued map F has a critical point in X then (X, λ) is not necessarily λ-complete.*

Example 3.1. *Let $E = \mathbb{R}$, $P = \mathbb{R}_+$ and $X = [0, 1)$. Define $\lambda : X \times X \to E$ by*

$$\lambda(x, y) =\mid x - y \mid,$$

then (X, λ) is a a λ-space. If we define a multi-valued map F on X by

$$F(x) = \begin{cases} [0, x] & if x \neq 0, \\ \{0\} & if x = 0, \end{cases}$$

then, all the conditions of Theorem (3.12) [18] *are satisfied, and there exists a critical point 0 of F in (X, λ) ($F(0) = \{0\}$). In addition, $\{\frac{n}{n+1}\}_{n\in\mathbb{N}}$ is a λ-Cauchy sequence in (X, λ), but since $\frac{n}{n+1} \xrightarrow{\lambda} 1 \notin X$, $\{\frac{n}{n+1}\}_{n\in\mathbb{N}}$ is not λ-converges. Therefore, (X, λ) is not λ-complete.*

Corollary 3.1. [18] *Let (X, d) be a cone metric space, E a normed space ordered by a normal cone P, and $F : X \to 2^X$ a multi-valued map with non-empty closed values in (X, d). Suppose that*

(i) *for all $x, y \in X$, $y \in F(x)$ implies $F(y) \subseteq F(x)$,*
(ii) *for every sequence $\{x_n\}$ with $x_{n+1} \in F(x_n)$,one has $\delta(F(x_n)) \xrightarrow{d} 0$ of the second type in (X, d),*

Then, the multi-valued map F has a critical point in X if and only if $X, d)$ is d-complete.

Remark 3.7. *According to the above explanation and Example 3.1, if F as a multi-valued map has a critical point in X, then X is not necessarily d-complete.*

Now we present another critical point theorem that generalizes the main result in [17].

Remark 3.8. *If we replaced the triangle inequality by the condition (iii) in Theorem 3.14* [18] *and define $\delta(F(x_n)) \xrightarrow{\lambda} 0$ of the first type w.r.t. $\{y_n\} \subseteq X$ if for every $c \in E$ with $c > 0$, there exists a positive integer N such that for each $n \geq N$, we have $\lambda(y_n, u) < c$ and $\lambda(u, y_n) < c$ for all $u \in A_n$, then for each $\widehat{x} \in X$, there exists $x^* \in F(\widehat{x})$ such that $F(x^*) = \{x^*\}$.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be the sequences given by condition (ii). Since $\delta(F(x_n)) \xrightarrow{\lambda} 0$ of the first type w.r.t. $\{y_n\}$ in (X, λ), for every $c \in E$ with $c > 0$, there exists a positive integer N such that for $n \geq N$, we have

$$\lambda(y_n, u) < \frac{c}{2} \quad \text{and} \quad \lambda(x, y_n) < \frac{c}{2} \quad \forall x, u \in F(x_n).$$

According to the triangle inequality, for all $x, y \in F(x_n)$

$$\lambda(x, u) \leq \lambda(x, y_n) + \lambda(y_n, u) < c.$$

Therefore, $\delta(F(x_n)) \xrightarrow{\lambda} 0$ of the first type in (X, λ) and Theorem 3.1 is satisfied. Thus, for each $\widehat{x} \in X$, there exists $x^* \in F(\widehat{x})$ such that $F(x^*) = \{x^*\}$. □

Lemma 3.3. *Let (X, λ) be a λ-space, E a normed space ordered by a normal cone P with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ quasi-λ-converges to x, then $\lambda(x_n, x) \underset{\|\cdot\|_E}{\longrightarrow} 0$ $(n \longrightarrow \infty)$.*

Proof. Suppose that $\{x_n\}$ quasi-λ-converges to x. For every real $\varepsilon > 0$, choose $c \in E$ with $0 < c$ and $K\|c\| < \varepsilon$. Then there is a positive integer N, for all $n \geq N$, $\lambda(x_n, x) < c$, therefore, $n \geq N$, $\| \lambda(x_n, x) \| \leq K \| c \| < \varepsilon$. This means $\lambda(x_n, x) \underset{\|\cdot\|_E}{\longrightarrow} 0$ $(n \longrightarrow \infty)$. □

Remark 3.9. [18] *The uniqueness assumption of the quasi-λ-limit in Theorem 3.14* [18] *holds if for each $x \in X$, one of the following conditions is satisfied:*

(A) $\lim_{n\longrightarrow\infty} \lambda(x_n, x) = 0$ *implies* $\lim_{n\longrightarrow\infty} \lambda(x, x_n) = 0$ *and for each $y \in X$, either* $\lambda(x, y) \leq \lambda(x, z) + \lambda(z, y)$ *or* $\lambda(y, x) \leq \lambda(y, z) + \lambda(z, x)$ *for all $z \in X$,*

(B) $\lambda(x, x) = 0$ *and for each $y \in X$, $\lambda(x, y) \leq \lambda(z, x) + \lambda(z, y)$ for all $z \in X$.*

Indeed, it is obvious that condition (B) implies the symmetricity of λ and thus implies condition (A). If condition (A) holds and both x and y are quasi-λ-limit points of sequence $\{x_n\}_{n\in\mathbb{N}}$. We have $\lambda(x_n, x) \overset{q-\lambda}{\longrightarrow} 0$ and $\lambda(x_n, y) \overset{q-\lambda}{\longrightarrow} 0$. Further, either

$$\lambda(x, y) \leq \lambda(x, x_n) + \lambda(x_n, y) \overset{q-\lambda}{\longrightarrow} 0 \quad \text{as } n \longrightarrow \infty, \tag{6}$$

or

$$\lambda(y, x) \leq \lambda(y, x_n) + \lambda(x_n, x) \overset{q-\lambda}{\longrightarrow} 0 \quad \text{as } n \longrightarrow \infty. \tag{7}$$

In either cases, we have $x = y$.

In the proof of Remark 3.9, since $\lambda(x, x_n)$ and $\lambda(y, x_n)$ are not quasi-λ-converges to 0, therefore, one cannot result in the relations

$$\lambda(x, y) \leq \lambda(x, x_n) + \lambda(x_n, y) \overset{q-\lambda}{\longrightarrow} 0 \quad \text{as } n \longrightarrow \infty, \tag{8}$$

or

$$\lambda(y, x) \leq \lambda(y, x_n) + \lambda(x_n, x) \overset{q-\lambda}{\longrightarrow} 0 \quad \text{as } n \longrightarrow \infty. \tag{9}$$

If we assume that in Remark 3.9, E is a normal vector space ordered by a normal cone P, then by Lemma 3.3 we will have the following correct version of the proof.

Proof. It is obvious that condition (B) implies the symmetricity of λ and thus implies condition (A). If condition (A) holds and both x and y are quasi-λ-limit points of sequence $\{x_n\}_{n\in\mathbb{N}}$. By Lemma 3.3

$$\lambda(x_n, x) \underset{\|\cdot\|_E}{\longrightarrow} 0 \quad \text{and} \quad \lambda(x_n, y) \underset{\|\cdot\|_E}{\longrightarrow} 0.$$

By condition (A)

$$\lambda(x, x_n) \underset{\|\cdot\|_E}{\longrightarrow} 0 \quad \text{and} \quad \lambda(y, x_n) \underset{\|\cdot\|_E}{\longrightarrow} 0.$$

Further, either

$$\lambda(x, y) \leq \lambda(x, x_n) + \lambda(x_n, y) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{10}$$

or

$$\lambda(y, x) \leq \lambda(y, x_n) + \lambda(x_n, x) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{11}$$

In either cases, we have $x = y$. □

Remark 3.10. *In the proof of Corollary 3.19* [18]*, we may have $x_n \notin G(x_n)$, in the other words, since $G(x_n) \subseteq F(x_n)$, we should get $x_n \notin G(x_n)$ which does not hold true in general. Also, Theorem 3.14* [18] *does not hold for the map G, because Condition (ii) of the Theorem 3.14* [18] *for G is established when $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in (X, d) with $x_{n+1} \in G(x_n)$ and for $y_n \in G(x_n)$ there exists $\{y_n\}_{n\in\mathbb{N}}$ such that $\delta(G(x_n)) \overset{d}{\longrightarrow} 0$ of the first type w.r.t $\{y_n\}_{n\in\mathbb{N}}$ in (X, d). While in the proof of Theorem 3.14* [18]*, it is shown that $\delta(G(x_n)) \overset{d}{\longrightarrow} 0$ of the first type w.r.t $\{x_n\}_{n\in\mathbb{N}}$ in (X, d) and it is may that $x_n \notin G(x_n)$. Also, according to Theorem 3.14* [18] *it is necessary that (X, d) be complete and G have closed values. In the following we give a short proof for Corollary Corollary 3.19* [18].

Proof. According to condition (iii), $\overline{x} \in \bigcap_{n\in\mathbb{N}} F(x_n)$ for all $n \in \mathbb{N}$. If $w \in \bigcap_{n\in\mathbb{N}} F(x_n)$ then according Condition (ii), $\lim_{n\longrightarrow\infty} p(x_n, w) = 0$. Also, because of condition (i) we have

$$\lim_{n\longrightarrow\infty} \sup\{p(x_n, x_m) : m > n\} = 0.$$

So with replacing $y_n \equiv w$ in (iii) we have

$$\lim_{n\longrightarrow\infty} d(x_n, w) = 0 \Longrightarrow x_n \to w.$$

The uniqueness of the limit in the hypothesis result that $w = \bar{x}$. Hence $\bigcap_{n\in\mathbb{N}} F(x_n) = \{\overline{x}\}$. Moreover, if condition (iv) holds, then

$$0 \neq F(\bar{x}) \subseteq \bigcap_{n\in\mathbb{N}} F(x_n) = \{\bar{x}\} \Longrightarrow F(\bar{x}) = \{\bar{x}\}.$$

□

4. Ekeland Type Variational Principles

Definition 4.1. [23] *Let (X, d) be a metric space. An extended real-valued function $f : X \to (-\infty, +\infty]$ is said to be lower semicontinuous from above (in short, lsca) at $x_0 \in X$ if for any sequence $\{x_n\}$ in X with $x_n \longrightarrow x_0$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ imply that $f(x_0) \leq \lim_{n\longrightarrow\infty} f(x_n)$. The function f is said to be lsca on X if f is lsca at every point of X.*

Definition 4.2. [18] *Let E be a normed vector space ordered by a cone P. Then E is called well-normed with respect to P if there exists $S \in \mathbb{R}_+$ such that*

$$\sum_{k=1}^{n} \| v_k \| \leq S \| \sum_{k=1}^{n} v_k \| \quad \forall n \in \mathbb{N}, v_k \in P \ \forall k \in \mathbb{N}. \tag{12}$$

The least positive number S satisfying inequality (12) is called well-normed constant of P. Also, E satisfies condition (L) if for all $n \in \mathbb{N}$, $\sum_{k=1}^{n} v_k \leq v$ for some $v \in E$ where

$$v_k \in P, \quad \forall \quad k \in \mathbb{N} \Rightarrow \lim_{n\longrightarrow\infty} \| v_n \| = 0.$$

Example 4.1. *For each $n \in \mathbb{N}$, let $\mathbb{R}^n$ be ordered by the cone $P = {\mathbb{R}_+}^n$. Then, $\mathbb{R}^n$ is not only well-normed with well norm constant $s = 1$ but also satisfies condition (L).*

The following lemma plays an important role in this section.

Lemma 4.1. [18] *Let X be a topological space, E a topological vector space ordered by a cone P and G, $H : X \to 2^E$ multi-valued maps with non-empty values. If G is a lower $(-P)$-continuous map and H is an upper P-continuous map with compact values, then the set $S = \{x \in X : G(x) \subseteq H(x) + P\}$ is a closed.*

Definition 4.3. [24] *Let X be a non-empty set and E be a topological vector space ordered by a cone P. A mapping $F : X \to 2^E$ is said to be P-bounded from below if there exists $l \in E$ such that $f(x) \subseteq l + P$ for all $x \in X$.*

Now we present a version of Ekeland type variational principle in the setting of complete cone metric spaces.

Remark 4.1. *To apply Lemma 4.1 in the proof of Theorem (4.7) in [18], it is necessary F be a multi-valued map with non-empty values. So, we will have the rectified theorem as follows.*

Theorem 4.1. *Let E be a well-normed vector space ordered by a normal cone P. Let (X, d) be a complete cone metric space, and let $f : X \to 2^E$ be a lower $(-P)$-continuous multi-valued map with non-empty values, bounded from below by l. Then, for every $\varepsilon > 0$ and for every $\widehat{x} \in X$, there exists $x^* \in X$ such that*

(i) $f(x^*) + \varepsilon d(x^*, \widehat{x}) \leq f(\widehat{x})$,

(ii) $\varepsilon d(x^*, x) \not\leq f(x^*) - f(x)$ *for all* $x \in X \setminus \{x^*\}$.

Definition 4.4. [25] *Let X be a topological space and E be a topological vector space ordered by a cone P. A set-valued mapping $F : X \times X \to 2^E$ is said to be bounded below on $X \times X$, if there exists $z \in E$ such that*

$$F(x, y) - z \subseteq P, \quad \forall x, y \in X$$

Remark 4.2. *In the proof of Theorem 4.8* [18], *to verify that each fixed $x \in X$, $d(x, .)$ is an upper P-continuous function on X, it is necessary that $(X, .)$ to be continuous w.r.t second component and F be a multi-valued maps with non-empty values.*

If $H(x) = X$ for all $x \in X$, then we deduce the following corollary from Theorem 4.8 [18].

Corollary 4.1. *Let (X, d) be a complete cone metric space, E be a normed vector space with an ordering normal cone P that satisfies condition (L),and let $F : X \times X \to 2^E$ be multi-valued map with non-empty values. For each $x \in X$, suppose that the following conditions hold:*

(i) *There exists $y \in X$ such that $F(x, y) + d(x, y) \subseteq -P$,*

(ii) *Either $F(x, .)$ is bounded from below on X or $\cap_{y \in X} F(x, y) \neq \emptyset$,*

(iii) *$F(x, z) \subseteq F(x, y) + F(y, z) - P$ for all $y, z \in X$,*

(iv) *the map $F(x, .)$, is lower P-continuous on X.*

Then, for every $\varepsilon > 0$ and for every $\widehat{x} \in X$, there exists $x^* \in X$ such that

(a) $F(\widehat{x}, x^*) + \varepsilon d(\widehat{x}, x^*) \subseteq -P$

(b) $F(x^*, x) + \varepsilon d(x^*, x) \not\subseteq -P$ *for all* $x \in X \setminus \{x^*\}$.

Definition 4.5. [20] *Let p be a τ-function on X. A sequence $\{x_n\}$ in X is asymptotic by p, if $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.*

As an application of Theorem 2.1, we improve Theorem 2.1 in [21] as follows.

Theorem 4.2. *Let (X, d) be a metric space, $f : X \to \mathbb{R}$ a lsca function and bounded from below on X, $\varphi : (-\infty, \infty] \longrightarrow \mathbb{R}_+$ a nondecreasing function, and p a τ-function on X with $p(x, x) = 0$ for all $x \in X$. Define a binary relation on X by*

$$y \Re x \Longleftrightarrow p(x, y) \leq \varphi(f(x))(f(x) - f(y)).$$

Suppose that X is $\Re$-complete and $H : X \to 2^X$ is a multi-valued map with nonempty values which for each $x \in X$, there exists $y \in H(x)$ such that $p(x, y) \leq \varphi(f(x))(f(x) - f(y))$. Then, for each $u \in X$, there exists $v \in X$ such that

(i) $p(v, y) > \varphi(f(v))(f(v) - f(y))$ *for all* $y \in X$, $y \neq v$,

(ii) $v \in H(v)$.

Proof. Define $F(x) = \{y \in X : y \Re x\}$ for all $x \in X$. We construct sequence $\{x_n\}$ in $F(x)$ as following: putting $x_1 = u$ and choose $x_{n+1} \Re x_n$ for all $n \in \mathbb{N}$ ($\{x_n\}$ is $\Re$-monotone). Then, it is easy to verify that $\Re$ is a transitive relation and hence condition (ii) of Theorem 2.1 is satisfied. Since $p(x, x) = 0$ for all $x \in X$, we have $x \in F(x)$ for all $x \in X$. We first prove that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \Re x_n$, we have

$$p(x_n, x_{n+1}) \leq \varphi(f(x_n))(f(x_n) - f(x_{n+1}))$$

which implies that $f(x_{n+1}) \leq f(x_n)$ for each $n \in \mathbb{N}$ and so $\{f(x_n)\}_{n\in\mathbb{N}}$ is a decreasing sequence. Also since f is bounded from below, $\lim_{n\longrightarrow\infty} f(x_n)$ exists. Let

$$r = \lim_{n\longrightarrow\infty} f(x_n) = \inf_{n\in\mathbb{N}} f(x_n),$$

then $r \leq f(x_n)$ for all $n \in \mathbb{N}$. We claim that $\lim_{n\longrightarrow\infty} \sup\{p(x_n, x_m) : m > n\} = 0$. Since φ is nondecreasing, if $m > n$, then we have

$$p(x_n, x_m) \leq \sum_{j=n}^{m-1} p(x_j, x_{j+1}) \leq \varphi(f(x_n))(f(x_n) - r) = \alpha_n$$

where $\alpha_n = \varphi(f(x_n))(f(x_n) - r)$. Then $0 \leq \sup\{p(x_n, x_m) : m > n\} \leq \alpha_n$ for all $n \in \mathbb{N}$. Since $r \leq f(x_n)$ for all $n \in \mathbb{N}$, we always have $f(x_n) - r \geq 0$ and $\lim_{n\longrightarrow\infty} f(x_n) = r$ implying that $\{\alpha_n\}$ is a sequence in $[0, \infty)$ converging to zero and

$$\lim_{n\longrightarrow\infty} \sup\{p(x_n, x_m) : m > n\} = 0. \tag{13}$$

Then by Lemma 2.1, $\{x_n\}$ is a Cauchy sequence in X.
By the $\Re$-completness of X, there exists $z \in X$ such that $\lim_{n\longrightarrow\infty} x_n = z$. Since f is $lsca$, it follows that $f(z) \leq \lim_{n\longrightarrow\infty} f(x_n) = r \leq f(x_k)$ for all $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed and for all $m \in \mathbb{N}$ with $m > n$,we have

$$p(x_n, x_m) \leq \sum_{j=n}^{m-1} p(x_j, x_{j+1}) \leq \varphi(f(x_n))(f(x_n) - f(z)).$$

From $(\tau 2)$,we have

$$p(x_n, z) \leq \varphi(f(x_n))(f(x_n) - f(z)) \quad for\ all\ n \in \mathbb{N}$$

which implies that $z\Re x_n$ for all $n \in \mathbb{N}$. Therefore, $\Re$ is lower closed. Since

$$0 \leq p(x_n, x_{n+1}) \leq \sup\{p(x_n, x_m) : m > n\}$$

by (13), we have $\lim_{n\longrightarrow\infty} p(x_n, x_{n+1}) = 0$ and thus, $\{x_n\}$ is asymptotic by p. Let $y_n = x_{n+1}$ for all $n \in \mathbb{N}$, we have $\lim_{n\longrightarrow\infty} p(x_n, y_n) = 0$. By $(\tau 3)$, we obtain $\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = 0$. Therefore, condition (iii) of Theorem 2.1 is satisfied and by Theorem 2.1, F has a critical point, that is, there exists $v \in X$ such that $F(v) = \{v\}$. for all $y \in X$ with $y \neq v$, we have $y \notin F(v)$, that is

$$p(v, y) \nleq \varphi(f(v))(f(v) - f(y)) \Longrightarrow p(v, y) > \varphi(f(v))(f(v) - f(y)).$$

Since $v \in X$, by the hypothesis, there exists $w_v \in H(v)$ such that

$$p(v, w_v) \leq \varphi(f(v))(f(v) - f(w_v)).$$

Then $w_v = v$. Indeed, if $w_v \neq v$, then

$$p(v, w_v) \leq \varphi(f(v))(f(v) - f(w_v)) < p(v, w_v)$$

which leads to a contradiction. Hence $v = w_v \in H(v)$. □

5. Conclusions

In this paper, it is considered the λ-space which is weaker than a cone metric space in general, and also some critical point results in λ-spaces are rectified. In addition, an improved version of Ekeland type variational principle in complete cone metric spaces is given.

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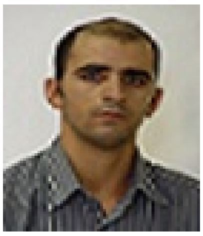

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