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INVARIANCES OF STRONGLY CONTINUOUS QUASI SEMIGROUPS AND DISTURBANCE DECOUPLING PROBLEMS

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Abstract. In this paper, invariances of a subspace of a Hilbert space under strongly continuous quasi semigroup (C_0-quasi semigroup) are characterized. The invariance-relationship between the C_0-quasi semigroups and its infinitesimal generator are also investigated including for the generator of Riesz-spectral operators. The invariant concepts for a non-autonomous system can also be characterized in the C_0-quasi semigroup term. Some relationships of the invariances are also identified. The system-invariance is applicable to solve a disturbance decoupling problem of the non-autonomous linear control systems. The sufficiency for the solvability is identified by the largest controlled invariant subspace of kernel of output operator. An example is simulated to confirm the disturbance decoupling problem of the non-autonomous linear control systems.

Keywords: invariant subspace, C_0-quasi semigroup, system-invariance, disturbance decoupling problem, solvable.

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1. Introduction

First, we consider an autonomous uncontrolled system

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \quad x(0) = x_0, \quad x_0 \in X, \tag{1}$$

where A is the infinitesimal generator of a C_0-semigroup on a Hilbert space X. Invariant subspaces have an important role in investigating dynamical properties of the system (1). For the system (1), there are two concepts of invariances, invariant semigroup and invariant generator. If the state space X is finite dimensional, then these invariances are equivalent. In general, if X is infinite dimensional, then they are not equivalent for unbounded generator [1, 2]. For a closed subspace V of X there are sufficient and necessary conditions for the invariances of semigroups and its generators [1, 3].

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Next, we consider the controlled version of system (1):

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_0, \quad x_0 \in X, \quad u \in U, \tag{2}$$

where X and U are Hilbert spaces, $u \in U$ is the control variable, A is the generator of a C_0-semigroup $T(t)$, and B is an injective bounded linear operator with finite dimensional $\operatorname{ran} B$, where $\operatorname{ran} F$ denotes the range of F. For the system (2), there are many types of system-invariance that have been characterized [1, 3, 4, 5, 6]. If X is finite dimensional, a subspace V, (A, B)-invariance is equivalent to the existence of a bounded linear feedback control law which achieves holdability in V. This equivalence also holds for infinite dimensions when A is bounded and subspace V, BU, and $V + BU$ are closed [3]. The system-invariance are applicable to solve a disturbance decoupling problem (DDP) [7, 5, 8]. In case, the largest open-loop invariant subspace is equal to the largest closed-loop invariant subspace, then the solvability of the DDP is equivalent to the solvability of a meromorphic matrix equation [6].

As a generalization of C_0-semigroup, C_0-quasi semigroup is a sophisticate tool for some non-autonomous problems. Initially, Leiva and Barcenas [9] introduced C_0-quasi semigroup and then Sutrima *et al.*[10, 11, 12] developed advanced properties and some theory stabilities. The continuity of the adjoint of a quasi semigroup was proved [13]. Moreover, relationships between spectrum of quasi semigroups and its generators were also characterized [14]. The applications of the C_0-quasi semigroups in various areas were also developed [15, 16, 17, 18]. These facts provide opportunities to investigate the invariant subspaces of the state space of the non-autonomous control systems using the C_0-quasi semigroups.

In this paper we are concern on the invariant subspaces of the Hilbert spaces under the non-autonomous version of linear control systems (1) and (2). The organization of this paper is as follows. In Section 2, we provide the sufficient and necessary conditions for the invariance under the C_0-quasi semigroups and its generators. Investigations of the invariance under the non-autonomous linear control systems are considered in Section 3. In Section 4, we apply the invariance to solve the disturbance decoupling problem of the non-autonomous linear control systems. The application is completed by an example.

2. Invariant concepts under C_0-quasi semigroups

The concepts of invariance for the C_0-quasi semigroups is split into two parts, respect to the quasi semigroup with its infinitesimal generator and respect to the related non-autonomous systems. These concepts have important applications in the non-autonomous control systems. We recall the definition of the C_0-quasi semigroup following [9] and [15].

Definition 2.1. *Let $\mathcal{L}(X)$ be the set of all bounded linear operators on a Hilbert space X. A two-parameter commutative family $\{R(t, s)\}_{s,t \geq 0}$ in $\mathcal{L}(X)$ is called a strongly continuous quasi semigroup (C_0-quasi semigroup) on X if for each $r, s, t \geq 0$ and $x \in X$:*

(a) $R(t, 0) = I$, *the identity operator on X,*
(b) $R(t, s + r) = R(t + r, s)R(t, r)$,
(c) $\lim_{s \to 0^+} \|R(t, s)x - x\| = 0$,
(d) *there exists a continuous increasing function $M : [0, \infty) \to [0, \infty)$ such that*

$$\|R(t, s)\| \leq M(t + s). \tag{3}$$

We denote $\mathcal{D}$ as the set of all $x \in X$ such that the following limits exist

$$\lim_{s \to 0^+} \frac{R(t, s)x - x}{s} = \lim_{s \to 0^+} \frac{R(t - s, s)x - x}{s}, \quad t \geq 0.$$

The infinitesimal generator of the C_0-quasi semigroup $\{R(t,s)\}_{s,t\geq 0}$ is defined as the family $\{A(t)\}_{t\geq 0}$ on $\mathcal{D}$ where

$$A(t)x = \lim_{s\to 0^+} \frac{R(t,s)x - x}{s}.$$

For the simplicity we denote the quasi semigroup $\{R(t,s)\}_{s,t\geq 0}$ and family $\{A(t)\}_{t\geq 0}$ by $R(t,s)$ and $A(t)$, respectively. Not like in the C_0-semigroups theory, the operators of the infinitesimal generator of the C_0-quasi semigroups must not be closed [10].

For a linear operator $T : \mathcal{D}(T) \subseteq X \to Y$, $\mathcal{D}(T)$, $\ker T$, $\rho(T)$, $\sigma(T)$, and $\mathcal{R}(\lambda, T)$ denote the domain, kernel, resolvent set, spectrum, and resolvent of T, respectively. First of all, we focus on developing the invariances of subspaces under the C_0-quasi semigroups and its infinitesimal generators.

Definition 2.2. *Let V be a subspace of a Hilbert space X, and let $R(t,s)$ be a C_0-quasi semigroup on X. The subspace V is said to be $R(t,s)$-invariant if for all $t,s \geq 0$ it satisfies*

$$R(t,s)V \subseteq V.$$

In case $A(t)$ is the infinitesimal generator of C_0-quasi semigroup $R(t,s)$, the related invariance is defined as follows.

Definition 2.3. *Let V be a subspace of a Hilbert space X and let $A(t)$ be an infinitesimal generator of a C_0-quasi semigroup $R(t,s)$ on X with domain $\mathcal{D}$. The subspace V is said to be $A(t)$-invariant if for all $t \geq 0$ it satisfies*

$$A(t)(V \cap \mathcal{D}) \subseteq V.$$

The following lemma explains that the $R(t,s)$-invariance implies the $A(t)$-invariance.

Lemma 2.1. *Let V be a closed subspace of the Hilbert space X and let $A(t)$ be an infinitesimal generator of a C_0-quasi semigroup $R(t,s)$ on X with domain $\mathcal{D}$. If V is $R(t,s)$-invariant, then:*

(a) *V is $A(t)$-invariant;*
(b) *$R(t,s)|_V$ is a C_0-quasi semigroup on V with the infinitesimal generator $A_V(t)$, where $A_V(t)v = A(t)v$ for $v \in \mathcal{D}(A_V(t)) = \mathcal{D} \cap V$.*

Proof. Proof follows the proof of Lemma 2.5.3 of [2]. □

Let $A(t)$ be the infinitesimal generator of the C_0-quasi semigroup $R(t,s)$ on X. For every $\lambda \in \mathbb{C}$ and $t,s \geq 0$, we define a bounded linear operator

$$D_\lambda(t,s)x := \int_0^s e^{\lambda(s-v)} R(t,v)x\,dv, \quad x \in X.$$

By this operator, Theorem 2.1 of [14] gives

$$(\lambda - A(t))D_\lambda(t,s)x = [e^{\lambda s} - R(t,s)]x, \quad x \in X, \tag{4}$$

and

$$D_\lambda(t,s)(\lambda - A(t))x = [e^{\lambda s} - R(t,s)]x, \quad x \in \mathcal{D}. \tag{5}$$

If $e^{\lambda s} \in \rho(R(t,s))$, (4) and (5) imply that $e^{\lambda s} - R(t,s)$ is invertible and $\lambda \in \rho(A(t))$. Therefore, if $F_\lambda(t,s) := [e^{\lambda s} - R(t,s)]^{-1}$, we have $D_\lambda(t,s)F_\lambda(t,s) = F_\lambda(t,s)D_\lambda(t,s)$ and

$$\mathcal{R}(\lambda, A(t))x = D_\lambda(t,s)F_\lambda(t,s)x, \quad x \in X. \tag{6}$$

We recall that a component of subset Z in X is the largest connected subset of Z. Let $\rho_\infty(A)$ denote the component of the resolvent set $\rho(A)$ containing an interval $[r,\infty)$.

Theorem 2.1. *Let $A(t)$ be the infinitesimal generator of C_0-quasi semigroup $R(t,s)$ on the Hilbert space X. For a closed subspace V the following statements are equivalent:*

(a) *V is $R(t,s)$-invariant.*
(b) *V is $\mathcal{R}(\lambda, A(t))$-invariant for some $\lambda \in \rho_\infty(A(t))$.*
(c) *V is $\mathcal{R}(\lambda, A(t))$-invariant for all $\lambda \in \rho_\infty(A(t))$.*

Proof. (a)⇒(b). From definitions of $D_\lambda(t,s)$, $F_\lambda(t,s)$, and (6), the hypothesis implies that $\mathcal{R}(\lambda, A(t))V \subseteq V$ for all $t \geq 0$.

(b)⇒(c). It follows from Lemma 2.5.5 of [2].

(c)⇒(a). From the hypothesis, for any $\lambda \in \rho_\infty(A(t))$ we have

$$\mathcal{R}(\lambda, A(t))v = D_\lambda(t,s)F_\lambda(t,s)v \in V,$$

for all $v \in V$. Definitions of $D_\lambda(t,s)$ and $F_\lambda(t,s)$ give $R(t,s)v \in V$ for all $v \in V$ and $t, s \geq 0$. Thus, V is $R(t,s)$-invariant. □

In general, Theorem 2.1 is not true if $\rho_\infty(A(t))$ is replaced by $\rho(A(t))$, see Example I.5 of [1]. As well, The converse of Lemma 2.1 is not always true for unbounded infinitesimal generator, as shown by the following example.

Example 2.1. *Let X be the Hilbert space $L_2[0,1]$ and $A(t)x(\xi) = \frac{1}{t+1}\frac{d^2x}{d\xi^2}$, $t \geq 0$, on*

$$\mathcal{D} = \{x \in X : x, \tfrac{dx}{d\xi} \text{ are absolutely continuous}, \ x(0) = x(1) = 0, \ \tfrac{d^2x}{d\xi^2} \in X\}.$$

If it is defined a subspace

$$V = \{x \in X : x(\xi) = 0 \text{ almost everywhere on } [0, \tfrac{1}{2}]\},$$

then V is $A(t)$-invariant, but not $R(t,s)$-invariant.

It is clear that V is $A(t)$-invariant and $\rho_\infty(A(t)) = \rho(A(t))$. Since 0 is not the eigenvalue of $A(t)$, so $A(t)$ is invertible, and

$$(A(t)^{-1}x)(\xi) = (t+1)\int_0^\xi (\xi - 1)\eta x(\eta)d\eta + (t+1)\int_\xi^1 \xi(\eta - 1)x(\eta)d\eta.$$

If we set $x_0(\xi) = \pi^2 \chi_{[\frac{1}{2},1]}(\xi)\sin \pi\xi$, then $x_0 \in V$, but

$$(A(t)^{-1}x_0)(\xi) = \begin{cases} -\xi, & 0 \leq \xi < 1/2 \\ 1 - \xi - \sin \pi\xi, & 1/2 \leq \xi \leq 1, \end{cases}$$

is not in V, where χ_E denotes the indicator function of the set E. Theorem 2.1 implies that V is not $R(t,s)$-invariant.

The converse of Lemma 2.1 remains true if each $A(t)$ is a non-autonomous Riesz-spectral operator i.e. $A(t) = a(t)A$ where A is a Riesz-spectral operator [2] and a is a positively continuous function for all $t \geq 0$. We see that for each $t \geq 0$, $A(t)$ and A have common eigenvectors [16].

Lemma 2.2. *Let $A(t)$ is a non-autonomous Riesz-spectral operator where A has the Riesz basis of the set of eigenvectors $\{\phi_n : n \in \mathbb{N}\}$ corresponding to the set of eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ satisfies $\sup_{n\in\mathbb{N}} \mathrm{Re}(\lambda_n) < \infty$. If $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup $R(t,s)$ on the Hilbert space X, V is a closed subspace of $\mathcal{D}$, and V is $A(t)$-invariant, then V is $R(t,s)$-invariant.*

Proof. It follows from Lemma 2.5.4 of [2] and Theorem 3 of [16]. □

Theorem 2.2. *Let $A(t)$ be the infinitesimal generator of a C_0-quasi semigroup $R(t,s)$ on a Hilbert space X. If for each $t \geq 0$, $A(t)$ is a non-autonomous Riesz-spectral operator with the basis Riesz of the set of the eigenvectors $\{\phi_n : n \in \mathbb{N}\}$, then $\rho_\infty(A(t)) = \rho(A(t))$. Moreover, if V is a closed subspace of X, then V is $R(t,s)$-invariant if and only if*

$$V = \overline{\operatorname{span}}_{n\in\mathbb{J}}\{\phi_n\}, \quad \textit{for some} \quad \mathbb{J} \subset \mathbb{N}.$$

Proof. The definition of the Riesz-spectral operator [2] and Theorem 3 of [16] imply that $\rho_\infty(A(t)) = \rho(A(t))$ for every $t \geq 0$.

Sufficiency. By the representation $R(t,s)$ in Theorem 3 of [16], we have that V is $R(t,s)$-invariant.

Necessity. It follows from the proof of Lemma 2.5.8 of [2] and Theorem 3 of [16] with

$$P_\Gamma(t)x := \frac{1}{2\pi i}\int_\Gamma (\lambda I - A(t))^{-1}x d\lambda = \frac{1}{a(t)}\sum_{\lambda_n\in\Gamma}\langle x, \varphi_n\rangle\phi_n,$$

where Γ is a simple, closed, positively oriented curve that encloses some eigenvalues. □

3. System-Invariance Concepts

We consider the non-autonomous linear control system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad t \geq 0, \\ x(0) &= x_0, \quad x(t) \in X, \quad u(t) \in U, \end{aligned} \tag{7}$$

where X and U are Hilbert spaces, $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup, and each $B(t) \in \mathcal{L}_s(U,X)$ with finite-dimensional $\operatorname{ran} B(t)$, and u is the control, where $\mathcal{L}_s(V,W)$ denotes the space of linear bounded operators from V to W equipped with strong operator topology. We also assume that $U = \mathbb{C}^m$ and $B(t)$ injective.

There are many concepts of the system invariance that can be developed from the autonomous systems. We begin with the strongest. We denote $R_{A+BF}(t,s)$ as the C_0-quasi semigroup generated by the family $A(t) + B(t)F(t)$. The existence of the quasi semigroup is guaranteed by Theorem 3 of [16]. Moreover, we see that $A(t) + B(t)F(t)$ and $A(t)$ have common domain.

Definition 3.1. *A subspace V of X is said to be closed-loop invariant if there exists a bounded feedback control $F(t)$ such that*

$$R_{A+BF}(t,s)V \subseteq V, \quad t,s \geq 0. \tag{8}$$

The definition says that V is closed-loop invariant if it is $R_{A+BF}(t,s)$-invariant for the system $\dot{x}(t) = [A(t) + B(t)F(t)]x(t)$ for some F with $F(t) \in \mathcal{L}_s(X,U)$, $t \geq 0$.

Lemma 3.1. *Let V be a closed subspace of X and $R_{A+BF_1}(t,s)$-invariant for some F_1 with $F_1(t) \in \mathcal{L}_s(X,U)$, $t \geq 0$. The subspace V is $R_{A+BF_2}(t,s)$-invariant for some F_2 if and only if $\operatorname{ran} B(t)(F_1(t) - F_2(t))|_{V\cap\mathcal{D}} \subseteq V$.*

Proof. Necessity. By the hypothesis we have

$$R_{A+BF_1}(t,s)V \subseteq V \qquad \text{and} \qquad R_{A+BF_2}(t,s)V \subseteq V,$$

for all $t,s \geq 0$. By differentiating both with respect to s and setting $s = 0$, we obtain

$$[A(t) + B(t)F_1(t)](V \cap \mathcal{D}) \subseteq V \qquad \text{and} \qquad [A(t) + B(t)F_2(t)](V \cap \mathcal{D}) \subseteq V.$$

These imply that $\operatorname{ran} B(t)(F_1(t) - F_2(t))|_{V\cap\mathcal{D}} \subseteq V$.

Sufficiency. Let V be $R_{A+BF_1}(t,s)$-invariant and $\operatorname{ran} B(t)(F_1(t) - F_2(t))|_{V\cap\mathcal{D}} \subseteq V$. The generator $A(t) + B(t)F_1(t)$ is densely defined on V and $\overline{V \cap \mathcal{D}} \subseteq V$. From the perturbation theory for the quasi semigroups, Theorem 3 of [17], we have a C_0-quasi semigroup $R_{A+BF_2}(r,t)$ which is the uniform limit of the series $\sum_{n=0}^{\infty} R_n(r,t)$, where

$$R_0(r,t) = R_{A+BF_1}(r,t)$$
$$R_n(r,t) = \int_0^t R_{A+BF_1}(r+s,t-s)B(r+s)(F_1-F_2)(r+s)R_{n-1}(r,s)ds,$$

for all $t,r,s \geq 0$ with $t \geq s$ and $n \in \mathbb{N}$. By an induction argument and the hypothesis, we have $R_n(r,t)V \subseteq V$ for all n. Since V is closed, it follows that $R_{A+BF_2}(r,t)V \subseteq V$. □

Definition 3.2. *Let λ be an element of $\rho(A(t))$. An operator $F(t)$ from X to U is said to be $A(t)$-bounded if $\mathcal{D} \subseteq \mathcal{D}(F(t))$ and $F(t)(\lambda I - A(t))^{-1} \in \mathcal{L}(X,U)$.*

We can show that if the operator $F(t)$ is bounded, then it is $A(t)$-bounded. Therefore, the family of $A(t)$-bounded operators is larger than the family of bounded operators. In case $A(t)$ is bounded, these families are equal.

Corollary 3.1. *Let V be a closed subspace of X and $\mathcal{B}$ be a subspace of $\operatorname{ran} B(t)$ such that $\mathcal{B} + (\operatorname{ran} B(t) \cap V) = \operatorname{ran} B(t)$. If V is closed-loop invariant, then there exists an $A(t)$-bounded feedback $F(t)$ such that V is $R_{A+BF}(t,s)$-invariant and $\operatorname{ran} B(t)F(t)|_{V\cap\mathcal{D}} \subseteq \mathcal{B}$.*

Proof. Assume that V is $R_{A+B\tilde{F}}(t,s)$-invariant for some $\tilde{F}$ with $\tilde{F}(t) \in \mathcal{L}_s(X,U)$, $t \geq 0$. Since the range of $\tilde{F}(t)$ is finite dimensional, $(B\tilde{F})(t)$ can be represented as

$$(B\tilde{F})(t)x = \sum_{i=1}^{q} b_i\langle (A(t)-\lambda I)x, f_i\rangle + \sum_{i=q+1}^{p} b_i\langle (A(t)-\lambda I)x, f_i\rangle,$$

where $\lambda \in \rho(A(t))$, $\operatorname{span}_{i=1,2,\dots,q}\{b_i\} = \mathcal{B}$, $\operatorname{span}_{i=q+1,\dots,p}\{b_i\} \subseteq \operatorname{ran} B(t) \cap V$, $x \in \mathcal{D}$, and $b_i, f_i \in X$. Setting F with

$$F(t)x = \sum_{i=1}^{q} \langle (A(t)-\lambda I)x, f_i\rangle, \quad x \in \mathcal{D},$$

we have $\operatorname{ran} B(t)(F(t)-\tilde{F}(t))|_{V\cap\mathcal{D}} \subseteq V$ and $B(t)F(t)|_{V\cap\mathcal{D}} \subseteq \mathcal{B}$. Lemma 3.1 gives that V is $R_{A+BF}(t,s)$-invariant with $F(t)$ is $A(t)$-bounded. □

Next we define the generator invariance corresponding to the closed-loop invariance. We use the class of $A(t)$-bounded operators.

Definition 3.3. *A subspace V of X is said feedback invariant if there exists a $A(t)$-bounded feedback law $F(t)$ such that*

$$[A(t) + B(t)F(t)](V \cap \mathcal{D}) \subseteq V. \tag{9}$$

Lemma 3.2. *If $V_1 \subseteq \mathcal{D}$ is a linear subspace, closed with respect to the graph norm of A and $V_2 \subseteq X$ is a closed linear subspace with*

$$A(t)V_1 \subseteq V_2 + \operatorname{ran} B(t),$$

then there exists an $A(t)$-bounded feedback law $F(t)$ such that

$$[A(t) + B(t)F(t)]V_1 \subseteq V_2.$$

Proof. Proof follows the proof of Theorem 4.1 of [6]. □

Definition 3.4. *A subspace V of X is said to be $(A(t),B(t))$-invariant if*

$$A(t)(V \cap \mathcal{D}) \subseteq V + \operatorname{ran} B(t). \tag{10}$$

For $u : [0, \infty) \to U$, the mild solution of the non-autonomous system (7) [16] defined by

$$x(t) = R(0,t)x_0 + \int_0^t R(s, t-s)B(s)u(s)ds, \quad t, s \geq 0. \tag{11}$$

We consider the operator $G^t_{R,B} : L_2([0,t], U) \to X$ defined by

$$G^t_{R,B}(u) := \int_0^t R(s, t-s)B(s)u(s)ds.$$

We see that $G^t_{R,B}$ is a continuous linear operator.

Definition 3.5. *The reachability subspace corresponding to the quasi semigroup $R(t,s)$ and operator $B(t)$ is defined by*

$$\mathcal{R}(R|\operatorname{ran} B(t)) := \bigcup_{t\geq 0} \operatorname{ran} G^t_{R,B}. \tag{12}$$

We see that if the closure of $\mathcal{R}(R|\operatorname{ran} B(t))$ equals X i.e. $\overline{\mathcal{R}(R|\operatorname{ran} B(t))} = X$, then system (7) is approximately controllable [16]. In general, $\overline{\mathcal{R}(R|\operatorname{ran} B(t))}$ is not in $\mathcal{D}$ [5], but we have the following results.

Theorem 3.1. *The reachability subspaces satisfy the followings.*

(a) $\overline{\mathcal{R}(R|\operatorname{ran} B(t))}$ *is the smallest closed, $R(t,s)$-invariant subspace containing* $\operatorname{ran} B(t)$.
(b) $\overline{\mathcal{R}(R|\operatorname{ran} B(t))}$ *is feedback invariant i.e.* $\overline{\mathcal{R}(R_{A+BF}|\operatorname{ran} B(t))} = \overline{\mathcal{R}(R|\operatorname{ran} B(t))}$.
(c) $\overline{\mathcal{R}(R|\operatorname{ran} B(t))}$ *is an $(A(t), B(t))$-invariant subspace.*
(d) *If* $\operatorname{ran} B(t)$ *is $R(t,s)$-invariant, then* $\overline{\mathcal{R}(R|\operatorname{ran} B(t))} = \overline{\operatorname{ran} B(t)}$.

Proof. (a) If $x \in \mathcal{R}(R|\operatorname{ran} B(t))$, there exists $u \in L_2([0,t], U)$ such that

$$x = \int_0^t R(r, t-r)B(r)u(r)dr.$$

We have that $R(t,s)x$ is also in $\mathcal{R}(R|\operatorname{ran} B(t))$, since it has the form

$$R(t,s)x = \int_0^t R(t,s)R(r,t-r)B(r)u(r)dr = \int_0^{t+s} R(r, t+s-r)B(r)u_0(r)dr,$$

where

$$u_0(r) = \begin{cases} u(r), & 0 \leq r \leq t \\ 0, & r > t. \end{cases}$$

This gives that $R(t,s)\mathcal{R}(R|\operatorname{ran} B(t)) \subseteq \mathcal{R}(R|\operatorname{ran} B(t))$. Since $R(t,s)$ is a linear bounded operator, it follows that $R(t,s)\overline{\mathcal{R}(R|\operatorname{ran} B(t))} \subseteq \overline{\mathcal{R}(R|\operatorname{ran} B(t))}$.

Since $\overline{\mathcal{R}(R|\operatorname{ran} B(t))}$ is closed, for $u \in L_2([0,t], U)$ we have

$$B(t)u(t) = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} R(s, t-s)B(s)u(s)ds \in \overline{\mathcal{R}(R|\operatorname{ran} B(t))}.$$

This shows that $\operatorname{ran} B(t) \subseteq \overline{\mathcal{R}(R|\operatorname{ran} B(t))}$.

Finally, we show that $\overline{\mathcal{R}(R|\operatorname{ran} B(t))}$ is the smallest closed, $R(t,s)$-invariant subspace which contains $\operatorname{ran} B(t)$. Suppose that V is another closed, $R(t,s)$-invariant subspace and $\operatorname{ran} B(t) \subseteq V \subseteq \overline{\mathcal{R}(R|\operatorname{ran} B(t))}$. If $z \in \overline{\mathcal{R}(R|\operatorname{ran} B(t))}$, there exists a sequence $t_n \in \mathbb{R}^+$, $u_n \in L_2([0,t_n], U)$ such that

$$z = \lim_{n\to\infty} \int_0^{t_n} R(s, t_n - s)B(s)u_n(s)ds.$$

By the hypothesis we have $R(s, t_n - s)B(s)u_n(s) \in V$. Moreover, by closedness of V it follows that $z \in V$. It completes that $V = \overline{\mathcal{R}(R|\operatorname{ran} B(t))}$.

(b) The quasi semigroup generated by $A(t) + B(t)F(t)$ is given by [16]

$$R_{A+BF}(r,t) = R(r,t) + \int_0^t R(r+s, t-s)B(r+s)F(r+s)R_{A+BF}(r,s)ds. \tag{13}$$

By transforming variables and changing the order of the integrals, (13) gives

$$\int_0^t R_{A+BF}(s, t-s)B(s)u(s)ds = \int_0^t R(s, t-s)B(s)u(s)ds + \int_0^t R(\xi, t-\xi)B(\xi)\Big(\int_0^\xi F(\xi)R_{A+BF}(s, \xi - s)B(s)u(s)ds\Big)d\xi \in \overline{\mathcal{R}(R|\operatorname{ran} B(t))}.$$

This gives $\overline{\mathcal{R}(R_{A+BF}|\operatorname{ran} B(t))} \subseteq \overline{\mathcal{R}(R|\operatorname{ran} B(t))}$. For the reverse inclusion, the quasi semigroup $R(r,t)$ can be consider as the perturbation $R_{A+BF}(r,t)$ with the feedback $-BF$, it follows that $\overline{\mathcal{R}(R|\operatorname{ran} B(t))} \subseteq \overline{\mathcal{R}(R_{A+BF}|\operatorname{ran} B(t))}$.

(c) Since $\overline{\mathcal{R}(R|\operatorname{ran} B(t))}$ is $R_{A+BF}(t,s)$-invariant, it follows that

$$R_{A+BF}(t,s)\overline{\mathcal{R}(R|\operatorname{ran} B(t))} \subseteq \overline{\mathcal{R}(R|\operatorname{ran} B(t))}, \quad t, s \geq 0. \tag{14}$$

Differentiating (14) with respect to s yields

$$[A(t) + B(t)F(t)]R_{A+BF}(t,s)(\overline{\mathcal{R}(R|\operatorname{ran} B(t))} \cap \mathcal{D}) \subseteq \overline{\mathcal{R}(R|\operatorname{ran} B(t))}.$$

Letting $s = 0$ we obtain

$$(A(t) + B(t)F(t))(\overline{\mathcal{R}(R|\operatorname{ran} B(t))} \cap \mathcal{D}) \subseteq \overline{\mathcal{R}(R|\operatorname{ran} B(t))}.$$

For $F = 0$ this implies (10).

(d) From (a) we have $\overline{\operatorname{ran} B(t)} \subseteq \overline{\mathcal{R}(R|\operatorname{ran} B(t))}$. On other hand, if $y \in \overline{\mathcal{R}(R|\operatorname{ran} B(t))}$ there exists a sequence $r_n \in \mathbb{R}^+$, $u_n \in L_2([0, r_n], U)$ such that

$$y = \lim_{n\to\infty} \int_0^{r_n} R(s, r_n - s)B(s)u_n(s)ds.$$

By the hypothesis we have $R(s, r_n - s)B(s)u_n(s) \in B(t)$. This implies that $y \in \overline{\operatorname{ran} B(t)}$. Thus $\overline{\mathcal{R}(R|\operatorname{ran} B(t))} \subseteq \overline{\operatorname{ran} B(t)}$. ☐

Definition 3.6. *A subspace V of X is said to be open-loop invariant if for each $x_0 \in V$ there exists a $u \in \mathcal{C}(\mathbb{R}^+, U)$ such that the mild solution* (11) *remains in V.*

We note that if the state space is finite-dimensional, then the function u in Definition 3.6 is measurable enough [1]. However, this condition no longer holds if the state space is infinite dimensional [6]. In the autonomous systems, a closed subspace V is closed-loop invariant if and only if it is open-loop invariant [1, 6]. In case the non-autonomous systems, this condition is only fulfilled when V is a subspace of $\mathcal{D}$.

Theorem 3.2. *Let V be a closed subspace of X and for each $t \geq 0$, $A(t)$ is a closed operator. The subspace V is $(A(t), B(t))$-invariant if and only if it is feedback invariant.*

Proof. Let V be $(A(t), B(t))$-invariant i.e.

$$A(t)(V \cap \mathcal{D}) \subseteq V + \operatorname{ran} B(t). \tag{15}$$

Lemma 3.2 implies that there exists an $A(t)$-bounded feedback law $F(t)$ such that

$$[A(t) + B(t)F(t)](V \cap \mathcal{D}) \subseteq V, \tag{16}$$

i.e. V is feedback invariant.

Conversely, let V be feedback invariant and (16) holds. Since V is closed and each $A(t)$ is a closed operator, $V \cap \mathcal{D}$ is a closed subspace with respect to the graph norm $\|(x, A(t)x)\| = \|x\| + \|A(t)x\|$. By (16) for any $x \in V \cap \mathcal{D}$ there exists $v \in V$ such that

$$A(t)x = v - B(t)F(t)x = v + B(t)w, \quad w = F(t)(-x).$$

This gives (15) i.e. V is $(A(t), B(t))$-invariant. □

Theorem 3.3. *Let V be a closed subspace of X. The subspace V is closed-loop invariant if and only if there exists an $\lambda_0 \in \mathbb{R}$ such that for for all $\lambda > \lambda_0$*

$$[\lambda I - A(t)](V \cap \mathcal{D}) + \mathcal{B}^0 = V + \mathcal{B}^0. \tag{17}$$

Proof. Let V be $R_{A+BF}(t,s)$-invariant for an $A(t)$-bounded feedback law $F(t)$. By Corollary 3.1 we may assume that $\operatorname{ran} B(t)F(t)|_{V\cap\mathcal{D}} \subseteq \mathcal{B}^0$. Since V is closed-loop invariant, by Theorem 2.1 (c) there exists a real number λ_0 such that

$$[\lambda I - A(t) - B(t)F(t)](V \cap \mathcal{D}(A(t) + B(t)F(t))) = V, \quad \lambda \geq \lambda_0. \tag{18}$$

Since $\mathcal{D}(A(t) + B(t)F(t)) = \mathcal{D}$ and $\operatorname{ran} B(t)F(t)|_{V\cap\mathcal{D}} \subseteq \mathcal{B}^0$, (18) implies (17).

Conversely, assume that (17) holds. This implies that

$$A(t)(V \cap \mathcal{D}) \subseteq V + \mathcal{B}^0 = V + \operatorname{ran} B(t). \tag{19}$$

By Theorem 3.2, there exists an $A(t)$-bounded feedback law $F(t)$ such that

$$[A(t) + B(t)F(t)](V \cap \mathcal{D}) \subseteq V. \tag{20}$$

Similar to the proof of Corollary 3.1, we may assume that V satisfies $\operatorname{ran} B(t)F(t)|_{V\cap\mathcal{D}} \subseteq \mathcal{B}^0$ or $\operatorname{ran} B(t)F(t)|_{V\cap\mathcal{D}} \cap V = \{0\}$. Let $\lambda \in \mathbb{R}$ with $\lambda > \lambda_0$ such that $[\lambda, \infty) \subseteq \rho(A(t) + B(t)F(t))$. In virtue (17), every $x \in V$ can be written as

$$x = (\lambda I - A(t))v + B(t)u \tag{21}$$

for some $v \in V \cap \mathcal{D}$ and $B(t)u \in \mathcal{B}^0$. From (21), we have

$$B(t)u + B(t)F(t)v = x - (\lambda I - A(t) - B(t)F(t))v.$$

From (20), we have that $B(t)u + B(t)F(t)v \in V$, but this is only possible if $B(t)u + B(t)F(t)v = 0$. Since $\operatorname{ran} B(t)F(t)|_{V\cap\mathcal{D}} \subseteq \mathcal{B}^0$ and $B(t)u \in \mathcal{B}^0$, we have

$$x = (\lambda I - A(t) - B(t)F(t))v. \tag{22}$$

Multiplying two sides of (22) by $(\lambda I - A(t) - B(t)F(t))^{-1}$ gives $(\lambda I - A(t) - B(t)F(t))^{-1}V \subseteq V$. Theorem 3.2 concludes that V is $R_{A+BF}(t,s)$-invariant. □

Definition 3.7. *A closed subspace V of a Hilbert space X is said to be controlled invariant if V satisfies condition* (17)*.*

The controlled invariance is a property of the system operator $A(t)$ and the input operator $B(t)$. The dual concept is called *conditioned invariance* i.e. a property of the system operator $A(t)$ and the output operator $C(t)$.

Definition 3.8. *A closed subspace V of a Hilbert space X is said to be conditioned invariant if there exists a $G(\cdot)$ where $G(t) \in \mathcal{L}_s(Y, X)$ such that for all $t, s \geq 0$ satisfies $R_{A+GC}(t,s)V \subseteq V$.*

The following theorem shows the precise meaning of the duality of the controlled and conditioned invariance.

Theorem 3.4. *A closed subspace V of a Hilbert space X is controlled invariant for the system $(A(t), B(t))$ if and only if $V^\perp$ is conditioned invariant for the system $(B^*(t), A^*(t))$, where T^* denotes the adjoint operator of T.*

Proof. We have that $R_{A+BF}(t,s)V \subseteq V$ if and only if $R^*_{A+BF}(t,s)(V^\perp) \subseteq V^\perp$ and $R^*_{A+BF}(t,s) = R_{A^*+F^*B^*}(t,s)$ for all $t, s \geq 0$. This provides the assertion. □

4. An Application in Disturbance Decoupling Problem

In this section we give an application of the invariant theory in disturbance decoupling problem (DDP) of the non-autonomous linear control systems. We consider

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + D(t)q(t), \\ y(t) &= C(t)x(t), \end{aligned} \tag{23}$$

where $A(t)$, $B(t)$, $u(t)$ are as before, q is a disturbance, $q(.) \in L_2([0,t],Q)$, $y(t) \in Y$ is the output to be decoupled, $D(t)$ and $C(t)$ are bounded operators such that $D(t) \in \mathcal{L}_s(Q,X)$, and $C(t) \in \mathcal{L}_s(X,Y)$, respectively. The DDP is depicted by Figure 1. The DDP is to find, if possible, for the non-autonomous system (23) a feedback system of form $u(t) = F(t)x(t)$ such that in the closed-loop system y does not depend on the disturbance input q. In this case we call that the DDP (23) is solvable. Thus, the DDP is to design a feedback law $u(t) = F(t)x(t)$ such that the transfer from q to y is zero, i.e. $C(t)[\lambda I - (A(t) + B(t)F(t))]^{-1}D(t) = 0$. By (11) the later is equivalent to

$$C(t)\int_0^t R_{A+BF}(s, t-s)D(s)q(s)ds = 0, \quad q \in L_2([0,t],Q), \quad t \geq 0. \tag{24}$$

From Definition 3.5, it is clear that (24) holds if and only if

$$\overline{\mathcal{R}(R_{A+BF}|\operatorname{ran} D(t))} \subseteq \ker C(t). \tag{25}$$

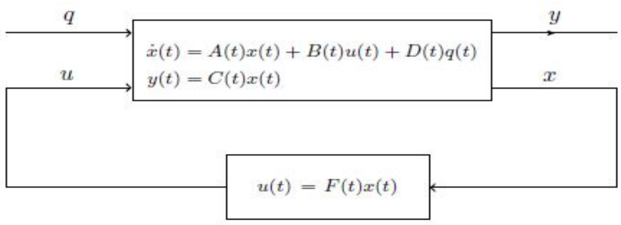

Figure 1. Disturbance Decoupling Problem

Lemma 4.1. *The DDP* (23) *is solvable if and only if there exists a controlled invariant subspace V such that* $\operatorname{ran} D(t) \subseteq V \subseteq \ker C(t)$ *for all $t \geq 0$.*

Proof. The DDP is solvable if and only if the equation (25) holds. By choosing $V = \overline{\mathcal{R}(R_{A+BF}|\operatorname{ran} D(t))}$, this subspace satisfies the assertions. □

For the closed subspace V of X, $\mathcal{V}^*(V)$ denotes the largest controlled invariant subspace contained in V. Existence of this subspace guarantees the solvability of DDP (23).

Theorem 4.1. *If $\mathcal{V}^*(\ker C(t))$ exists, then DDP* (23) *is solvable if and only if*

$$\operatorname{ran} D(t) \subseteq \mathcal{V}^*(\ker C(t)). \tag{26}$$

Proof. Necessity. Suppose that for every $t \geq 0$, $F(t) \in \mathcal{L}_s(X,U)$ satisfies (25). Notice that $\mathcal{R}_{A+BF}$ is $R_{A+BF}(t,s)$-invariant and is contained in $\ker C(t)$. Therefore

$$\operatorname{ran} D(t) \subseteq \overline{\mathcal{R}(R_{A+BF}|\operatorname{ran} D(t))} \subseteq \mathcal{V}^*(\ker C(t)).$$

Sufficiency. Since $\mathcal{V}^*(\ker C(t))$ exists, there exists an $F(t) \in \mathcal{L}_s(X, U)$ such that

$$[A(t) + B(t)F(t)](\mathcal{V}^*(\ker C(t)) \cap \mathcal{D}) \subseteq \mathcal{V}^*(\ker C(t)).$$

Part (d) of Theorem 3.1 and (26) provide

$$\overline{\mathcal{R}(R_{A+BF}|\operatorname{ran} D(t))} \subseteq \overline{\mathcal{R}(R_{A+BF}|\mathcal{V}^*(\ker C(t))} = \mathcal{V}^*(\ker C(t)) \subseteq \ker C(t).$$

This proves (25). □

In general, the subspace $\mathcal{V}^*(V)$ need not always exist, see Example E.9 of [1]. Moreover, since $R_{A+BF}(t,s)V \subseteq V$ implies $R_{A+BF}(t,s)\overline{V} \subseteq \overline{V}$, $\mathcal{V}^*(V)$ must be closed when it exists. We end this section with an example about the solvability of DDP of the non-autonomous heat system which is modified from Example 8 of [5].

Example 4.1. *Consider a non-autonomous heated rod which is heated around one point and due to some experimental setup is subject to disturbances in another region with the temperature at a certain measurement point be independent of the disturbances. The configuration is schematized in Figure 2 and the mathematical model is given by*

$$\begin{aligned} \frac{\partial x}{\partial t} &= a(t)\frac{\partial^2 x}{\partial \xi^2} + b(\xi,t)u(t) + d(\xi,t)q(t), \quad 0 < \xi < 1, \quad t \geq 0 \\ y(t) &= \int_0^1 c(\xi,t)x(\xi,t)d\xi, \quad x(0,t) = x(1,t) = 0, \end{aligned} \tag{27}$$

where for each t there are various shape-functions b, d, and c to approximate the sensor and control actuators.

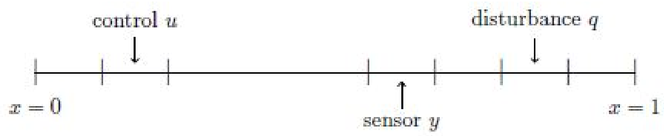

FIGURE 2. Configuration of Disturbance

The problem (27) can be formulated as a system of the form (23) on the Hilbert space $X = L_2(0,1)$, where

$$A(t) = a(t)\frac{d^2}{dx^2}, \quad B(t) = b(\cdot,t), \quad D(t) = d(\cdot,t), \quad C(t) = \langle \cdot, c(\cdot,t)\rangle,$$

with $A(t)$ is self-adjoint on $\mathcal{D} = \{x \in X : x, \frac{dx}{d\xi}$ absolutely continuous $\frac{d^2x}{d\xi^2} \in X, x(0) = x(1) = 0\}$ and eigenvectors $\phi_n(\xi) = \sqrt{2}\sin n\pi\xi$, $n \in \mathbb{N}$. For each $t \geq 0$, set $b = b(\cdot,t)$, $d = d(\cdot,t)$, and $c = c(\cdot,t)$ are elements of X. From Corollary 6.6 of [5], $\mathcal{V}^*(\ker C(t))$ will exist if either of the following conditions hold

$$\langle R(t,s)b, c\rangle = 0, \; t, s \geq 0, \tag{28}$$

$$\overline{\mathcal{R}(R|\operatorname{ran} C(t))} = \overline{\mathcal{R}(A(t)|\operatorname{ran} C(t))}, \qquad \langle B(t), A(t)^k c\rangle = 0, \quad k \geq 0 \tag{29}$$

$$\langle B(t), A(t)^k c\rangle = 0, \; k = 0, \ldots, p-1 \qquad \langle B(t), A(t)^p c\rangle \neq 0. \tag{30}$$

In case the system $(A(t), B(t))$ is approximately controllable, (28) implies that $d = 0$. Thus, (28) is not true in general. The condition (29) is not fulfilled in general, since $\overline{\mathcal{R}(R|\operatorname{ran} C(t))} = \overline{\mathcal{R}(A(t)|\operatorname{ran} C(t))}$ holds only if d is an eigenvector of $A(t)$. This provides that (30) with $p = 0$ is the possibility satisfying the situation. Moreover, in virtue of Theorem 4.1, the sufficiency for disturbance decoupling are

$$\langle b, c\rangle \neq 0, \quad \langle d, c\rangle = 0, \quad c \in \mathcal{D}. \tag{31}$$

The appropriate feedback law is given by $u(t) = F(t)x(t)$,

$$F(t) = \langle \cdot, f(t) \rangle, \quad f(t) = (\alpha c - A(t)c)/\langle d, c \rangle, \quad \alpha \quad \text{constant.}$$

The physical interpretation of the conditions (31) is that for each time t the shape function c is smooth and bounded on the interior of $[0, 1]$, the shape functions d and c do not overlap, and b and c must overlap.

5. Conclusions

The sufficient and necessary conditions for the invariance under the C_0-quasi semigroups and its generator can be identified. The relation-invariance of both can also be investigated. Closed loop invariant, open-loop invariant, feedback invariant, $(A(t), B(t))$-invariant, conditioned invariant, and controlled invariant are the types of the invariance of non-autonomous control systems with respect to the C_0-quasi semigroups. The system-invariance is applicable to solve the DPP of the non-autonomous control systems. The sufficiency for the solvability of the DPP is identified by the largest controlled invariant subspace of kernel of output operator.

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