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ON THE ULAM-HYERS-RASSIAS STABILITY FOR A BOUNDARY VALUE PROBLEM OF IMPLICIT ψ-CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION

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Abstract. The main purpose of this paper is to study the existence and uniqueness of a nonlinear implicit ψ-Caputo fractional order integro-differential boundary value problem using Schauder's and Banach's fixed point theorems. Besides, we study its stability using Ulam-Hyers-Rassias stability type. Finally, we demonstrate our main findings, with a particular case example included to show the significance of our results.

Keywords: Implicit fractional-orders differential equation, ψ-Caputo derivative, existence results, boundary value problems, Green's function, Ulam stability.

AMS Subject Classification: Primary 26A33; Secondary 34K45, 47G10.

1. Introduction

The topic of fractional calculus generalizes the integer-order integration and differentiation concepts to an arbitrary (real or complex) order. Over the most recent couple of many years, fractional-order models were observed to be more sufficient than integer-order models and had extensive applications in the mathematical modeling of real-world phenomena occurring in scientific and engineering disciplines, such as physics, biophysics, chemistry, biology, medical sciences, ecology, monetary financial aspects, and so on. Though, a large portion of researches have been led by utilizing fractional derivatives that for the most part depend on Riemann-Liouville, Hadamard, Katugampola, Grunwald Letnikov, and Caputo approaches. For more information about the developments of the theory of fractional differential equations, one can investigate to the monographs of Kilbas et al [15],

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Miller and Ross [19], Oldham [20], Pudlubny [22], Sabatier et al [25], and the references therein.

Fractional derivatives of a function with respect to function ψ have been considered in the classical articles as a generalization of Riemann-Liouville derivative. This fractional derivative is called the ψ-fractional derivative and it differs from the classical one as the kernel appears in terms of ψ. Recently, this derivative has been reassessed by Almeida in [7], where the Caputo-type regularization of the existing definition and some interesting properties are provided. Several properties and applications of the ψ-caputo fractional derivatives could be found and realized in ([1], [5], [15], [16], [17], [21]), and the references therein.

On the other hand, the stability problem of differential equations was discussed by Ulam in [26]. From there on, Hyers in [13] fostered the idea of Ulam stability in the case of Banach spaces. Rassias gave an impressive speculation of the Ulam–Hyers (UH) stability of mappings by considering variables. His methodology was alluded to as Ulam–Hyers–Rassias (UHR) stability [23]. Recently, the Ulam stability problem of implicit differential equations was extended into fractional implicit differential equations by some authors [27]. A progression of papers was committed to the examination of existence, uniqueness and (UH) stability of solutions of the fractional differential equations within different kinds of fractional derivatives.

Inspired by the recent progresses in ψ–fractional calculus, we study in this work the existence, uniqueness, and Ulam-Hyers type stability for the following nonlinear implicit ψ-Caputo fractional order integro-differential boundary value problem *CIFDP*:

$$\begin{cases} {}^cD^{\alpha,\psi}y(t) = f(t, y(t), {}^cD^{\beta,\psi}y(t), \int_0^t k(t,s) {}^cD^{\alpha,\psi}y(s)ds), \ t \in I = [0;T], \\ y(0) = y_\circ, \ y(T) = y_T, \end{cases} \tag{1}$$

where ${}^cD^{\alpha,\psi}$ is the ψ-Caputo fractional derivative of order $\alpha \in (0,1]$, $f : I \times R^3 \to R$, and y_o, y_T are constant real numbers.

First, we show the existence and uniqueness of the solution of our model by using Schauder's and Banach's fixed point theorems. Then, we study the Ulam-Hyers stability of solution. In addition, numerical example is given to demonstrate the application of our main results. Finally, we present a conclusion that summarizes what is done in this paper.

The results we just established concerning existence of solution and its stability also hold to special cases. These fractional derivative classes are created by selecting an appropriate value for $\psi(t)$ and taking into account the value of β.
In particular, we can deduce some existence results from our approach as follows:

- When $\psi(t) = t$, the obtained outcomes in the current paper incorporates the investigation of [11] relating to

$$ {}^cD^{\alpha}y(t) = f\big(t, y(t), {}^cD^{\beta}y(t), \int_0^t k(t,s) {}^cD^{\alpha}y(s)ds\big), \ t \in I = [0;T], \tag{2}$$

$$y(0) = y_\circ, \ y(T) = y_T.$$

- Also, if $f(t,x,y,z) = f(t,x,y)$ and $\beta = \alpha$, in Equation (2), then we have the implicit fractional-order differential equation

$$ {}^cD^{\alpha}y(t) = f\big(t, y(t), {}^cD^{\alpha}y(t)\big), \ t \in I = [0;T], \tag{3}$$

$$y(0) = y_\circ, \ y(T) = y_T.$$

which is the same result obtained in [9].

2. Preliminaries

In the following, we introduce some notations, definitions, lemmas, and theorems that are important in developing our results throughout this paper.

Definition 2.1. [7] *For any real number $\alpha > 0$, the left-sided ψ-Riemann-Liouville fractional integral of order α for an integrable function $u : I \to R$ with respect to another function $\psi : I \to R$, which is an increasing differentiable function such that $\psi'(t) \neq 0$ for all $t \in I$ is defined by:*

$$I^{\alpha,\psi}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s)\,ds,$$

where Γ is the classical Euler Gamma function.

Definition 2.2. [7] *If $n \in N$ and $\psi, u \in C^n(I,R)$ are two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in I$, then the left-sided ψ-Caputo fractional derivative of a function u of order α is defined by:*

$$\begin{aligned} {}^cD^{\alpha,\psi}u(t) &= I^{n-\alpha,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}u_\psi^{[n]}(s)\,ds, \end{aligned}$$

where $u_\psi^{[n]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t)$ and $n = [\alpha]+1$ for $\alpha \notin N$, and $n = \alpha$ for $\alpha \in N$.

Remark 2.1. *Let $\alpha > 0$, then the differential equation $({}^cD^{\alpha,\psi}_{a+}h)(t) = 0$ has solution*

$$h(t) = c_0 + c_1(\psi(t)-\psi(0)) + c_2(\psi(t)-\psi(0))^2 + ... + c_{n-1}(\psi(t)-\psi(0))^{n-1},$$

where $c_i \in R$, $i = 0,1,2,...,n-1$, $n = [\alpha]+1$.

Lemma 2.1. [8] *Let $\alpha, \beta \in R^+$, and $f(t) \in L_1(I)$. Then, $I^{\alpha,\psi}_{a+}I^{\beta,\psi}_{a+}f(t) = I^{\beta,\psi}_{a+}I^{\alpha,\psi}_{a+}f(t) = I^{\alpha+\beta,\psi}_{a+}f(t)$, and $(I^{\alpha,\psi}_{a+})^n f(t) = I^{n\alpha,\psi}_{a+}f(t)$, where $n \in N$.*

Definition 2.3. [3] *Let X be any space and let $f : X \to X$. A point $x \in X$ is called a fixed point for mapping f if $x = f(x)$.*

Theorem 2.1. [6] ***Banach fixed point theorem.***
Let C be a non-empty closed subset of a Banach space X. Then any contraction mapping T of C into itself has a unique fixed point.

Theorem 2.2. [18] ***Schauder's fixed-point Theorem***
Let S be a convex subset of a Banach space B, let the mapping $T : S \to S$ be compact and continuous. Then, T has at least one fixed-point in S.

3. Main Results

Consider the *CIFDP* (1) under the following assumptions:

(H_1) The nonlinear function $f : I \times R^3 \to R$ is continuous and there exist $\lambda \in C(I, R_+)$ with norm $\|\lambda\|$ such that:

$$|f(t,u_1,u_2,u_3) - f(t,v_1,v_2,v_3)| \le \lambda(t)\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right),$$

$\forall\ t \in I,\ u_i, v_i \in R,\ (i = 1, 2, 3)$.

(H_2) The function $k(t, s)$ is continuous for all $(t, s) \in I \times I$, and there is a positive constant K such that:

$$\max_{t,s\in[0,T]} |k(t, s)| = K.$$

(H_3) The function $\Phi \in C(I, R_+)$ is increasing and there exists $\lambda_\Phi > 0$ such that, for each $t \in J$

$$I^{\alpha}\ \Phi(t) \le \lambda_\Phi\ \Phi(t).$$

Remark 3.1. *From assumption* (H_1)*, we have*

$$\begin{aligned}|f(t,u_1,u_2,u_3)| - |f(t,0,0,0)| &\le |f(t,u_1,u_2,u_3) - f(t,0,0,0)|\\ &\le \lambda(t)(|u_1| + |u_2| + |u_3|),\end{aligned}$$

If $F = \sup_{t\in I} |f(t, 0, 0, 0)|$*, then*

$$|f(t,u_1,u_2,u_3)| \le F + \lambda\,(t)\,(|u_1| + |u_2| + |u_3|).$$

Lemma 3.1. *If the solution of the CIFDP* (1) *exists, then it can be represented by the following integral equation*

$$y(t) = h(t) + \int_0^T \psi'(s)G(t,s)u(s)ds, \tag{4}$$

where u is the solution of the following ψ-Caputo fractional integral equation

$$u(t) = f(t, h(t) + \int_0^T \psi'(s)G(t,s)u(s)ds, I^{\alpha-\beta,\psi}u(t), \int_0^t k(t,s)u(s)ds), \tag{5}$$

$G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)}\left[(\psi(t)-\psi(s))^{\alpha-1} - \frac{\psi(t)-\psi(0)}{\psi(T)-\psi(0)}(\psi(T)-\psi(s))^{\alpha-1}\right] & if\ \ 0 \le s \le t \le T,\\ -\frac{(\psi(t)-\psi(0))}{\Gamma(\alpha)(\psi(T)-\psi(0))}(\psi(T)-\psi(s))^{\alpha-1} & if\ 0 \le t \le s \le T\end{cases} \tag{6}$$

with

$$G_\circ := \max\{|G(t,s)|,\ (t,s) \in I \times I\},$$

and

$$h(t) = y_\circ + \frac{(y_T - y_\circ)(\psi(t) - \psi(0))}{\psi(T) - \psi(0)}. \tag{7}$$

Proof. It is clear that ${}^cD^{\beta,\psi}y(t) = I^{\alpha-\beta,\psi}\ {}^cD^{\alpha}y(t)$ for all $t \in I$. So, if $y\,(t)$ is a solution of equation (1), then $\forall t \in I$, we have

$${}^cD^{\alpha,\psi}y(t) = f(t, y(t), I^{\alpha-\beta,\psi}\ {}^cD^{\alpha,\psi}y(t), \int_0^t k(t,s){}^cD^{\alpha,\psi}y(s)ds).$$

Let ${}^cD^{\alpha,\psi}y(t) = u(t)$, then equation (1) becomes:

$$u(t) = f(t, y(t), I^{\alpha-\beta,\psi}u(t), \int_0^t k(t,s)u(s)ds)$$

and

$$y(t) = c_\circ + c_1(\psi(t) - \psi(0)) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left[\psi(t) - \psi(s)\right]^{\alpha-1} u(s)ds$$

from (1), we get that $c_o = y_o$, and

$$c_1 = -\frac{1}{\Gamma(\alpha)(\psi(T)-\psi(0))}\int_0^T \psi'(s)\left[\psi(T)-\psi(s)\right]^{\alpha-1}u(s)ds + \frac{(y_T - y_\circ)}{(\psi(T)-\psi(0))}.$$

Hence, the integral solution of (1) is given by:

$$\begin{aligned} y(t) &= y_\circ + \frac{(\psi(t)-\psi(0))}{\psi(T)-\psi(0)}(y_T - y_\circ) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left[\psi(t)-\psi(s)\right]^{\alpha-1}u(s)ds \\ &\quad - \frac{\psi(t)-\psi(0)}{\Gamma(\alpha)\left[\psi(T)-\psi(0)\right]}\int_0^T \psi'(s)\left[\psi(T)-\psi(s)\right]^{\alpha-1}u(s)ds \\ &= y_\circ + \frac{(\psi(t)-\psi(0))}{\psi(T)-\psi(0)}(y_T - y_\circ) \\ &\quad - \frac{\psi(t)-\psi(0)}{\Gamma(\alpha)(\psi(T)-\psi(0))}\int_t^T \psi'(s)\left[\psi(T)-\psi(s)\right]^{\alpha-1}u(s)ds. \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left[\left[\psi(t)-\psi(s)\right]^{\alpha-1} - \frac{\psi(t)-\psi(0)}{\psi(T)-\psi(0)}\left[\psi(T)-\psi(s)\right]^{\alpha-1}\right]u(s)ds. \end{aligned}$$

If $h(t) = y_\circ + \frac{(y_T - y_\circ)(\psi(t)-\psi(0))}{\psi(T)-\psi(0)}$, then the solution of (1) is also a solution of (4). □

Definition 3.1. *By a mild solution of the CIFDP* (1)*, we mean a function* $u \in C(I,R)$ *that satisfies the integral equation* (5)*.*

Definition 3.2. *Let* $\mathcal{A} : C(I,R) \to C(I,R)$ *be an operator defined by:*

$$\mathcal{A}(y(t)) = h(t) + \int_0^T \psi'(s)G(t,s)v(s)ds,$$

where $v(s) \in C(I,R)$ *satisfies the implicit fractional equation*

$$v(t) = f(t, h(t) + \int_0^T \psi'(s)G(t,s)v(s)ds, I^{\alpha-\beta,\psi}v(t), \int_0^t k(t,s)v(s)ds),$$

where $h(t) = y_\circ + \frac{(y_T - y_\circ)(\psi(t)-\psi(0))}{\psi(T)-\psi(0)}$ *for all* $t \in I = [0,T]$*, and* $G(t,s)$ *is the Green's function defined in* (6)*.*

3.1. **Existence result via Schauder's fixed point theorem.** Our first result is based on the existence of at least one mild solution for the *CIFDP* (1) using Shauder's fixed point theorem.

Lemma 3.2. *The operator* $\mathcal{A}$ *is continuous.*

Proof. Define the nonempty, bounded, closed and convex ball

$$B_r = \{y \in C(I,R) : \|y\| \le r\}, \text{ with } r \ge \frac{|y_T| + \frac{G_\circ\, TF}{1 - \left[\frac{\|\lambda\|(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \|\lambda\|KT\right]}}{1 - \frac{G_\circ F\|\lambda\|}{1 - \left[\frac{\|\lambda\|(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \|\lambda\|KT\right]}}$$

Consider a sequence $\{y_n\} \subset B_r$ that converges to $y \in B_r$, i.e., $y_n \to y$ in B_r as $n \to \infty$. Then,

$$|\mathcal{A}(y_n(t)) - \mathcal{A}(y(t))| \le \int_0^T \psi'(t)\; |G(t,s)|\; |u_n(s) - u(s)|ds,$$

where $u_n, u \in C(I, R)$, such that

$$u_n(t) = f(t, y_n(t), I^{\alpha-\beta,\psi} u_n(t), \int_0^t k(t,s) u_n(s) ds),$$
$$u(t) = f(t, y(t), I^{\alpha-\beta,\psi} u(t), \int_0^t k(t,s) u(s) ds).$$

By assumption (H_1), we have

$$\begin{aligned}
&|u_n(t) - u(t)| \\
&= |f(t, y_n(t), I^{\alpha-\beta} u_n(t), \int_0^t k(t,s) u_n(s) ds) - f(t, y(t), I^{\alpha-\beta} u(t), \int_0^t k(t,s) u(s) ds)| \\
&\leq \ \lambda(t) \ \Bigg[|y_n(t) - y(t)| + \int_0^t \frac{\psi(s)'(\psi(t) - \psi(s))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |u_n(s) - u(s)| \ ds \\
&\qquad\qquad + \int_0^t |k(t,s)| \ |u_n(s) - u(s)| ds \Bigg].
\end{aligned}$$

Taking supremum for all $t \in I$, we get

$$||u_n - u|| \leq \|\lambda\| \left[\|y_n - y\| + \frac{(\psi(T) - \psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \|u_n - u\| + KT \|u_n - u\| \right].$$

Thus,

$$\|u_n - u\| \leq \frac{\|\lambda\|}{1 - \|\lambda\| \left[K \ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right]} \|y_n - y\|.$$

Since, $y_n \to y$, then we get $u_n(t) \to u(t)$ as $n \to \infty$ for each $t \in I$. And there exist $\varepsilon > 0$ such that $|u_n(t)| \leq \frac{\varepsilon}{2}$ and $|u(t)| \leq \frac{\varepsilon}{2}$ for each $t \in I$. By applying Lebesgue Dominated Convergence Theorem we get

$$\begin{aligned}
|\mathcal{A}(y_n(t)) - \mathcal{A}(y(t))| &\leq \int_0^T \psi'(t) \ |G(t,s)| \ |u_n(s) - u(s)| ds \\
&\leq |G(t,s)| (\psi(T) - \psi(0)) |u_n(s) - u(s)| \\
&\leq |G(t,s)| (\psi(T) - \psi(0)) [|u_n(s)| + |u(s)|] \\
&\leq \varepsilon |G(t,s)| (\psi(T) - \psi(0))
\end{aligned}$$

Hence, $\|\mathcal{A}(y_n) - \mathcal{A}(y)\| \to 0$ as $n \to \infty$ and consequently $\mathcal{A}$ is continuous. □

Lemma 3.3. *The operator $\mathcal{A}$ maps bounded sets in B_r into bounded sets in B_r.*

Proof. By assumption (H_2), we have

$$|\mathcal{A}(y(t))| = |h(t) + \int_0^T \psi'(t) G(t,s) v(s) ds| \leq |h(t)| + \int_0^T \psi'(t) |G(t,s)||v(s)| ds,$$

where $v(t) = f(t, y(t), I^{\alpha-\beta,\psi} v(t), \int_0^t k(t,s) v(s) ds)$ such that:

$$\begin{aligned}
|v(t)| &= |f(t, y(t), I^{\alpha-\beta,\psi} v(t), \int_0^t k(t,s) v(s) ds)| \\
&\leq \ F + \lambda(t) \left(|y(t)| + \int_0^t \frac{\psi(s)'(\psi(t) - \psi(s))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |v(s)| \ ds + \int_0^t |k(t,s)||v(s)| ds \right).
\end{aligned}$$

Taking supremum for all $t \in I$, we have

$$\|v\| \leq \ F + \|\lambda\| \left[\|y\| + \frac{(\psi(T) - \psi(0))^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} \|v\| + K\|v\|T \right].$$

Thus,

$$\|v\| \leq \ \frac{F + \|\lambda\| r}{1 - \left[\frac{\|\lambda\|(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \|\lambda\|KT \right]},$$

and

$$|h(t)| = \left| y_\circ + \frac{(\psi(t) - \psi(0))}{\psi(T) - \psi(0)} (y_T - y_\circ) \right| \leq |y_T|.$$

Thus, for each $t \in I$ we have

$$|\mathcal{A}(y(t))| \leq |y_T| + \frac{G_\circ\, T(F + \|\lambda\| r)}{1 - \left[\frac{\|\lambda\|(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \|\lambda\|KT \right]} \leq r.$$

Taking supermum for $t \in I$, we have $||\mathcal{A}(y)|| \leq r$ for any $y \in B_r$. Hence, $\mathcal{A}(B_r) \subset B_r$. □

Lemma 3.4. *The operator $\mathcal{A}$ is relatively compact.*

Proof. Suppose that for every $\varepsilon > 0$, there exist $\delta > 0$ and $t_1, t_2 \in I$, with $t_1 < t_2$ and $|t_2 - t_1| < \delta$. Then, we have

$$\begin{aligned}
&|\mathcal{A}(y(t_2)) - \mathcal{A}(y(t_1))| \\
&\leq |h(t_2) - h(t_1)| + \int_0^T \psi'(s) \left| G(t_2, s) - G(t_1, s) \right| |v(s)| ds \\
&\leq |h(t_2) - h(t_1)| + \|v\| \int_0^T \psi'(s) \left| G(t_2, s) - G(t_1, s) \right| ds \\
&\leq |h(t_2) - h(t_1)| + \frac{F + \|\lambda\| r}{1 - \left[\frac{\|\lambda\|(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \|\lambda\|KT \right]} \int_0^T \left| G(t_2, s) - G(t_1, s) \right| d\psi(s).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality is not dependent on y and tends to zero. Consequently,

$$|\mathcal{A}(y(t_2)) - \mathcal{A}(y(t_1))| \to 0, \ \forall \ |t_2 - t_1| \to 0.$$

Thus, $\{Ay\}$ is equi-continuous on B_r, and T is a compact operator by the Arzela-Ascoli Theorem [10]. □

Now, we present our first result which is based on Shauder's fixed point theorem.

Theorem 3.1. *If assumptions (H_1) & (H_2) hold, and if*

$$\frac{\|\lambda\|}{1 - \frac{(\psi(T)-\psi(0))^{\alpha-\beta}\|\lambda\|}{\Gamma(\alpha-\beta+1)} + \|\lambda\|KT} < 1,$$

then the CIFDP (1) has at least one mild solution on $I = [0, T]$.

Proof. By using Lemmas 3.1, 3.2, 3.3, 3.4 and applying Shauder's fixed point theorem on the continuous and completely continuous operator $\mathcal{A} : C(I; R) \to C(I; R)$, we deduce that the *CIFDP* (1) has at least one mild solution on $I = [0, T]$. □

3.2. **Uniqueness result via Banach's fixed point theorem.** Our second result is based on the uniqueness of the mild solution for *CIFDP* (1) by using Banach's fixed point theorem.

Lemma 3.5. *The operator* $\mathcal{A} : C(I, R) \to C(I, R)$ *defined as*

$$\mathcal{A}(y(t)) = h(t) + \int_0^T \psi'(s)G(t,s)v(s)ds$$

is a contraction.

Proof. Suppose that the assumptions of Theorem (3.1) hold, and consider the continuous functions $x, y \in C(I, R)$. Then, for any $t \in I$, we have

$$\mathcal{A}(x(t)) - \mathcal{A}(y(t)) = \int_0^T \psi'(t)\, G(t,s)u(s)ds - \int_0^T \psi'(t)\, G(t,s)v(s)ds, \tag{8}$$

where $u, v \in C(I, R)$ such that

$$u(t) = f(t, x(t), I^{\alpha-\beta,\psi}u(t), \int_0^t k(t,s)u(s)ds),$$
$$v(t) = f(t, y(t), I^{\alpha-\beta,\psi}v(t), \int_0^t k(t,s)v(s)ds).$$

Then, for any $t \in I$ we have

$$|\mathcal{A}(x(t)) - \mathcal{A}(y(t))| \leq \int_0^T \psi'(t)\, |G(t,s)|\ |u(s) - v(s)|ds.$$

But, by conditions (H_1) and (H_2), we have

$$\begin{aligned}
|u(t) - v(t)| &= \Big| f(t, x(t), I^{\alpha-\beta,\psi}u(t), \int_0^t k(t,s)u(s)ds) \\
&\qquad - f(t, y(t), I^{\alpha-\beta,\psi}v(t), \int_0^t k(t,s)v(s)ds) \Big| \\
&\leq \lambda(t) \left(|x(t) - y(t)| + \int_0^t \frac{\psi(s)'(\psi(t) - \psi(0))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |u(s) - v(s)|\ ds \right. \\
&\qquad \left. + \int_0^t |k(t,s)|\ |u(s) - v(s)|ds \right) \\
&\leq \|\lambda\| \left(\|x - y\| + \frac{(\psi(T) - \psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \|u - v\| + K\|u - v\|\ T \right).
\end{aligned}$$

Taking supremum for all $t \in T$, we have

$$\|u - v\| \leq \frac{\|\lambda\|}{1 - \|\lambda\| \left[K\ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right]} \|x - y\|.$$

Thus,

$$|\mathcal{A}(x(t)) - \mathcal{A}(y(t))| \leq \frac{G_\circ\ \|\lambda\|\ T}{1 - \|\lambda\| \left[K\ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right]} \|x - y\|.$$

Also, if we take supermum for $t \in I$, we get

$$\|\mathcal{A}(x) - \mathcal{A}(y)\| \leq \left(\frac{G_\circ\ \|\lambda\|\ T}{1 - \|\lambda\| \left[K\ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right]} \right) \|x - y\|.$$

Now, if $\frac{G_\circ \ \|\lambda\| \ T}{1-\|\lambda\|\left[K \ T+\frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]} < 1$, then the operator $\mathcal{A}$ is a contraction. □

Theorem 3.2. *If assumptions* (H_1) *and* (H_2) *hold, and if*

$$\frac{G_\circ \ \|\lambda\| \ T}{1-\|\lambda\|\left[K \ T+\frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]} < 1, \tag{9}$$

then the CIFDP (1) *has a unique mild solution on* $I = [0, T]$.

Proof. It is already proven in Theorem (3.1) that *CIFDP* (1) has at least one mild solution. In addition, Lemma 11 proves that the operator $\mathcal{A}$ is a contraction. Then, by Banach's fixed point theorem, we deduce that operator $\mathcal{A}$ has a unique fixed point which is also a unique mild solution of the *CIFDP* (1) on $I = [0, T]$. □

3.3. **Stability results via Ulam-Hyers type.** In the following, we consider the Ulam stability for *CIFDP* (1). Let $\varepsilon > 0$, $\Phi : I \to R_+$ be a continuous function, and consider the following inequalities:

$$|{}^cD^{\alpha,\psi}y(t) - f(t, y(t), {}^cD^{\beta,\psi}y(t), \int_0^t k(t,s){}^cD^{\alpha,\psi}y(s)ds)| \leq \varepsilon(t), \ t \in I \tag{10}$$

$$|{}^cD^{\alpha,\psi}y(t) - f(t, y(t), {}^cD^{\beta,\psi}y(t), \int_0^t k(t,s){}^cD^{\alpha,\psi}y(s)ds)| \leq \Phi(t), \ t \in I \tag{11}$$

$$|{}^cD^{\alpha,\psi}y(t) - f(t, y(t), {}^cD^{\beta,\psi}y(t), \int_0^t k(t,s){}^cD^{\alpha,\psi}y(s)ds)| \leq \varepsilon \ \Phi(t), \ t \in I. \tag{12}$$

Definition 3.3. [13] *The CIFDP* (1) *is Ulam-Hyers stable if there exists a real number* $c_f > 0$ *such that there exists a solution* $x \in C(I, R)$ *of* (1) *such that*

$$|y(t) - x(t)| \leq \varepsilon \ c_f \quad \forall \ t \in I.$$

for each solution $y \in C(I, R)$ *of the inequality (10).*

Definition 3.4. [13] *The CIFDP* (1) *is generalized to be Ulam-Hyers stable if there is* $c_f \in C(R_+, R_+)$ *with* $c_f(0) = 0$ *so that there is a solution* $x \in C(I, R)$ *of CIFDP* (1) *with*

$$|y(t) - x(t)| \leq c_f(\varepsilon), \ \forall \ t \in I.$$

for each $\varepsilon > 0$ *and for each solution* $y \in C(I, R)$ *of the inequality (11).*

Definition 3.5. [23] *The CIFDP* (1) *is Ulam-Hyers-Rassias stable with respect to* Φ *if there exists a real number* $c_{f,\Phi} > 0$ *such that there is a solution* $x \in C(I, R)$ *of* (1) *with*

$$|y(t) - x(t)| \leq \varepsilon \ c_{f,\Phi}\Phi(t), \ \forall \ t \in I.$$

for each $\varepsilon > 0$ *and for each solution* $y \in C(I, R)$ *of the inequality (12).*

Definition 3.6. [23] *The CIFDP* (1) *is generalized Ulam-Hyers-Rassias stable with respect to* Φ *if the actual number* $c_{f,\Phi} > 0$ *exists in such a way that for each solution* $y \in C(I, R)$ *of the inequality (12) there is a solution* $x \in C(I, R)$ *of* (1) *with the solution* $x \in C(I, R)$ *of the inequality.*

$$|y(t) - x(t) > | \leq c_{f,\Phi}\Phi(t), \ \forall \ t \in I.$$

3.3.1. *Ulam-Hyers stability.* In the following, we present the Ulam-Hyers stability result for *CIFDP* (1).

Theorem 3.3. *Suppose that the assumptions of Theorem* (3.2) *are satisfied. Then, CIFDP* (1) *is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and let $z \in C(I, R)$ be a function which satisfies inequality (10), such that

$$|{}^cD^{\alpha,\psi}z(t) - f(t, z(t), {}^cD^{\beta,\psi}z(t), \int_0^t k(t,s){}^cD^{\alpha,\psi}z(s)ds)| \leq \varepsilon,\ \forall\ t \in I, \tag{13}$$

and let $y \in C(I, R)$ be the unique solution of $CIFDP$ (1) which is by Lemma 3.1 is equivalent to the fractional order integral equation

$$y(t) = h(t) + \int_0^T \psi'(t)\, G(t,s)u(s)ds,$$

where u is the solution of the functional integral equation

$$u(t) = f(t, h(t) + \int_0^T \psi'(t)\, G(t,s)u(s)ds, {}^cI^{\alpha-\beta,\psi}u(t), \int_0^t k(t,s)u(s)ds).$$

Taking the left-sided ψ-Riemann-Liouville fractional integral $I^{\alpha,\psi}$ on both sides of inequality (13), and then integrating, we get

$$|z(t) - h(t) - \int_0^T \psi'(t)\, G(t,s)v(s)ds| \leq \frac{\varepsilon\ (\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)}. \tag{14}$$

For each $t \in I$, we have

$$\begin{aligned}
|z(t) - y(t)| &= |z(t) - h(t) - \int_0^T \psi'(t)\, G(t,s)u(s)ds| \\
&\leq |z(t) - h(t) - \int_0^T \psi'(t)\, G(t,s)v(s)ds + \int_0^T \psi'(t)\, G(t,s)v(s)ds \\
&\qquad - \int_0^T \psi'(t)\, G(t,s)u(s)ds| \\
&\leq \frac{\varepsilon\ (\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} + \int_0^T \psi'(t)\, |G(t,s)|\ |v(s) - u(s)|ds \\
&\leq \frac{\varepsilon\ (\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} + G_\circ \|u - v\| \left(\psi(T) - \psi(0)\right).
\end{aligned}$$

But, from Lemma 3.5, we have

$$\|u - v\| \leq \frac{\|\lambda\|}{1 - \|\lambda\| \left[K\ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]} \|z - y\|,$$

which implies that for each $t \in I$

$$\|z - y\| \leq \frac{\varepsilon\ (\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} + \frac{G_0\|\lambda\| \left(\psi(T) - \psi(0)\right)}{1 - \|\lambda\| \left[K\ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]} \|z - y\|.$$

Hence,

$$\|z - y\| \leq \frac{\varepsilon\ (\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} \left[1 - \frac{G_0\|\lambda\| \left(\psi(T) - \psi(0)\right)}{1 - \|\lambda\| \left[K\ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]}\right]^{-1} = \varsigma\ \epsilon,$$

where $\varsigma = \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\left[1-\frac{G_0\|\lambda\|(\psi(T)-\psi(0))}{1-\|\lambda\|\left[K\ T+\frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]}\right]^{-1}$. Therefore, the *CIFDE* (1) is Ulam-Hyers stable. □

Indeed, if we put $\Phi(\varepsilon)=\varsigma\ \varepsilon$, then we get $\Phi(0)=0$ which yields that the *CIFDP* (1) is generalized Ulam-Hyers stable.

3.3.2. *Ulam-Hyers-Rassias stability.* In the following, we study the Ulam-Hyers-Rassias stability of *CIFDP* (1).

Theorem 3.4. *Assume that assumptions (H_1), (H_2), and (H_3) hold. Then, CIFDP* (1) *is Ulam-Hyers-Rassias stable with respect to Φ.*

Proof. Let $z \in C(I,R)$ be a solution of the inequality (12), i.e.,

$$|^{c}D^{\alpha,\psi}z(t)-f(t,z(t),^{c}D^{\beta,\psi}z(t),\int_0^t k(t,s)^{c}D^{\alpha,\psi}z(s)ds)|\le\varepsilon\ \Phi,\ \ t\in I.$$

In addition, let y be a solution of *CIFDP* (1), and let $u\in C(I,R)$ such that:

$$y(t)=h(t)+\int_0^T\psi'(s)\,G(t,s)u(s)ds,$$

where

$$u(t)=f(t,y(t),^{c}I^{\alpha-\beta,\psi}u(t),\int_0^t k(t,s)u(s)ds).$$

Operating $I^{\alpha,\psi}$ on both sides of inequality (12) and then integrating, we get

$$|z(t)-h(t)-\int_0^T\psi'(t)\,G(t,s)v(s)ds|\le\frac{\varepsilon}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\ \Phi(s)ds\le\varepsilon\mu_\Phi\Phi(t),$$

where $v\in C(I,R)$ such that

$$v(t)=f(t,z(t),I^{\alpha-\beta,\psi}v(t),\int_0^t k(t,s)v(s)ds).$$

Hence, for each $t\in I$, we have

$$\begin{aligned}|z(t)-y(t)|&=|z(t)-h(t)-\int_0^T\psi'(t)\,G(t,s)u(s)ds|\\&\le|z(t)-h(t)-\int_0^T\psi'(t)\,G(t,s)v(s)ds|\\&\qquad+|\int_0^T\psi'(t)\,G(t,s)v(s)ds-\int_0^T\psi'(t)\,G(t,s)u(s)ds|\\&\le\varepsilon\mu_\Phi\ \Phi(t)+\int_0^T\psi'(t)\,|G(t,s)|\ \ |v(s)-u(s)|ds\\&\le\varepsilon\mu_\Phi\ \Phi(t)+\int_0^T\psi'(t)\,|G(t,s)|\,|v(s)-u(s)|ds\\&\le\varepsilon\mu_\Phi\ \Phi(t)+G_o\,(\psi(T)-\psi(0))\,\|v-u\|\end{aligned}$$

But, from proof of Theorem 3.2, we have

$$\|u-v\|\le\frac{\|\lambda\|}{1-\|\lambda\|\left[K\ T+\frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]}\|z-y\|.$$

Then, for each $t \in I$

$$\|z-y\| \leq \varepsilon\mu_\Phi\ \Phi(t) + \frac{\|\lambda\| G_o\left(\psi\left(T\right)-\psi\left(0\right)\right)}{1-\|\lambda\|\left[K\ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]}\|z-y\|.$$

Thus,

$$\|z-y\| \leq \left[1-\frac{\|\lambda\| G_o\left(\psi\left(T\right)-\psi\left(0\right)\right)}{1-\|\lambda\|\left[K\ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]}\right]^{-1}\varepsilon\mu_\Phi\Phi(t) = c_\Phi\varepsilon\Phi(t),$$

where

$$c_\Phi = \left[1-\frac{\|\lambda\| G_o\left(\psi\left(T\right)-\psi\left(0\right)\right)}{1-\|\lambda\|\left[K\ T + \frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]}\right]^{-1}\mu_\Phi.$$

Therefore, the problem $CIFDP$ (1) is Ulam-Hyers-Rassias stable with respect to Φ. □

4. Numerical Example

Given the following *CIFDP*:

$$\begin{cases} {}^cD^{\frac{7}{5},e^{\sqrt{t}+1}}y\left(t\right) = \frac{\sqrt{t+1}}{6e^{t+1}}\left[\frac{3+y(t)+{}^cD^{\frac{6}{5},e^{\sqrt{t}+1}}y(t)+\int_0^1 e^{(t-s)}\ {}^cD^{\frac{7}{5},e^{\sqrt{t}+1}}y(s)ds}{1+y(t)+{}^cD^{\frac{6}{5},e^{\sqrt{t}+1}}y(t)+\int_0^1 e^{(t-s)}\ {}^cD^{\frac{7}{5},e^{\sqrt{t}+1}}y(s)ds}\right] \text{ for all } t\in[0,1], \\ y(0)=1, \quad \text{and } y(1)=1, \end{cases} \tag{15}$$

where $\alpha = \frac{7}{5}$, $\beta = \frac{6}{5}$, $T = 1$, and $\psi\left(t\right) = e^{\sqrt{t}+1}$, which is an increasing differentiable function such that $\psi'\left(t\right) = \frac{e^{\sqrt{t}+1}}{2\sqrt{t}} \neq 0\ \forall\ t \in [0,1]$.

Set

$$f(t,u,v,w) = \frac{\sqrt{t+1}}{6e^{t+1}}\left[\frac{3+|u|+|v|+|w|}{1+|u|+|v|+|w|}\right].$$

Obviously, f is a mutually continuous function. Besides, for any $u_1, v_1, w_1, u_2, v_2, w_2 \in R$, and $t \in [0,1]$ we have

$$|f(t,u,v,w) - f(t,u_1,v_1,w_1)| \leq \frac{1}{6e}\left[|u_1-u_2| + |v_1-v_2| + |w_1-w_2|\right].$$

Thus,

$$|f(t,u,v,w)| = \frac{\sqrt{t+1}}{6e^{t+1}}\left(3+|u|+|v|+|w|\right),\ F = \frac{1}{2e}, \text{ and } \|\lambda\| = \frac{1}{6e}.$$

Hence, condition (H_2) is satisfied with

$$\lambda(t) = \frac{\sqrt{t+1}}{2e^{t+1}},\ \|\lambda\| = \frac{1}{6e}, \text{ and } K = e.$$

It clear from Theorem 3.1 that the *CIFDP* has at least one mild solution on $[0,1]$ since the condition

$$\frac{\|\lambda\|}{1-\left[\frac{(\psi(T)-\psi(0))^{\alpha-\beta}\|\lambda\|}{\Gamma(\alpha-\beta+1)} + \|\lambda\| KT\right]} \approx \frac{\left(\frac{1}{6e}\right)}{\frac{5}{6}+\left(\frac{(e^2-e)^{1/5}}{6e\Gamma\left(\frac{6}{5}\right)}\right)} \approx 0.0825829 < 1.$$

In addition, the Green's function $G\left(t,s\right)$ is given by:

$$G\left(t,s\right)=\begin{cases}\dfrac{\left(e^{\sqrt{t}+1}-e^{\sqrt{s}+1}\right)^{2/5}-\frac{\left(e^{\sqrt{t}+1}-e\right)\left(e^{\sqrt{t}+1}-e^{\sqrt{s}+1}\right)^{2/5}}{e^{2}-e}}{\Gamma\left(\frac{7}{5}\right)} & \text{if } \ 0\leq s\leq t\leq 1\\ \dfrac{\left(e^{2}-e^{\sqrt{s}+1}\right)^{2/5}\left(e^{\sqrt{t}+1}-e\right)}{(e^{2}-e)\Gamma\left(\frac{7}{5}\right)} & \text{if } \ 0\leq t\leq s\leq 1\end{cases}$$

and it is clear that $G_{\circ}=\max\{|G(t,s)|,\ (t,s)\in I\times I\}$, then $G_{o}<1$.

Thus, if we check the condition (9) we get that

$$\frac{G_{\circ}\ \|\lambda\|\ T}{1-\|\lambda\|\left[K\ T+\frac{(\psi(T)-\psi(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]}\approx 0.0743246<1.$$

It follows from Theorem 3.2 that problem (15) has a unique solution on $I=[0,1]$.

5. Conclusion

In this research paper. First, we proved the equivalence between *CIFDP* (1) and the Volterra integration equation (5). Then, the existence and uniqueness of mild solutions for boundary value problems of implicit fractional order differential equations were established based on Schauder's fixed point theorem and Banach's contraction principle. In addition, we studied both the Ulam-Hayers and the Ulam-Hyers-Rassias stability types, and their generalizations for *CIFDP* (1), which is an implicit integro-differential equation of fractional order, supplemented with fractional integral type boundary conditions. At the end of the article, we gave a numerical example proving the applicability of the obtained results. Moreover, we mentioned some remarks about the importance of our results. In these remarks we found that if $\psi(t)=t$ then the obtained outcomes in the current paper incorporates the investigation of [11]. Besides, if we take $f(t,x,y,z)=f(t,x,y)$ and $\beta=\alpha,$ in Equation (2), then we have the implicit fractional-order differential equation, which is the same result obtained in [9]. As a conclusion, our results can be considered a step forward in the development of qualitative analysis of fractional differential equations, and this article proposes a generalized nonlocal boundary condition to study Ulam-Hyers stability in the frame of the ψ-Caputo fractional derivatives, and other coupled systems will be provided in the near future.

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