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GENERALIZATION OF PICTURE FUZZY MATRIX

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Abstract. In this paper, we introduce the concept of k-regular Picture Fuzzy Matrix (PiFuM) as a generalization of regular matrix and investigate some basic properties of a k-Regular Picture Fuzzy Matrix (k-RPiFuM). Moreover we analyze the characterization of a matrix for which the regularity index and the index are identical. Furthermore the relation between regular, k-regular and regularity of powers of picture fuzzy matrices are discussed.

Keywords: Picture Fuzzy Set(PiFuS), Picture Fuzzy Matrices(PiFuMs), K-Regular Picture Fuzzy Matrix (k-RPiFuM), Powers of Picture Fuzzy Matrices (PPiFuMs).

AMS Subject classification: Primary 03E72; Secondary 15B15.

1. Introduction

As long as fuzzy set theory is expanded by Zadeh [32], it becomes a gate to solve more problems in uncertain ambits. This concept was established well by allocating membership degrees to elements of a set. Folowing that to clear up any ambiguities, Atanassov [1] introduced non-membership degree and defined intuitionistic fuzzy set as, the sum of degrees which is not greater than 1. This set is used in image processing, robotic system, decision making, medical diagnosis, etc. In 1982 Pawlak [17] familiarised the notion of rough sets. Although both fuzzy set and rough set have their individual adv antages,the soft set introduced by Molodtsov [14] generalizes both of the above theories. The notions of fuzzy soft sets was introduced by Maji et.al[9]. In 1994, Zhang [33] introduced bipolar fuzzy set theory which is an inflation of fuzzy set theory. After that Bipolar soft set is formulated by two soft sets, one of them provides us the positive information and the other provides us the negative information. The Philosophy of bipolarity is that human judgement is based on two sides, positive and negative, and choose the one which is stronger. For historical background see [7, 27, 28, 29, 19, 20, 21, 22, 23, 24]. Though intuitionistic fuzzy sets, bipolar fuzzy sets, etc. have two parameters, we have limitations. But, in occasion like voting, medical diagnosis, etc. we affront with concept neutrality degree.

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Smarandache [6] managed this situation by introducing Neutrosophic set which involves three parameters such as membership degree, non-membership degree and neutral degree whose sum less than or equal to 3. Cuong and Kreinevich [5] introduced the concept of picture fuzzy set as conaction of Neutrosophic set and intuitionistic fuzzy set. Any element in this set include three degrees as membership, neutral and non-membership. Where, the sum of degrees is not greater than 1. Some properties of picture fuzzy set is studied by Cuong [5]. Phong et al. [2] have surveyed some composition of picture fuzzy relations. Viet and Hal [26] proposed fuzzy inference system on picture fuzzy sets. Some results about PiFuMs are explored in [15, 16].

Throughout this paper we deal with Picture Fuzzy Matrices (PiFuMs), that is matrices over the fuzzy algebra $\mathcal{P} = [0, 1]$ under max-min operations $(+, .)$ denoted as $a + b = \max\{a, b\}$ and $a.b = \min\{a, b\}$ for all $a, b \in \mathcal{P}$ and the standard order $' \leq'$ of real numbers. Let $\mathcal{P}_{mn}$ represent the collection of all $m \times n$ PiFuMs over $\mathcal{P}$. In short $\mathcal{P}_n$ stands for $\mathcal{P}_{nn}$. For $A = \langle a_{ij\mu}, a_{ij\eta}, a_{ij\gamma}\rangle \in \mathcal{P}_n$, $A^T, R(A), C(A), \rho_r(A), \rho_c(\mathrm{A})$ and $\rho(A)$ denotes the transpose, row space, column space, row rank, column rank, and rank of A respectively. The algebraic operations on PiFuMs are max-max-min operations, which are different from that of the standard operations on real matrices. In practice, PiFuMs have been proposed to represent the fuzzy relation in a system based on fuzzy sets theory [8], the behavior of the dynamic fuzzy systems depends heavily on the products of PiFuMs in the matrices representations of the system. $\langle a_{ij\mu}, a_{ij\eta}, a_{ij\gamma}\rangle = A \in \mathcal{P}_{mn}$ is a regular if there exists X such that $AXA = A$; $X = \langle x_{ij\mu}, x_{ij\eta}, x_{ij\gamma}\rangle$ is called a generalized (g^-) inverse of A and is denoted by $A^- = \langle a_{ij\mu}, a_{ij\eta}, a_{ij\gamma}\rangle^-$. $A\{1\}$ denote the set of all g-inverse of a regular- PiFuMs A. The regular matrix, which has a generalised inverse, serves as the foundation for studying fuzzy relational equtions. Regular picture fuzzy matrices play an important role in estimation and inverse problem in picture fuzzy relational equations [18] and in fuzzy optimization problems [6].

Multi attribute decision making problem under picture fuzzy environment was studied by Wei [30, 31]. Then picture fuzzy matrix and its application were presented by Shovan Dogra and Pal [25].

This motivates us to develop the study on generalized regular picture fuzzy matrices. The powers of a picture fuzzy matrix are either convergent to a picture fuzzy matrix or oscillating with finite period. For a PiFuMs A, $A^{k+d} = A^k$ for some integers $k, d > 0$. Therefore, all PiFuMs have an index and a period. On the other hand, most matrices over the non-negative real numbers will not have an index and a period [8]. Spectral inverse, such as group inverse and Drazin inverse are defined for fuzzy matrices, analogous to that for complex matrices [3]. For $A \in \mathcal{P}_n$, the Drazin inverse of A is a solution of the equations: $A^kXA = A^k, XAX = X, AX = XA$, for some positive integer k. Group inverse is the solution of the equations: $AXA = A, XAX = X, AX = XA$. Hence Drazin inverse and group inverse are identical when $k = 1$.

In this paper we introduced the concepts of k-regular PiFuMs as a generalization of regular matrices. The row and column ranks of k-regular PiFuMs are determined. Conditions for products of k-regular PiFuM to be regular are obtained. The relation between regular, k-regular and k^{th} power of a PiFuM are discussed. We defined the regularity index of a PiFuMs $A \in \mathcal{P}_n$ as a generalization of the index of A. A characterization of a matrix whose regularity index coincides with the index of A is established. It is shown that for a PiFuMs, regularity index is less than (or) equal to the index of the PiFuM. Furthermore some examples are provided to illustrate the relation between the regularity index and index of PiFuMs.

2. Preliminaries

Definition 2.1. [25] A PiFuM of size $x \times y$ is defined as $(\langle a_{ij\mu}, a_{ij\eta}, a_{ij\gamma}\rangle)$, where $a_{ij\mu} \in [0,1], a_{ij\eta} \in [0,1]$ and $a_{ij\gamma} \in [0,1]$ are, respectively, the measure of positive, neutral and negative membership of a_{ij} for $i = 1,2,...,x$ and $j = 1,2,...,y$ satisfying $0 \leq a_{ij\mu} + a_{ij\eta} + a_{ij\gamma} \leq 1$.

Definition 2.2. A PiFuM is said to be square PiFuM if the number of rows is equal to the number of columns.

Definition 2.3. [25] Let $A = (\langle a_{ij\mu}, a_{ij\eta}, a_{ij\gamma}\rangle)$ and $B = (\langle b_{ij\mu}, b_{ij\eta}, b_{ij\gamma}\rangle)$ be two square PiFuM of order n. Then product $A.B$ is defined as $C = A.B = (\langle c_{ij\mu}, c_{ij\eta}, c_{ij\gamma}\rangle)$, where $c_{ij\mu} = \vee_k(a_{ik\mu} \wedge b_{kj\mu}), c_{ij\eta} = \vee_k(a_{ik\eta} \wedge b_{kj\eta})$ and $c_{ij\gamma} = \wedge_k(a_{ik\gamma} \vee b_{kj\gamma})$
for $i, j = 1,2,...,x$.

Definition 2.4. Let $A = (\langle a_{ij\mu}, a_{ij\eta}, a_{ij\gamma}\rangle)$ and $B = (\langle b_{ij\mu}, b_{ij\eta}, b_{ij\gamma}\rangle)$ be two square PiFuM of order n. Then $A \leq B$ if $a_{ij\mu} \leq b_{ij\mu}, a_{ij\eta} \leq b_{ij\eta}$ and $a_{ij\gamma} \geq b_{ij\gamma}$
for $i, j = 1,2,...,x$.

Definition 2.5. [25] If a special restricted square PiFuM has its diagonal entries $\langle \epsilon_1, \epsilon_2, 0\rangle$ and non-diagonal entries $\langle 0, 0, \epsilon_3\rangle$, then it is called identity special restricted PiFuM.

Definition 2.6. [25] If a special restricted square PiFuM has all entries $\langle 0, 0, \epsilon_3\rangle$ then it is called null special restricted square PiFuM.

3. K-regular Picture Fuzzy Matrices

Definition 3.1. A PiFuM $A \in \mathcal{P}_n$, is said to be right (left) k-regular if there exists a PiFuM $X \in \mathcal{P}_n$ such that $A^kXA(AXA^k) = A^k$, for some positive integer k. X is called a right(left) k-g-inverse of A. Let $A_r(A_l)\{1^k\} = \{X/A^kXA(AXA^k) = A^k\}$.

In general, right k-regular and left k-regular are not equal. Hence a right k-g- inverse need not be a left k-g-inverse (refer Example 3.3). Throughout this paper, we have proved the results for right k-regular and the results for the left k-regular are analogous to the right k-regular and hence ommited. Henceforth we call a right k-regular (or) left k-regular PiFuM as a k-regular PiFuM. Let $A\{1^k\} = A_r\{1^k\} \cup A_l\{1^k\}$.

Remark 3.1. Each element of the set $A\{1^k\}$ is called a k-g-inverse of A. If A is k-regular then A is q-regular for all integers $q \geq k$. For $k = 1$, $A\{1^k\}$ reduces to the set of all g-inverses of a regular PiFuM A.

The following results are very important in this paper.

Lemma 3.1. For PiFuMs $A, B \in \mathcal{P}_n$, $R(B) \subseteq R(A) \Leftrightarrow B = XA$ for some $X \in \mathcal{P}_n$,
$C(B) \subseteq C(A) \Leftrightarrow B = AY$ for some $Y \in \mathcal{P}_n$.

Lemma 3.2. For PiFuMs $A \in \mathcal{P}_{mn}$ and $B \in \mathcal{P}_{np}$, $R(AB) \subseteq R(B), C(AB) \subseteq C(A)$.

Lemma 3.3. For PiFuMs $A, B \in \mathcal{P}_n$, and a positive integer k, the following hold.
(i) If A is right k-regular and $R(B) \subseteq R(A^k)$ then $B = BXA$ for each right k-g-inverse X of A.
(ii) If A is left k-regular and $C(B) \subseteq C(A^k)$ then $B = AYB$ for each left k-g-inverse Y of A.
Proof:
(i) Since $R(B) \subseteq R(A^k)$ by Lemma 3.1, there exists Y such that $B = YA^k$. By Definition 2.1, $A^kXA = A^k$. Hence, $B = YA^k = YA^kAX = BXA$. Thus (i) holds.
The proof of (ii) is similar to (i).

Remark 3.2. In Lemma 3.1, $R(B) \subseteq R(A)$ and $C(B) \subseteq C(A)$ need not imply $R(B) \subseteq R(A^k)$ and $C(B) \subseteq C(A^k)$ (see the Example 3.1).

Howsoever, in particular for $k = 1$, we get,

Lemma 3.4. For $A, B \in \mathcal{P}_{mn}$, if A is regular then
(i) $R(B) \subseteq R(A) \Leftrightarrow B = BA^{-}A$ for each A^{-} of A.
(ii) $C(B) \subseteq C(A) \Leftrightarrow B = AA^{-}B$ for each A^{-} of A.

Theorem 3.1. Let $A \in \mathcal{P}_n$ and k be a positive integer,
then $X \in A_r\{1^k\} \Leftrightarrow X^T \in A_l^T\{1^k\}$.
Proof:

$$\begin{aligned} X \in A_r\{1^k\} &\Leftrightarrow A^k XA = A^k \\ &\Leftrightarrow (A^k XA)^T = (A^k)^T \\ &\Leftrightarrow A^T X^T (A^T)^k = (A^T)^k \\ &\Leftrightarrow X^T \in A_l^T\{1^k\} \end{aligned}$$

Theorem 3.2. Let $A \in \mathcal{P}_n$ and k be a positive integer,
(i) If $X \in A_r\{1^k\}$ then $\rho_c(A^k) = \rho_c(A^k X)$ and $\rho_r(A^k) \leq \rho_r(XA) \leq \rho_r(A)$.
(ii) If $X \in A_l\{1^k\}$ then $\rho_r(A^k) = \rho_r(XA^k)$ and $\rho_c(A^k) \leq \rho_c(AX) \leq \rho_c(A)$.
Proof:
(i) Since $X \in A_r\{1^k\}, A^k XA = A^k$, by Lemma 3.1, $C(A^k) = C(A^k XA) \subseteq C(A^k X) \subseteq C(A^k)$. Therefore, $C(A^k) = C(A^k X)$ and $\rho_c(A^k) = \rho_c(A^k X)$. Since, $A^k XA = A^k$, we have $A^k = A^k XA = A^k(XA)^2 = ... = A^k(XA)^k$. Therefore, $A^k = A^k(XA)^k$.
Hence, by Lemma 3.1, $R(A^k) = R(A^k(XA)^k) \subseteq R((XA)^k) \subseteq R(XA) \subseteq R(A)$.
Therefore, $R(A^k) \subseteq R(XA) \subseteq R(A)$ and $\rho_r(A^k) \leq \rho_r(XA) \leq \rho_r(A)$.
(ii) Proof is similar to (i) so omitted.

Theorem 3.3. Let $A \in \mathcal{P}_n$ and k be a positive integer, The following conditions are equivalent:
(i) A is k-regular.
(ii) λA is k-regular for $\lambda \neq 0 \in \mathcal{P}$.
(iii) PAP^T is k-regular for some permutation PiFuM P.
Proof:

$$\begin{aligned} A\ is\ k-regular &\Rightarrow A^k AX = A^k \\ &\Rightarrow (\lambda A)^k X(\lambda A) = (\lambda A)^k for \lambda \neq 0 \in \mathcal{P} \\ &\Rightarrow \lambda A\ is\ k-regular \end{aligned}$$

If λA is k-regular, then for $\lambda = 1$, A is k-regular. Thus, (i) $\Leftrightarrow$ (ii) hold.

$$\begin{aligned} A\ is\ k-regular &\Leftrightarrow A^k XA = A^k\ for\ some\ X \in \mathcal{P}_n \\ &\Leftrightarrow (PA^k P^T)(PXP^T)(PAP^T) = PA^k P^T \\ &\quad for\ some\ permutation\ matrix\ P\ and\ some\ X \in \mathcal{P}_n \\ &\Leftrightarrow (PAP^T)^k (PXP^T)(PAP^T) = (PAP^T)^k \\ &\Leftrightarrow PAP^T is\ k-regular \end{aligned}$$

Thus (i) $\Leftrightarrow$ (iii) hold. Hence complete the Proof.

Theorem 3.4. If $Y, Z \in A_r\{1^k\}$, then $YAZ \in A_r\{1^k\}$.
Proof:
Since $Y, Z \in A_r\{1^k\}, A^kYA = A^k$ and $A^kZA = A^k$.
Now, take $X = YAZ$, then we have
$A^kXA = A^k(YAZ)A = (A^kYA)ZA = A^kZA = A^k$.
Hence, $X = YAZ \in A_r\{1^k\}$.

Theorem 3.5. For $A, B \in \mathcal{P}_n$, with $R(A) = R(B)$ and $R(A^k) = R(B^k)$ then, A is right k-regular $\Leftrightarrow B$ is right k-regular.
Proof:
Let A be a right k-regular PiFuM, satisfying $R(B^k) \subseteq R(A^k)$ and $R(A) \subset R(B)$.
Since $R(B^k) \subseteq R(A^k)$, by Lemma 3.3, $B^k = B^kXA$ for each k-g-inverse X of A.
Since $R(A) \subseteq R(B)$, by Lemma 3.1, $A = YB$ for some $Y \in \mathcal{P}_n$. Substituting for A in $B^k = B^kXA$, we get $B^k = B^kXA = B^kXYB = B^kZB$ where $XY = Z$.
Hence B is right k-regulr.
Conversely, if B is a right k-regular picture fuzzy matrix satisfying $R(A^k) \subseteq R(B^k)$ and $R(B) \subseteq R(A)$, then A is right k- regular can be proved in same manner. Hence the theorem.

Theorem 3.6. For $A, B \in \mathcal{P}_n$, with $C(A) = C(B)$ and $C(A^k) = C(B^k)$ then, A is left k-regular $\Leftrightarrow B$ is left k-regular.
Proof:
This is similar to Theorem 3.5 and hence omitted.

Theorem 3.7. For $A, B \in \mathcal{P}_n$, if A is right k-regular, $R(B) \subseteq R(A^k)$ and B is idempotent then AB is right k-regular and $A_r\{1^k\} \subseteq (AB)_r\{1^k\}$.
Proof:
Since $R(B) \subseteq R(A^k)$ by Lemma 3.3, $B = BXA$, for each right k-g-inverse X of A. Since B is idempotent, $B^2 = B$.

$$\begin{aligned}(AB)^k &= (AB)^{k-1}(AB)\\ &= (AB)^{k-1}(AB^2)\\ &= (AB)^{k-1}(ABB)\\ &= (AB)^{k-1}A(BXA)B\\ &= (AB)^kX(AB)\end{aligned}$$

Thus AB is right k-regular. X is a right k-g-inverse of AB.
Hence $A_r\{1^k\} \subseteq (AB)_r\{1^k\}$.

Theorem 3.8. For $A, B \in \mathcal{P}_n$, if A is left k-regular, $C(A) \subseteq C(A^k)$ and B is idempotent then BA is left k-regular and $A_l\{1^k\} \subseteq (BA)_l\{1^k\}$.
Proof:
Proof is similar to the above Theorem 3.7 and hence ommitted.

Remark 3.3. In particular for $k = 1$ and A commutes with B, it reduces to the following result.

Theorem 3.9. For a Regular Picture Fuzzy Matrix (RPiFuM) A if $R(B) \subseteq R(A)$ and B is idempotent then AB is RPiFuM and $A\{1^k\} \subseteq (AB)\{1^k\}$.
Below we analyze the relation between regular, k-regular and k^{th} power of a PiFuM A in $\mathcal{P}_n$, We assume $A^0 = I$. Thought k is a positive integer.
If A is a regular, then $AXA = A$ for some $X \in \mathcal{P}_n$, for $k \geq 1$, pre(post) multiplying by

A^{k-1} on both side, we get $A^kXA = A^k(AXA^k = A^k)$. Thus A is k-regular for all $k \geq 1$. If A^k is a regular then $A^kYA^k = A^k$ for some Y in $\mathcal{P}_n$. This can written as $A^kZA = A^k$ where $Z = YA^{k-1}$, hence A is k-regular. On the other hand if A is k-regular for some positive integer k, need not imply A^k is regular or A is regular.
The processes are illustrated here.

Example 3.1. Let us consider a non-regular PiFuM

$$A = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle \epsilon_1, \epsilon_2, 0\rangle \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle \epsilon_1, \epsilon_2, 0\rangle \end{bmatrix}$$

$$\text{For } X = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0, 0, \epsilon_3\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0, 0, \epsilon_3\rangle & \langle 0, 0, \epsilon_3\rangle & \langle \epsilon_1, \epsilon_2, 0\rangle \end{bmatrix},$$

$A^2XA = A^2$ holds. A is 2-regular, but A is not regular.

Example 3.2. Let us consider PiFuM $A = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0, 0, \epsilon_3\rangle & \langle 0, 0, \epsilon_3\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle 0, 0, \epsilon_3\rangle & \langle 0, 0, \epsilon_3\rangle \end{bmatrix}$

For this A, $A^2 = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle 0, 0, \epsilon_3\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0, 0, \epsilon_3\rangle \end{bmatrix}$

$$A^3 = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle \end{bmatrix}$$

For PiFuM $X = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0, 0, \epsilon_3\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle 0, 0, \epsilon_3\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0.5, 0.3, 0\rangle \end{bmatrix}$,

$A^3XA = A^3 \neq AXA^3$ holds. Therefore A is 3-regular. $A^5 = A^4$.
Put $A^3 = B$, now we prove that B is not regular. If B is regular then $BYB = B$ for some $Y \in \mathcal{P}_3$. The (1,1) entry of B is $\langle b_{11\mu}, b_{11\eta}, b_{11\gamma}\rangle = \langle \epsilon_1, \epsilon_2, 0\rangle$.

$$\begin{aligned} Therefore, \langle b_{11\mu}, b_{11\eta}, b_{11\gamma}\rangle &= \sum_{j,k=1}^{3} (\langle b_{1j\mu}, b_{1j\eta}, b_{1j\gamma}\rangle \langle y_{jk\mu}, y_{jk\eta}, y_{jk\gamma}\rangle \langle b_{k1\mu}, b_{k1\eta}, b_{k1\gamma}\rangle \\ &\Rightarrow \langle \epsilon_1, \epsilon_2, 0\rangle \\ &= \langle b_{11\mu}, b_{11\eta}, b_{11\gamma}\rangle \langle y_{1k\mu}, y_{1k\eta}, y_{1k\gamma}\rangle \langle b_{k1\mu}, b_{k1\eta}, b_{k1\gamma}\rangle \\ &+ \langle b_{12\mu}, b_{12\eta}, b_{12\gamma}\rangle \langle y_{2k\mu}, y_{2k\eta}, y_{2k\gamma}\rangle \langle b_{k1\mu}, b_{k1\eta}, b_{k1\gamma}\rangle \\ &+ \langle b_{13\mu}, b_{13\eta}, b_{13\gamma}\rangle \langle y_{3k\mu}, y_{3k\eta}, y_{3k\gamma}\rangle \langle b_{k1\mu}, b_{k1\eta}, b_{k1\gamma}\rangle \\ &\Rightarrow \langle b_{11\mu}, b_{11\eta}, b_{11\gamma}\rangle \langle y_{1k\mu}, y_{1k\eta}, y_{1k\gamma}\rangle \langle b_{11\mu}, b_{k1\eta}, b_{k1\gamma}\rangle \\ &= \langle \epsilon_1, \epsilon_2, 0\rangle. \end{aligned}$$

Since $\langle b_{12\mu}, b_{12\eta}, b_{12\gamma}\rangle, \langle b_{13\mu}, b_{13\eta}, b_{13\gamma}\rangle = \langle 0.5, 0.3, 0\rangle$ and $\langle b_{11\mu}, b_{11\eta}, b_{11\gamma}\rangle = \langle \epsilon_1, \epsilon_2, 0\rangle$, the only possibility is $\langle y_{1k\mu}, y_{1k\eta}, y_{1k\gamma}\rangle\langle b_{k1\mu}, b_{k1\eta}, b_{k1\gamma}\rangle = \langle \epsilon_1, \epsilon_2, 0\rangle$ for all k. Now for $k = 1$, $\langle y_{11\mu}, y_{11\eta}, y_{11\gamma}\rangle\langle b_{11\mu}, b_{11\eta}, b_{11\gamma}\rangle = \langle \epsilon_1, \epsilon_2, 0\rangle \Rightarrow \langle y_{11\mu}, y_{11\eta}, y_{11\gamma}\rangle = \langle \epsilon_1, \epsilon_2, 0\rangle$.
For $k = 2$, $\langle y_{12\mu}, y_{12\eta}, y_{12\gamma}\rangle\langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle = \langle \epsilon_1, \epsilon_2, 0\rangle$
For $k = 3$, $\langle y_{13\mu}, y_{13\eta}, y_{13\gamma}\rangle\langle b_{31\mu}, b_{31\eta}, b_{31\gamma}\rangle = \langle \epsilon_1, \epsilon_2, 0\rangle$ are not possible,

since $\langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle = \langle b_{31\mu}, b_{31\eta}, b_{31\gamma}\rangle = \langle 0.5, 0.3, 0\rangle$.
Hence $\langle y_{11\mu}, y_{11\eta}, y_{11\gamma}\rangle = \langle \epsilon_1, \epsilon_2, 0\rangle$.
The $(2,3)^{th}$ entry of B is $\langle b_{23\mu}, b_{23\eta}, b_{23\gamma}\rangle = \langle 0, 0, \epsilon_3\rangle$

$$\begin{aligned} Therefore, \langle b_{23\mu}, b_{23\eta}, b_{23\gamma}\rangle &= \sum_{j,k=1}^{3} ((\langle b_{2j\mu}, b_{2j\eta}, b_{2j\gamma}\rangle\langle y_{jk\mu}, y_{jk\eta}, y_{jk\gamma}\rangle\langle b_{k3\mu}, b_{k3\eta}, b_{k3\gamma}\rangle \\ &\Rightarrow \langle 0, 0, \epsilon_3\rangle \\ &= \langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle\langle y_{1k\mu}, y_{1k\eta}, y_{1k\gamma}\rangle\langle b_{k3\mu}, b_{k3\eta}, b_{k3\gamma}\rangle \\ &+ \langle b_{22\mu}, b_{22\eta}, b_{22\gamma}\rangle\langle y_{2k\mu}, y_{2k\eta}, y_{2k\gamma}\rangle\langle b_{k3\mu}, b_{k3\eta}, b_{k3\gamma}\rangle \\ &+ \langle b_{23\mu}, b_{23\eta}, b_{23\gamma}\rangle\langle y_{3k\mu}, y_{3k\eta}, y_{3k\gamma}\rangle\langle b_{k3\mu}, b_{k3\eta}, b_{k3\gamma}\rangle \\ &\Rightarrow \langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle\langle y_{1k\mu}, y_{1k\eta}, y_{1k\gamma}\rangle\langle b_{k3\mu}, b_{k3\eta}, b_{k3\gamma}\rangle \\ &= \langle 0, 0, \epsilon_3\rangle. \end{aligned}$$

Suppose $\langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle\langle y_{1k\mu}, y_{1k\eta}, y_{1k\gamma}\rangle\langle b_{k3\mu}, b_{k3\eta}, b_{k3\gamma}\rangle = \langle 0, 0, \epsilon_3\rangle$
$\Rightarrow \langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle\langle y_{11\mu}, y_{11\eta}, y_{11\gamma}\rangle\langle b_{13\mu}, b_{13\eta}, b_{13\gamma}\rangle = \langle 0, 0, \epsilon_3\rangle$,
$\langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle\langle y_{12\mu}, y_{12\eta}, y_{12\gamma}\rangle\langle b_{23\mu}, b_{23\eta}, b_{23\gamma}\rangle = \langle 0, 0, \epsilon_3\rangle$,
$\langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle\langle y_{13\mu}, y_{13\eta}, y_{13\gamma}\rangle\langle b_{33\mu}, b_{33\eta}, b_{33\gamma}\rangle \neq \langle 0, 0, \epsilon_3\rangle$.
but $\langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle\langle y_{11\mu}, y_{11\eta}, y_{11\gamma}\rangle\langle b_{13\mu}, b_{13\eta}, b_{13\gamma}\rangle \neq \langle 0, 0, \epsilon_3\rangle$.
Since $\langle y_{11\mu}, y_{11\eta}, y_{11\gamma}\rangle = \langle \epsilon_1, \epsilon_2, 0\rangle$ and $\langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle = \langle b_{13\mu}, b_{13\eta}, b_{13\gamma}\rangle = \langle 0.5, 0.3, 0\rangle$,
therefore $\langle b_{21\mu}, b_{21\eta}, b_{21\gamma}\rangle\langle y_{1k\mu}, y_{1k\eta}, y_{1k\gamma}\rangle\langle b_{k3\mu}, b_{k3\eta}, b_{k3\gamma}\rangle \neq \langle 0, 0, \epsilon_3\rangle$.
Hence $\sum_{j,k=1}^{3} ((\langle b_{2j\mu}, b_{2j\eta}, b_{2j\gamma}\rangle\langle y_{jk\mu}, y_{jk\eta}, y_{jk\gamma}\rangle\langle b_{k3\mu}, b_{k3\eta}, b_{k3\gamma}\rangle \neq \langle 0, 0, \epsilon_3\rangle$
$\Rightarrow \langle b_{23\mu}, b_{23\eta}, b_{23\gamma}\rangle \neq \langle 0, 0, \epsilon_3\rangle$.
Therefore, $BYB \neq B$ for any $Y \in \mathcal{P}_3 \Rightarrow B = A^3$ is not regular.

Example 3.3. The following example shows that, for $A \in \mathcal{P}_n$ and k is positive integer, $A_r\{1^k\} \neq A_l\{1^k\}$.

Let us consider PiFuM $A = \begin{bmatrix} \langle 0,0,\epsilon_3\rangle & \langle 0.5,0.3,0\rangle & \langle 0,0,\epsilon_3\rangle \\ \langle 0,0,\epsilon_3\rangle & \langle \epsilon_1,\epsilon_2,0\rangle & \langle 0.5,0.3,0\rangle \\ \langle 0.5,0.3,0\rangle & \langle 0,0,\epsilon_3\rangle & \langle 0,0,\epsilon_3\rangle \end{bmatrix}$.

For this A, $A^2 = \begin{bmatrix} \langle 0,0,\epsilon_3\rangle & \langle 0.5,0.3,0\rangle & \langle 0.5,0.3,0\rangle \\ \langle 0.5,0.3,0\rangle & \langle \epsilon_1,\epsilon_2,0\rangle & \langle 0.5,0.3,0\rangle \\ \langle 0,0,\epsilon_3\rangle & \langle 0.5,0.3,0\rangle & \langle 0,0,\epsilon_3\rangle \end{bmatrix}$,

$A^3 = \begin{bmatrix} \langle 0.5,0.3,0\rangle & \langle 0.5,0.3,0\rangle & \langle 0.5,0.3,0\rangle \\ \langle 0.5,0.3,0\rangle & \langle \epsilon_1,\epsilon_2,0\rangle & \langle 0.5,0.3,0\rangle \\ \langle 0,0,\epsilon_3\rangle & \langle 0.5,0.3,0\rangle & \langle 0.5,0.3,0\rangle \end{bmatrix}$.

For PiFuM $X = \begin{bmatrix} \langle 0.5,0.3,0\rangle & \langle 0,0,\epsilon_3\rangle & \langle 0.5,0.3,0\rangle \\ \langle 0.5,0.3,0\rangle & \langle \epsilon_1,\epsilon_2,0\rangle & \langle 0,0,\epsilon_3\rangle \\ \langle 0,0,\epsilon_3\rangle & \langle 0.5,0.3,0\rangle & \langle 0,0,\epsilon_3\rangle \end{bmatrix}$,

$A^3XA = A^3$. Hence A is 3-regular. For $k = 3$, $A^3XA = A^3$ but $AXA^3 \neq A^3$.
Hence $A_r\{1^k\} \neq A_l\{1^k\}$.

4. Regularity Index of PFM

Every $A \in \mathcal{P}_n$ has index k and period $'d'$ that is least positive integers k and d exist such that $A^{k+d} = A^k$. Let $i(A)$ and $p(A)$, denote the index and period of A rspectively. It is clearly that, if $i(A) = 1$ then A is regular but the converse is not true(refer Example

4.1).
In this section first we shall prove that every PiFuM of index k is k-regular.

Theorem 4.1. For PiFuM $A \in \mathcal{P}_n$ with $i(A) = k$, we have the following:
(i) The small exponent for which $R(A^k) = R(A^{k+1})$ and $C(A^k) = C(A^{k+1})$ hold is k.
(ii) The smallest positive integer k for which $\rho_r(A^k) = \rho_r(A^{k+1})$ and $\rho_c(A^k) = \rho_c(A^{k+1})$ holds is k
(iii) The smallest exponent for which $YA^{k+1} = A^K$ and $A^{k+1}X = A^K$ have solution is k.
(iv) A is right k-regular and left k-rgular.
(v) $A_r\{1^k\} = A_l\{1^k\} = A\{1^k\} \neq \phi$.
(vi) A is q-regular for all integers $q \geq k$.
Proof:
By definition of index of A, k is the smallest positive integer and $d > 0$ such that $A^{k+d} = A^k$. For $d \geq 1$ this can be written as,

$$A^{d-1}A^{k+1} = A^k = A^{k+1}A^{d-1} \qquad (4.1)$$

Therefore by Lemma 3.1, we have $R(A^k) \subseteq R(A^{k+1})$ and $C(A^k) \subseteq C(A^{k+1})$. By Lemma 3.2, $R(A^{k+1}) \subseteq R(A^k)$ and $C(A^{k+1}) \subseteq C(A^k)$ always holds, hence k is smallest exponent for which, $R(A^k) \subseteq R(A^{k+1})$ and $C(A^k) \subseteq C(A^{k+1})$. This (i) holds. (ii) Automatically follows from (i). Again by Lemma 3.1, there exists $Y, X \in \mathcal{P}_n$ satisfying $YA^{k+1} = A^k$ and $A^{k+1}X = A^k$. Thus (iii) holds. In (4.1), put $X = A^{d-1}$, then $XA^{k+1} = A^k = A^{k+1}X$, premultiplying by A, $AXA^{k+1} = A^{k+1}$ and post multiplying by A, $A^{k+1}XA = A^{k+1}$. By multiplying these equations suitably by A^{d-1}, yield $AXA^{k+d} = A^{k+d}$ and $A^{k+d}XA = A^{k+d}$. Again by using $A^{k+d} = A^k$, we get $AXA^k = A^k = A^kXA$. Thus A is both right k-regular and left k-regular and $X \in A_r\{1^k\} = A_l\{1^k\} \neq \phi$. Thus (iv) and (v) holds. (vi) follows from Remark 3.1. Hence the theorem

Remark 4.1. In particular for $i(A) = 1, A$ is regular and the group inverse exists. From Theorem 4.1, we observe that every PiFuM of index k is k-regular but k is not the least positive integer for which A is k-regular.
Hence $A_r\{1^k\} = A_l\{1^k\} = A\{1^k\} \neq \phi$, but k is not the smallest positive integer satisfying $A^kXA = A^k$ or $AYA^k = A^k$.
Further $R(A^{h-1})$ is not contained in $R(A^h)$ and $C(A^{h-1})$ is not contained in $C(A^h)$ for all $h \leq k$.

Definition 4.1. The smallest positive integer k for which PiFuM A is k-regular, is called the regular index of A and is denoted as reg.$i(A)$.

Remark 4.2. It is clear that from the Definition 2.3 and Remark 3.1, $reg.i(A) = k_0 \Leftrightarrow k_0$ is the smallest positive integer for which A has a $k_0 - g - inverse \Leftrightarrow A\{1^{k_0}\} \neq \phi$ and $A\{1^h\} = \phi$ for all $h < k_0$. Further is regular $\Leftrightarrow reg.i(A) = 1$. From Theorem 4.1, it follow that $A\{1^{k_0}\} \subseteq A\{1^k\}$. Therefore $reg.i(A) \leq i(A)$.
This is illustrted below.

Example 4.1. $A = \begin{bmatrix} \langle\epsilon_1, \epsilon_2, 0\rangle & \langle\epsilon_1, \epsilon_2, 0\rangle \\ \langle 0\epsilon_1, \epsilon_2, 0\rangle & \langle 0, 0, \epsilon_3\rangle \end{bmatrix}$.
is regular and hence k-regular for all $k \geq 1; reg.i(A) = 1$. By using Algorithm (1) in [8],for PiFuMs
$A\{1\} = \left\{ X = \begin{bmatrix} \langle 0, 0, \epsilon_3\rangle & \langle\epsilon_1, \epsilon_2, 0\rangle \\ \langle\epsilon_1, \epsilon_2, 0\rangle & \langle\alpha_\mu, \alpha_\eta, \alpha_\gamma\rangle \end{bmatrix}, \alpha = \langle\alpha_\mu, \alpha_\eta, \alpha_\gamma\rangle \in \mathcal{P} \right\}$ is the set of all g-inverse of A. It can be verified that $AX \neq XA$ for all $X \in A\{1\}$. Therefore the group inverse of A and Drazin inverse of A does not exist. $R(A)$ is not contained in $R(A^2), C(A)$ is not contained in $C(A^2)$.

Here $A^2 = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle \epsilon_1, \epsilon_2, 0\rangle \\ \langle 0\epsilon_1, \epsilon_2, 0\rangle & \langle \epsilon_1, \epsilon_2, 0\rangle \end{bmatrix} = A^3$,
therefore $i(A) = 2$ and $A^2XA = A^2 = AXA^2$ holds, but 2 is not the smallest positive integer. Hence $reg.i(A) < i(A)$.

Corollary 4.1. For a non-regular PiFuM $A \in \mathcal{P}_n$ if $i(A) = 2$, then $reg.i(A) = 2$.
Proof:
Since $i(A) = 2$, by Theorem 4.1, A is 2-regular and $A^2XA = A^2 = AXA^2$, for $X \in \mathcal{P}_n$. Since A is not regular, $reg.i(A) \neq 1$. Therefore $k = 2$ is the smallest integer satisfying $A^kXA = A^k = AXA^k$.
Hence $reg.i(A) = i(A) = 2$. Relation between regularity index and index of a PiFuMs are illustrated below.

Example 4.2. Let us consider PiFuM $A = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.8, 0.2, 0\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0.8, 0.2, 0\rangle & \langle 0.7, 0.2, 0\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0.7, 0.2, 0\rangle & \langle 0.6, 0.1, 0\rangle & \langle 0, 0, \epsilon_3\rangle \end{bmatrix}$.
Here, $\rho_r(A) = 3, \rho_c(A) = 2$.
Therefore by Theorem 3.1 of [4] A is not regular. Hence, $reg.i(A) \neq 1$.
Since $A^2 = \begin{bmatrix} \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0.8, 0.2, 0\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0.8, 0.2, 0\rangle & \langle 0.8, 0.2, 0\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0.7, 0.2, 0\rangle & \langle 0.7, 0.2, 0\rangle & \langle 0, 0, \epsilon_3\rangle \end{bmatrix} = A^3$,
$i(A) = 2, 1 < reg.i(A) \leq i(A) = 2$. Thus $reg.i(A) = i(A) = 2$.

Example 4.3. Let us consider the A in Example 3.3.
For $X = \begin{bmatrix} \langle 0.5, 0.3, 0\rangle & \langle 0, 0, \epsilon_3\rangle & \langle 0.5, 0.3, 0\rangle \\ \langle 0.5, 0.3, 0\rangle & \langle \epsilon_1, \epsilon_2, 0\rangle & \langle 0, 0, \epsilon_3\rangle \\ \langle 0, 0, \epsilon_3\rangle & \langle 0.5, 0.3, 0\rangle & \langle 0, 0, \epsilon_3\rangle \end{bmatrix}$,
$A^3XA = A^3$ holds but $A^2XA \neq A^2, AXA \neq A$ and 3 is the smallest positive integer for which X is a 3-g-inverse of A. $X \in A\{1^3\}, X \notin A\{1^2\}, X \notin A\{1\}$. Hence $A\{1^3\} \neq \phi$. Therefore $reg.i(A) = 3$. For this X, $AX \neq XA$. Hence X is not a Drazin inverse of A. Further $A^5 = A^4$. Hence $i(A) = 4$. Thus $reg.i(A) = 3 < 4 = i(A)$.

Theorem 4.2. For $A \in \mathcal{P}_n$ with $i(A) = k > 1$ and $\rho(A) = d/(k-1)$ then A^k is regular.
Proof:
Since $i(A) = k$, by Theorem 4.1, A is k-regular. Therefore, $A^kXA = A^k$.......(4.2)
Post multiplying Equation (4.2) by A^{k-1} on both sides, we get $A^kXA^k = A^{k+(k-1)}$.
Since $d/(k-1)$, $A^kXA^k = A^{k+(k-1)} = A^k$. Hence A^k is regular.

Theorem 4.3. For $A \in \mathcal{P}_n$ and k a positive integer, the following are equivalent:
(i) $reg.i(A) = i(A) = k$.
(ii) The smallest exponent for which $A^kXA = A^k$ holds is k.
(iii) The smallest exponent for which $AXA^k = A^k$ holds is k.
(iv) k is smmalest positive integer such that A is k-regular.
(v) k is smmalest positive integer such that A^T is k-regular.
(vi) The Drazin inverse A_D exists and unique.
Proof:
This theorem proof that follows from Theorem 4.1 and the definition of regularity index.

5. Conclusions

In this paper, the authors added the idea of k-Regular Picture Fuzzy Matrix (k-RPiFuM) as a generalization of RPiFuM and a few primary houses of a k-RPiFuM are established.

The characterizations of a PiFuM for which the regularity index and the index are identical are explored. Finally we investigated approximately the relation among normal, k-normal and regularity of powers of PiFuMs. As a future work the application of k-regularity in medical field is under process.

References

[1] Atanassov, K., (1983), Intuitionistic Fuzzy Sets, Fuzzy Sets and System, 20, pp. 87-96.

[2] Adiga, C., and Smitha, M., (2009), On the skew laplacian energy of a digraph, International Mathematical Forum, 4(39), pp. 1907-1914.

[3] Ben-Israel, A., and Greville, T. N. E., (1974), Generalized inverses, Theory and Applications, Wiley, Newyork.

[4] Cho, H., (1999), Regular fuzzy matrices and fuzzy equations, Fuzzy sets and systems, 105, pp. 445-451.

[5] Cuong, B. C., and Kreinovich, V., (2013), Picture fuzzy sets-a new concept for computational intelligence problems, In Proceeding of the 2013 Third Word Congress on Information and Communication Technologies (WICT), Honoi, Vietnam, pp.1-6.

[6] Florentin Smarandache, (2004), Neutrosophic Set- A generalization of the Intuitionistic fuzzy set, International Journal of Pure and Applied Mathematics, 24(3), pp. 1-29.

[7] Hayat, K., Ali, M. I., Alcantud, J. C. R, Cao, B.-Y. Tariq, K. U., (2018), Best concept selection in design process: An application of generalized intuitionistic fuzzy soft sets, Journal of intelligent and fuzzy system, 35(5), pp. 5707-5720.

[8] Kim, K.H., and Roush, F.W., (1980), Generalized Fuzzy Matrices, Fuzzy Sets and System, 4(3), pp. 293-315.

[9] Maji, P.K., Bismas, R., and Roy, A.R., (2003), Soft Set Theory, Computers and Mathematics with applications, 45, pp. 555-562.

[10] Meenakshi, A.R., (2004), On regularity of block triangular fuzzy matrices, Applied Math and Computing, 15, pp. 207-220.

[11] Meenakshi, A.R., and Sriram. S., (2003), On regularity of sums of fuzzy matrices, Bulletin of Pure and Applied sciences, 22E(2), pp. 395-403.

[12] Meeakshi, A.R., and Jenita, P., (2011), Generalized regular Fuzzy Matrices, Indian journal od Fuzzy System, 8(2), pp. 133-141.

[13] Meenakshi, A.R., and Poongodi, P., (2012), Generalized regular interval valued Fuzzy Matrices, International Journal of Fuzzy Mathematics and Systems, 2(1), pp. 29-36.

[14] Molodtsov, D.A., (1999), Soft set theory-first result, Computers and Mathematics with applications, 37(4-5), pp. 19-31.

[15] Murugadas, P., (2021), Implication operation on picture fuzzy matrices (communicated and accepted in AIP Journal)

[16] Murugadas, P., (2021), Powers of picture fuzzy matrix (communicated to Int.J. of Group theory)

[17] Pawlak, Z., (1982), Rough Sets, International Journal of Computing and Information Science, 11, pp. 341-356.

[18] Pedrycz, W., (1998), Fuzzy relational calculus, Hand Book of Fuzzy Computations, IOP.4d.

[19] Shahzaib Ashraf, tahiMahmood, Saleem Abdullah, and Qaisar Khan, (2019), Different approaches to multi-criteria group decision making problems for picture fuzzy environment, 50, pp. 373-397.

[20] Silambarasan, I., and Sriram, S., (2018), Algebraic operations on Pythagorean Fuzzy Matrices, Mathematical Sciences International Research Journal, 7(2), pp. 406-414.

[21] Silambarasan, I., and Sriram, S., (2019), Commutative monoid of Pythagorean Fuzzy Matrices, International Journal of Computer Sciences and Engineering, 7(4), pp. 637-643.

[22] Silambarasan, I., (2020), Some algebraic structures of Picture fuzzy matrices. World Scientific News, 150, pp. 78-91.

[23] Silambarasan, I., (2021), Some Algebraic properties of Picture fuzzy sets. Bull. Int. Math. Virtual Inst., 11(3), pp. 429-442.

[24] Silambarasan, I., and Sriram, S., (2019), New operations for Pythagorean Fuzzy Matrices, Indian Journal of Science and Technology, 12(20), pp. 1-7.

[25] Dogra, S., and Pal, M., (2020), Picture fuzzy matrix and its application, Soft Computing (springgger-Verlag GmbH Germany, part of springer Nature), 24, pp. 9413-9428.

[26] Son, L.H., Viet, P., and Hai, P., (2017), Picture inference system: a new fuzzy inference system on picture fuzzy set, Appl. Intell., 46, pp. 652-669.

[27] Mahmood, T. and Rehman, U., (2021), A novel approach towards Bipolar complex fuzzy sets and their applications in generalized similarity measures, International Journal of Intelligent Systems, 37(1), pp. 535-567.

[28] Mahmood, T., (2020), A novel approach towards Bipolar soft sets and their applications, Journal of Mathematics, Article ID 4690808.

[29] Mahmood, T., (2020), The Fuzzy cross-entropy for picture hesitant fuzzy sets and their application in multi criteria decisionmaking, Punjab University Journal of Mathematics, 52(10), pp. 55-82.

[30] Wei, G., (2016), Picture fuzzy cross-entropy for multiple attribute decision making problems, J. Bus. Econ. Manage, 17(4), pp. 491-502.

[31] Wei, G., (2017), Picture fuzzy aggregation operators and their application to multiple attribute decision making, J. Intell. Fuzzy System, 33, pp. 713-724.

[32] Zadeh, L.A., (1965), Fuzzy Sets, Information and Control, 8, pp. 338-353.

[33] Zhang, W.R., (1994), Bipolar fuzzy sets and reations: a computational framework for cognitive modeling and multiagent decision analysis. Proceedings of the industrial fuzzy control and intelligent systems conference,and the NASA joint technology workshop on neural networks and fuzzy logicand information processing society biannual confernece , San Antonio, Tex, USA, 305-309.

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