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ON ALMOST CONVERGENCE OF FUZZY VARIABLES

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Abstract. In this paper, within framework credibility theory, we examine several notions of convergence and almost convergence of fuzzy variable sequences. We investigate relations between these notions. Utilizing fuzzy variables, the almost convergence with regards to (w.r.t.) almost surely, credibility, mean, distribution, and uniformly almost surely are investigated. We also examine almost Cauchy sequence types in credibility theory and obtain significant results.

Keywords: Credibility measure, credibility theory, almost convergence, mean, distribution.

AMS Subject Classification: 40A35, 03E72.

1. Introduction

Several authors involving Duran [6], King [13], Lorentz [24], Moricz and Rhoades [30], and Schafer [39] have researched in the space of almost convergent sequences. The notion of strong almost convergence was considered by Maddox [26]. Related articles with almost convergence and strong almost convergence can be seen in [2, 4, 5].

Fuzzy theory is well advanced on the mathematical foundations of fuzzy set theory, initiated by Zadeh [46], established in 1965. The fuzzy approach can be utilized in a comprehensive variety of real problems. For instance, the possibility theory has been selected by several researchers, such as Dubois and Prade [9] and Nahmias [32]. A fuzzy variable is a function from a credibility space (denoted with the credibility measure) to the set of real numbers. The convergence of fuzzy variables is significant component of credibility theory, which can be applied in actual problems in engineering and mathematical Finance. Kaufmann has examined the fuzzy variable, possibility distribution, and membership function [12]. Possibility measure, which is generally determined as supremum preserving set function on the power set of a nonempty set, is a fundamental notion in

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possibility theory, but it is not self-dual. Since a self-dual measure is required in theory and practice, Liu and Liu [15] have introduced a self-duality credibility measure. The credibility measure plays the role of the possibility measure in the fuzzy world because it shares some fundamental features with the possibility measure. Especially since Liu began the survey of credibility theory, many specific contents have been examined (see [16, 18, 19, 20, 21, 22, 34, 35, 36, 43, 47]). Contemplating sequence convergence plays a crucial role in credibility theory. Liu [17] presented four kinds of convergence notions for fuzzy variables: convergence almost surely, convergence in credibility, convergence in mean, and convergence in distribution. In addition, based on credibility theory, several convergence features of credibility distribution for fuzzy variables were worked on by Jiang [10] and Ma [25].

Wang and Liu [43] thought the relationships among convergence in mean, convergence in credibility, convergence almost uniformly, convergence in distribution, and convergence almost surely. Besides, numerous researchers emphasized convergence notions in classical measure theory, credibility theory, and probability theory and examined their connections. The interested readers may examine Chen et al. [3], Lin [14], Liu and Wang [23], Xia [44], and You [45].

Statistical convergence was first presented by Fast [7] as a generalization of ordinary convergence for real sequences. Statistical convergence turned out to be one of the most active research areas in the summability theory after the studies of Fridy [8]. Statistical convergence has also been studied in more general abstract spaces such as the fuzzy number space [33]. More investigations in this direction and more applications of statistical convergence can be seen in [27, 28, 29, 33, 37, 38]. Also, the readers should refer to the monographs [1], and [31], and recent papers [11], [40], [41], and [42] and for the background on the sequence spaces. This work presented a new kind of convergence for fuzzy variable sequences. Section 2 recalls some definitions and theorems in uncertainty theory, some preliminary definitions and theorems related to fuzzy variables sequences, credibility space, and almost convergence are presented. In Section 3, we also plan to investigate on the notion of almost convergence of fuzzy variables and to construct fundamental properties of the almost convergence in credibility.

2. Preliminaries

A set function Cr is credibility measure if it supplies the subsequent axioms: Let Θ be a nonempty set, and $\mathcal{P}(\Theta)$ the power set of Θ (i.e., the larggest algebra over Θ). Each element in $\mathcal{P}$ is said an event. For any $A \in \mathcal{P}(\Theta)$, Liu and Liu [15] presented a credibility measure $\mathrm{Cr}\{A\}$ to express the chance that fuzzy event A occurs. Li and Liu [20] proved that a set function $\mathrm{Cr}\{.\}$ a credibility measure iff

Axiom i. $\mathrm{Cr}\{\Theta\} = 1$;

Axiom ii. $\mathrm{Cr}\{A\} \leq \mathrm{Cr}\{B\}$ whenever $A \subset B$;

Axiom iii. Cr is self-dual, i.e., $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$, for any $A \in \mathcal{P}(\Theta)$;

Axiom iv. $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$ with $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

The triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is named a credibility space. A fuzzy variable is investigated by Liu and Liu [15] as function from the credibility space to the set of real numbers.

Definition 2.1. *([15]) The expected value of fuzzy variable μ is given by*

$$E[\mu] = \int_0^{+\infty} \mathrm{Cr}\{\mu \geq r\}\, dr - \int_{-\infty}^0 \mathrm{Cr}\{\mu \leq r\}\, dr$$

provided that at least one of the two integrals is finite.

Wang and Liu [43] proved that convergence in credibility means convergence almost surely for a sequence of fuzzy variable sequences. Also, Liu [17] examined the convergence in mean means convergence in credibility.

Theorem 2.1. *Let μ be a fuzzy variable. Then, for any given numbers $t > 0$ and $p > 0$, we have*

$$Cr\left\{|\mu| \geq t\right\} \leq \frac{E\left[|\mu|^p\right]}{t^p}.$$

3. Main results

In this section, based on almost convergence, we study the almost convergence in credibility and the almost Cauchy sequence in credibility. To better explain our results, we first, present some crucial definitions.

Definition 3.1. *The double sequence $\{\mu_{\alpha,\beta}\}$ of fuzzy variable is known as almost convergent w.r.t. almost surely to the fuzzy variable μ if there is a $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ and for any $\eta > 0$, there exists $t_0 \in \mathbb{N}$ such that*

$$\left\| \frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}(\phi) - \mu(\phi) \right\| < \eta,\ \forall u,v > t_0 \text{ and uniformly } \forall \alpha,\beta \in \mathbb{N}.$$

Definition 3.2. *The sequence $\{\mu_{\alpha,\beta}\}$ is known as almost convergent in credibility to μ if for any $\eta, \sigma > 0$, there exists $t_0 \in \mathbb{N}$ such that*

$$\mathrm{Cr}\left\{ \left\| \frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s} - \mu \right\| \geq \sigma \right\} < \eta,\ \forall u,v > t_0 \text{ and uniformly } \forall \alpha,\beta \in \mathbb{N}.$$

Definition 3.3. *The sequence $\{\mu_{\alpha,\beta}\}$ is known as almost convergent in mean to μ if for any $\eta > 0$, there exists $t_0 \in \mathbb{N}$ such that*

$$E\left[\left\| \frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s} - \mu \right\| \right] < \eta,\ \forall u,v > t_0 \text{ and uniformly } \forall \alpha,\beta \in \mathbb{N}.$$

Definition 3.4. *Assume Φ and $\Phi_{\alpha,\beta}$ be the credibility distributions of the fuzzy variables μ, $\mu_{\alpha,\beta}$ respectively. Then, the double fuzzy variable sequence $\{\mu_{\alpha,\beta}\}$ is said almost convergent in distribution provided that for any $\eta > 0$, there exists $t_0 \in \mathbb{N}$ such that*

$$\left\| \frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\Phi_{\alpha+r,\beta+s}(y) - \Phi(y) \right\| < \eta,$$

for all y at which Φ is continuous and for all $u, v > t_0$ and $\alpha, \beta \in \mathbb{N}$.

Theorem 3.1. *If the double sequence of fuzzy variable $\{\mu_{\alpha,\beta}\}$ is almost convergent in mean to μ, then it is almost convergent in credibility. However, the converse is not generally true.*

Proof. Presume the sequence $\{\mu_{\alpha,\beta}\}$ almost converges in mean to μ. Then, for any $\eta > 0$, there exists $t_0 \in \mathbb{N}$ such that

$$E\left[\left\| \frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s} - \mu \right\| \right] < \eta,\ \forall u,v > t_0 \text{ and uniformly } \forall \alpha,\beta \in \mathbb{N}.$$

With the aid of Markov inequality, we acquire

$$\mathrm{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\eta\right\}\leq\frac{E\left[\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\right]}{\eta}\rightarrow 0$$

as $u,v\rightarrow\infty$. So, the sequence $\{\mu_{\alpha,\beta}\}$ is almost convergent w.r.t. credibility to μ. For the contrary, we can examine the subsequent example:

Take $(\Theta,\mathcal{P}(\Theta),\mathrm{Cr})$ to be $\{\phi_1,\phi_2,...\}$ with the fuzzy variable credibility Cr determined by

$$\mathrm{Cr}\{A\}=\begin{cases}\sup_{\phi_{\alpha+\beta}\in A}\frac{1}{(\alpha+\beta)^2+5}, & \text{if } \sup_{\phi_{\alpha+\beta}\in A}\frac{1}{(\alpha+\beta)^2+5}<0.5,\\ 1-\sup_{\phi_{\alpha+\beta}\in A^C}\frac{1}{(\alpha+\beta)^2+5}, & \text{if } \sup_{\phi_{\alpha+\beta}\in A^C}\frac{1}{(\alpha+\beta)^2+5}<0.5,\\ 0.5, & \text{otherwise.}\end{cases}$$

Take the fuzzy variable $\{\mu_{\alpha,\beta}\}$ as follows:

$$\mu_{\alpha,\beta}(\phi)=\begin{cases}(\alpha+\beta)^2+5, & \text{if } \phi=\phi_{\alpha+\beta},\\ 0, & \text{otherwise.}\end{cases}$$

for all $\alpha,\beta\in\mathbb{N}$ and $\mu(\phi)\equiv 0$, $\forall\phi\in A$.

For any given $\eta>0$, there exist $t_0\in\mathbb{N}$ and $\alpha,\beta\geq t_0$ such that

$$\frac{1}{(\alpha+\beta)^2+5}<0.5.$$

Then, we acquire

$$\begin{aligned}&\mathrm{Cr}\left\{\left\|\tfrac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\sigma\right\}\\ &=\mathrm{Cr}\left\{\phi:\left\|\tfrac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}(\phi)-\mu(\phi)\right\|\geq\sigma\right\}\\ &=\mathrm{Cr}\{\phi=\phi_{\alpha+\beta}\}=\sup_{\phi_{\alpha+\beta}\in A}\tfrac{1}{(\alpha+\beta)^2+5}<\eta,\text{ as }\alpha,\beta\rightarrow\infty\end{aligned}$$

As a result, the sequence $\{\mu_{\alpha,\beta}\}$ almost converges in credibility to μ.

Estimating to the distribution function $\Phi_{\alpha,\beta}$ for the fuzzy variable $\|\mu_{\alpha,\beta}-\mu\|=\|\mu_{\alpha,\beta}\|$ gives

$$\Phi_{\alpha,\beta}(y)=\begin{cases}0, & \text{if } y<0,\\ 1-\frac{1}{(\alpha+\beta)^2+5}, & \text{if } 0\leq y<(\alpha+\beta)^2+5,\\ 1, & \text{if } y\geq(\alpha+\beta)^2+5.\end{cases}$$

Then

$$E\left[\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\right]$$
$$=\int_{0}^{\infty}\left[1-\Phi_{\alpha,\beta}(y)\right]dy-\int_{-\infty}^{0}\Phi_{\alpha,\beta}(y)\,dy$$
$$=\int_{0}^{(\alpha+\beta)^2+5}\left(1-\left(1-\frac{1}{(\alpha+\beta)^2+5}\right)\right)dy+\int_{(\alpha+\beta)^2+5}^{+\infty}(1-1)\,dy-\int_{-\infty}^{0}0dy$$
$$=\int_{0}^{(\alpha+\beta)^2+5}\frac{1}{(\alpha+\beta)^2+5}dy=1.$$

So, the sequence $\{\mu_{\alpha,\beta}\}$ is not almost convergent in mean to μ. □

Theorem 3.2. *Presume that $\{\mu_{\alpha,\beta}\}$ and $\{\nu_{\alpha,\beta}\}$ be two sequences of fuzzy variables converge in credibility to μ and ν, respectively. If there are positive numbers P_1, P, T_1 and T such that $P_1\leq\|\mu_{\alpha,\beta}\|\leq P$ and $T_1\leq\|\nu_{\alpha,\beta}\|\leq T$ for any α,β, then*
(i) $\mu_{\alpha,\beta}+\nu_{\alpha,\beta}$ almost converges in credibility to $\mu+\nu$,
(ii) $\mu_{\alpha,\beta}-\nu_{\alpha,\beta}$ almost converges in credibility to $\mu-\nu$,
(iii) $\mu_{\alpha,\beta}\nu_{\alpha,\beta}$ almost converges in credibility to $\mu\nu$,
(iv) $\left\{\frac{\mu_{\alpha,\beta}}{\nu_{\alpha,\beta}}\right\}$ almost converges in credibility to $\left\{\frac{\mu}{\nu}\right\}$.

Proof. (i) Assume the sequence $\{\mu_{\alpha,\beta}\}$ and $\{\nu_{\alpha,\beta}\}$ be almost convergent in credibility. Then for any $\eta>0$,

$$\lim_{u,v\to\infty}\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{\eta}{2}\right\}=0,$$

and

$$\lim_{u,v\to\infty}\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\nu_{\alpha+r,\beta+s}-\nu\right\|\geq\frac{\eta}{2}\right\}=0,$$

$\forall u,v>t_0$ and uniformly $\forall\alpha,\beta\in\mathbb{N}$.

Since $P_1\leq\|\mu_{\alpha,\beta}\|\leq P$, we have $P_1\leq\|\mu\|\leq P$. In a similar way, if $T_1\leq\|\nu_{\alpha,\beta}\|\leq T$, then $T_1\leq\|\nu\|\leq T$. Then, we have

$$\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\left(\mu_{\alpha+r,\beta+s}+\nu_{\alpha+r,\beta+s}\right)-(\mu+\nu)\right\|\geq\eta\right\}$$
$$\subseteq\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{\eta}{2}\right\}\cup\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\nu_{\alpha+r,\beta+s}-\nu\right\|\geq\frac{\eta}{2}\right\}.$$

Using the Axiom ii (monotonicity), Axiom iii (self-duality) and Axiom iv (maximality) features of the credibility measure, it is easily observed that the credibility measure is subadditive. That is, $Cr\{A_1\cup A_2\}\leq Cr\{A_1\}+Cr\{A_2\}$ for any events A_1 and A_2. From the credibility subadditivity theorem in [19], it follows that

$$\begin{aligned}
&\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\left(\mu_{\alpha+r,\beta+s}+\nu_{\alpha+r,\beta+s}\right)-(\mu+\nu)\right\|\geq\eta\right\}\\
&=\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\left(\mu_{\alpha+r,\beta+s}-\mu\right)+\left(\nu_{\alpha+r,\beta+s}-\nu\right)\right\|\geq\eta\right\}\\
&\leq\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{\eta}{2}\right\}+\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\upsilon_{\alpha+r,\beta+s}-\upsilon\right\|\geq\frac{\eta}{2}\right\}\to 0,
\end{aligned}$$

as $u,v\to\infty$. Then, we get $\mu_{\alpha,\beta}+\nu_{\alpha,\beta}$ almost converges in credibility to $\mu+\nu$.

(ii) We see by Axiom iv that

$$\begin{aligned}
&\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\left(\mu_{\alpha+r,\beta+s}-\nu_{\alpha+r,\beta+s}\right)-(\mu-\nu)\right\|\geq\eta\right\}\\
&=\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\left(\mu_{\alpha+r,\beta+s}-\mu\right)+\left(\nu-\nu_{\alpha+r,\beta+s}\right)\right\|\geq\eta\right\}\\
&\leq\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{\eta}{2}\right\}+\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\nu-\nu_{\alpha+r,\beta+s}\right\|\geq\frac{\eta}{2}\right\}\to 0,
\end{aligned}$$

as $u,v\to\infty$. Then, we get $\mu_{\alpha,\beta}-\nu_{\alpha,\beta}$ almost converges in credibility to $\mu-\nu$.

(iii) We see by Axiom iv that

$$\begin{aligned}
&\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}\upsilon_{\alpha+r,\beta+s}-\mu\upsilon\right\|\geq\eta\right\}\\
&=\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}\upsilon_{\alpha+r,\beta+s}-\mu_{\alpha+r,\beta+s}\upsilon+\mu_{\alpha+r,\beta+s}\upsilon-\mu\upsilon\right\|\geq\eta\right\}\\
&\leq\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}\upsilon_{\alpha+r,\beta+s}-\mu_{\alpha+r,\beta+s}\upsilon\right\|\geq\frac{\eta}{2}\right\}\\
&+\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}\upsilon-\mu\upsilon\right\|\geq\frac{\eta}{2}\right\}\\
&\leq\operatorname{Cr}\left\{P\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\upsilon_{\alpha+r,\beta+s}-\upsilon\right\|\geq\frac{\eta}{2}\right\}\\
&+\operatorname{Cr}\left\{T\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{\eta}{2}\right\}\\
&\leq\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\upsilon_{\alpha+r,\beta+s}-\upsilon\right\|\geq\frac{\eta}{2P}\right\}\\
&+\operatorname{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{\eta}{2T}\right\}\to 0,
\end{aligned}$$

as $u,v\to\infty$. Then, we get $\mu_{\alpha,\beta}\nu_{\alpha,\beta}$ almost converges in credibility to $\mu\nu$.

(iv) We see by Axiom iv that

$$\begin{aligned}
&\mathrm{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\frac{\mu_{\alpha+r,\beta+s}}{\upsilon_{\alpha+r,\beta+s}}-\frac{\mu}{\upsilon}\right\|\geq\eta\right\}\\
&=\mathrm{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\frac{\mu_{\alpha+r,\beta+s}}{\upsilon_{\alpha+r,\beta+s}}-\frac{\mu_{\alpha+r,\beta+s}}{\upsilon}+\frac{\mu_{\alpha+r,\beta+s}}{\upsilon}-\frac{\mu}{\upsilon}\right\|\geq\eta\right\}\\
&=\mathrm{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\frac{\mu_{\alpha+r,\beta+s}\left(\upsilon-\upsilon_{\alpha+r,\beta+s}\right)}{\upsilon_{\alpha+r,\beta+s}\upsilon}+\frac{\mu_{\alpha+r,\beta+s}-\mu}{\upsilon}\right\|\geq\eta\right\}\\
&\leq\mathrm{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\frac{\mu_{\alpha+r,\beta+s}\left(\upsilon-\upsilon_{\alpha+r,\beta+s}\right)}{\upsilon_{\alpha+r,\beta+s}\upsilon}\right\|\geq\frac{\eta}{2}\right\}\\
&+\mathrm{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\frac{\mu_{\alpha+r,\beta+s}-\mu}{\upsilon}\right\|\geq\frac{\eta}{2}\right\}\\
&\leq\mathrm{Cr}\left\{\frac{P}{T_1^2}\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\upsilon-\upsilon_{\alpha+r,\beta+s}\right\|\geq\frac{\eta}{2}\right\}\\
&+\mathrm{Cr}\left\{\frac{1}{T_1}\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{\eta}{2}\right\}\\
&\leq\mathrm{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\upsilon-\upsilon_{\alpha+r,\beta+s}\right\|\geq\frac{\eta T_1^2}{2P}\right\}\\
&+\mathrm{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{\eta T_1}{2}\right\}\rightarrow 0,
\end{aligned}$$

as $u,v\rightarrow\infty$. Then, we get $\frac{\mu_{\alpha,\beta}}{\nu_{\alpha,\beta}}$ almost converges in credibility to $\frac{\mu}{\nu}$. □

A double sequence $\{\mu_{\alpha,\beta}\}$ of fuzzy variable is almost convergent w.r.t. almost surely does not necessarily give that it is almost convergent w.r.t. mean. The presentation of the same is given in the following example.

Example 3.1. *Contemplate the credibility space* $(\Theta,\mathcal{P}(\Theta),\mathrm{Cr})$ *to be* $\Theta=\{\phi_1,\phi_2,...\}$ *such that with* $\mathrm{Cr}\{A\}=\sum_{\phi_r,\phi_s\in A}\frac{1}{2^{(r+s)}}$. *Establish the fuzzy variables* $\mu_{\alpha,\beta}$ *and* μ *by*

$$\mu_{\alpha,\beta}(\phi)=\begin{cases}2^{\alpha+\beta}, & \text{if } \phi=\phi_{\alpha+\beta},\\ 0, & \text{otherwise,}\end{cases}$$

and $\mu\equiv 0$, $\forall\phi\in\Theta$ *and for* $\alpha,\beta\in\mathbb{N}$. *Calculating in the same way like above example, we get* $\{\mu_{\alpha,\beta}\}$ *almost converges to* μ *w.r.t. almost surely. Fuzzy variable distribution function of fuzzy variable sequence* $\{\mu_{\alpha,\beta}\}$ *is given by*

$$\Phi_{\alpha,\beta}(y)=\begin{cases}0, & \text{if } y<0,\\ 1-\frac{1}{2^{\alpha+\beta}}, & \text{if } 0\leq y<2^{\alpha+\beta},\\ 1, & \text{otherwise.}\end{cases}$$

for $\alpha, \beta \in \mathbb{N}$. *Now*

$$E\left[\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\right]=\int_{0}^{\infty}\left[1-\Phi_{\alpha,\beta}\left(y\right)\right]dy-\int_{-\infty}^{0}\Phi_{\alpha,\beta}\left(y\right)dy$$

$$=\int_{0}^{2^{\alpha+\beta}}\left(1-\left(1-\tfrac{1}{2^{\alpha+\beta}}\right)\right)dy+\int_{2^{\alpha+\beta}}^{+\infty}\left(1-1\right)dy-\int_{-\infty}^{0}0dy=\int_{0}^{2^{\alpha+\beta}}\tfrac{1}{2^{\alpha+\beta}}dy=1.$$

Hence, the sequence $\{\mu_{\alpha,\beta}\}$ *is not almost convergent in mean to* μ.

Proposition 3.1. *Take* $\{\mu_{\alpha,\beta}\}$ *as a double sequence of fuzzy variable. Then, it almost converges in almost surely to* μ *iff for any* $\eta > 0$ *there is a* $P \in \mathbb{N}$ *such that for all* $\alpha, \beta > P$,

$$\mathrm{Cr}\left(\bigcap_{P=1}^{\infty}\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}\left(\phi\right)-\mu\left(\phi\right)\right\|\geq\eta\right)=0.$$

Proof. As stated in the definition of almost convergence in almost surely of fuzzy variable double sequence provides us the existence of a $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{\alpha,\beta\to\infty}\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|=0.$$

Let $\eta > 0$. Then, there exists $P \in \mathbb{N}$ such that for any $\phi \in \Theta$, we get

$$\mathrm{Cr}\left(\bigcap_{P=1}^{\infty}\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}\left(\phi\right)-\mu\left(\phi\right)\right\|<\eta\right)=1.$$

From the duality axiom of credibility measure we acquire

$$\mathrm{Cr}\left(\bigcap_{P=1}^{\infty}\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}\left(\phi\right)-\mu\left(\phi\right)\right\|\geq\eta\right)=0.$$

□

Proposition 3.2. *Take* $\{\mu_{\alpha,\beta}\}$ *as a double sequence of fuzzy variable, where* $\alpha, \beta = 1, 2, \ldots$ *Then, the sequence* $\{\mu_{\alpha,\beta}\}$ *almost converges w.r.t. uniformly almost surely to* μ *iff for a given* $\eta > 0$, *there exists* $\delta > 0$ *and* $P \in \mathbb{N}$ *such that*

$$\mathrm{Cr}\left(\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\delta\right)<\eta.$$

Proof. Assume the sequence $\{\mu_{\alpha,\beta}\}$ of fuzzy variable almost converges w.r.t. almost surely to μ. Then, for any $\eta > 0$, there exists $\delta > 0$, and H with credibility measure less than ϑ and the sequence $\{\mu_{\alpha,\beta}\}$ converges uniformly to μ on $\mathcal{P}(\Theta) - H$. So, for any $\eta > 0$, there exists $P \in \mathbb{N}$ such that

$$\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}\left(\phi\right)-\mu\left(\phi\right)\right\|<\delta,$$

for all $\alpha, \beta \geq P$ and all $\phi \in \mathcal{P}(\Theta) - H$. Thus, we obtain

$$\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}\left(\phi\right)-\mu\left(\phi\right)\right\|\geq\delta\right\}\subseteq H.$$

Utilizing the subadditivity axiom of credibility measure, we have

$$\mathrm{Cr}\left(\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\delta\right\}\right)\leq\mathrm{Cr}\left(H\right)<\vartheta<\eta.$$

On the contrary, assume

$$\mathrm{Cr}\left(\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\delta\right\}\right)<\eta.$$

We take $\delta>0$. Then, for any $\vartheta>0$, $t\geq 1$, there exists $t_w>0$ such that

$$\mathrm{Cr}\left(\bigcup_{\alpha=t_w}^{\infty}\bigcup_{\beta=t_w}^{\infty}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{1}{t}\right\}\right)<\frac{\vartheta}{2^t}.$$

Think

$$H=\bigcup_{t=1}^{\infty}\bigcup_{\alpha=t_w}^{\infty}\bigcup_{\beta=t_w}^{\infty}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{1}{t}\right\}.$$

Then

$$\mathrm{Cr}\left(H\right)\leq\sum_{t=1}^{\infty}\mathrm{Cr}\left(\bigcup_{\alpha=t_w}^{\infty}\bigcup_{\beta=t_w}^{\infty}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\frac{1}{t}\right\}\right)\leq\sum_{t=1}^{\infty}\frac{\vartheta}{2^t}=\vartheta.$$

In addition,

$$\sup_{\phi\in\mathcal{P}(\Theta)-H}\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|<\frac{1}{q},$$

where $q=1,2,3,...$ and $\alpha,\beta>t_w$. Hence, the proposition is proved. □

Theorem 3.3. *Let $\{\mu_{\alpha,\beta}\}$ be an almost convergent w.r.t. uniformly almost surely to μ. Then, $\{\mu_{\alpha,\beta}\}$ is an almost convergent sequence in almost surely to μ.*

Proof. Assume $\{\mu_{\alpha,\beta}\}$ almost converges w.r.t. uniformly almost surely to μ. Then, we get

$$\mathrm{Cr}\left(\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\delta\right\}\right)<\eta.$$

Now, since

$$\mathrm{Cr}\left(\bigcap_{P=1}^{\infty}\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\{\left\|\tfrac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\delta\right\}\right)$$
$$\leq\mathrm{Cr}\left(\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\{\left\|\tfrac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\delta\right\}\right)<\eta.$$

Hence, $\{\mu_{\alpha,\beta}\}$ almost converges in almost surely to μ by the Proposition 3.2. □

Theorem 3.4. *The double fuzzy variable sequence $\{\mu_{\alpha,\beta}\}$ which almost converges w.r.t. uniformly almost surely to μ is also almost converges in credibility to μ.*

Proof. Presume $\{\mu_{\alpha,\beta}\}$ almost converges w.r.t. uniformly almost surely to μ. Then, for any $\eta > 0$ and $\delta > 0$ there exists $P \in \mathbb{N}$ such that

$$\mathrm{Cr}\left(\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\delta\right\}\right)<\eta$$

and we get

$$\mathrm{Cr}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\delta\right\}\leq\mathrm{Cr}\left(\bigcup_{\alpha=P}^{\infty}\bigcup_{\beta=P}^{\infty}\left\{\left\|\frac{1}{uv}\sum_{r=0}^{u-1}\sum_{s=0}^{v-1}\mu_{\alpha+r,\beta+s}-\mu\right\|\geq\delta\right\}\right)<\eta.$$

So, $\{\mu_{\alpha,\beta}\}$ almost converges in credibility to μ. □

Definition 3.5. *The double sequence of fuzzy variable $\{\mu_{\alpha,\beta}\}$ is almost Cauchy w.r.t. almost surely, provided that for any given $\eta > 0$, there are $w_0 \in \mathbb{N}$ and $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that*

$$\left\|\frac{1}{u_1v_1}\sum_{r=0}^{u_1-1}\sum_{s=0}^{v_1-1}\mu_{\alpha_1+r,\beta_1+s}(\phi)-\frac{1}{u_2v_2}\sum_{r=0}^{u_2-1}\sum_{s=0}^{v_2-1}\mu_{\alpha_2+r,\beta_2+s}(\phi)\right\|<\eta,$$

for all $u_1, v_1, u_2, v_2 > w_0$, $\phi \in A$ and uniformly for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$.

Definition 3.6. *The sequence $\{\mu_{\alpha,\beta}\}$ is almost Cauchy in credibility, provided that for any given $\eta, \delta > 0$, there exists $w_0 \in \mathbb{N}$ such that*

$$\mathrm{Cr}\left\{\left\|\frac{1}{u_1v_1}\sum_{r=0}^{u_1-1}\sum_{s=0}^{v_1-1}\mu_{\alpha_1+r,\beta_1+s}-\frac{1}{u_2v_2}\sum_{r=0}^{u_2-1}\sum_{s=0}^{v_2-1}\mu_{\alpha_2+r,\beta_2+s}\right\|\geq\delta\right\}<\eta,$$

for all $u_1, v_1, u_2, v_2 > w_0$ and uniformly for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$.

Definition 3.7. *The sequence $\{\mu_{\alpha,\beta}\}$ is almost Cauchy in mean, provided that for any given $\eta > 0$, there exists $w_0 \in \mathbb{N}$ such that*

$$E\left[\left\|\frac{1}{u_1v_1}\sum_{r=0}^{u_1-1}\sum_{s=0}^{v_1-1}\mu_{\alpha_1+r,\beta_1+s}-\frac{1}{u_2v_2}\sum_{r=0}^{u_2-1}\sum_{s=0}^{v_2-1}\mu_{\alpha_2+r,\beta_2+s}\right\|\right]<\eta,$$

for all $u_1, v_1, u_2, v_2 > w_0$ and uniformly for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$.

Definition 3.8. *Let $\Phi_{\alpha,\beta}$ be the fuzzy credibility distributions of fuzzy variables $\mu_{\alpha,\beta}$. Then, the sequence $\{\mu_{\alpha,\beta}\}$ is known as almost Cauchy in distribution if for any $\eta > 0$, there exists $w_0 \in \mathbb{N}$ such that*

$$\left\|\frac{1}{u_1v_1}\sum_{r=0}^{u_1-1}\sum_{s=0s}^{v_1-1}\Phi_{\alpha_1+r,\beta_1+s}(y)-\frac{1}{u_2v_2}\sum_{r=0}^{u_2-1}\sum_{s=0}^{v_2-1}\Phi_{\alpha_2+r,\beta_2+s}(y)\right\|<\eta,$$

for all $u_1, v_1, u_2, v_2 > w_0$, uniformly for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$ and for all y at which the distribution function is continuous.

Definition 3.9. *The sequence $\{\mu_{\alpha,\beta}\}$ is known as almost Cauchy w.r.t. uniformly almost surely if there is a $A'_i \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A'_i\} \to 0$ such that $\{\mu_{\alpha,\beta}\}$ is almost Cauchy in $\mathcal{P}(\Theta) - A_i$, for every fixed $i \in \mathbb{N}$.*

Theorem 3.5. *The fuzzy variable sequence sequence $\{\mu_{\alpha,\beta}\}$ is almost convergent w.r.t. almost surely iff it is an almost Cauchy sequence w.r.t. almost surely.*

Proof. Assume the sequence $\{\mu_{\alpha,\beta}\}$ is almost convergent w.r.t. almost surely to μ. Then, there is a $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ and for any $\eta > 0$, there is a $w_0 \in \mathbb{N}$ such that

$$\left\| \frac{1}{uv} \sum_{r=0}^{u-1} \sum_{s=0}^{v-1} \mu_{\alpha+r,\beta+s}(\phi) - \mu(\phi) \right\| < \frac{\eta}{2}, \ \forall u, v > t_0 \text{ and uniformly } \forall \alpha, \beta \in \mathbb{N}.$$

Therefore, we acquire

$$\begin{aligned} &\left\| \tfrac{1}{u_1 v_1} \sum_{r=0}^{u_1-1} \sum_{s=0}^{v_1-1} \mu_{\alpha_1+r,\beta_1+s}(\phi) - \tfrac{1}{u_2 v_2} \sum_{r=0}^{u_2-1} \sum_{s=0}^{v_2-1} \mu_{\alpha_2+r,\beta_2+s}(\phi) \right\| \\ &\le \left\| \tfrac{1}{u_1 v_1} \sum_{r=0}^{u_1-1} \sum_{s=0}^{v_1-1} \mu_{\alpha_1+r,\beta_1+s}(\phi) - \mu(\phi) \right\| + \left\| \tfrac{1}{u_2 v_2} \sum_{r=0}^{u_2-1} \sum_{s=0}^{v_2-1} \mu_{\alpha_2+r,\beta_2+s}(\phi) - \mu(\phi) \right\| \\ &< \tfrac{\eta}{2} + \tfrac{\eta}{2} = \eta, \end{aligned}$$

for all $u_1, v_1, u_2, v_2 > w_0$, $\phi \in \Theta$ and for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$. Hence, $\{\mu_{\alpha,\beta}\}$ almost Cauchy sequence w.r.t. almost surely.

Conversely, assume $\{\mu_{\alpha,\beta}\}$ be almost Cauchy sequence w.r.t. almost surely. Then, for any given $\eta > 0$, there exist $w_0 \in \mathbb{N}$ and $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\left\| \frac{1}{u_1 v_1} \sum_{r=0}^{u_1-1} \sum_{s=0}^{v_1-1} \mu_{\alpha_1+r,\beta_1+s}(\phi) - \frac{1}{u_2 v_2} \sum_{r=0}^{u_2-1} \sum_{s=0}^{v_2-1} \mu_{\alpha_2+r,\beta_2+s}(\phi) \right\| < \frac{\eta}{2},$$

for all $u_1, v_1, u_2, v_2 > w_0$, $\phi \in A$ and uniformly for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$. Taking $\alpha_1 = \alpha_2 = \alpha_0$ and $\beta = \beta_2 = \beta_0$ in the above equation, we acquire that $\left(\frac{1}{uv} \sum_{r=0}^{u-1} \sum_{s=0}^{v-1} \mu_{\alpha_0+r,\beta_0+s}(\phi) \right)_{u,v=1}^{\infty}$ becomes a Cauchy sequence and so convergent. For any $\eta > 0$, there exists $t_0 \in \mathbb{N}$ such that

$$\left\| \frac{1}{uv} \sum_{r=0}^{u-1} \sum_{s=0}^{v-1} \mu_{\alpha_0+r,\beta_0+s}(\phi) - \mu(\phi) \right\| < \frac{\eta}{2}, \ \forall u, v > t_0 \text{ and uniformly } \forall \alpha, \beta \in \mathbb{N}.$$

Then,

$$\begin{aligned} &\left\| \tfrac{1}{uv} \sum_{r=0}^{u-1} \sum_{s=0}^{v-1} \mu_{\alpha+r,\beta+s}(\phi) - \mu(\phi) \right\| \le \left\| \tfrac{1}{uv} \sum_{r=0}^{u-1} \sum_{s=0}^{v-1} \mu_{\alpha+r,\beta+s}(\phi) - \tfrac{1}{uv} \sum_{r=0}^{u-1} \sum_{s=0}^{v-1} \mu_{\alpha_0+r,\beta_0+s}(\phi) \right\| \\ &+ \left\| \tfrac{1}{uv} \sum_{r=0}^{u-1} \sum_{s=0}^{v-1} \mu_{\alpha_0+r,\beta_0+s}(\phi) - \mu(\phi) \right\| < \tfrac{\eta}{2} + \tfrac{\eta}{2} = \eta, \end{aligned}$$

for all $u, v > \max(t_0, t)$ and all $\alpha, \beta \in \mathbb{N}$. So, $\{\mu_{\alpha,\beta}\}$ is almost convergent w.r.t. almost surely to μ. □

Theorem 3.6. *If the sequence $\{\mu_{\alpha,\beta}\}$ is almost convergent in credibility then it is almost Cauchy in credibility.*

Proof. Presume the sequence $\{\mu_{\alpha,\beta}\}$ is almost convergent in credibility to μ. Then, for each $\eta, \delta > 0$, there exists $w_0 \in \mathbb{N}$ such that

$$\mathrm{Cr}\left(\left\| \frac{1}{uv} \sum_{r=0}^{u-1} \sum_{s=0}^{v-1} \mu_{\alpha_0+r,\beta_0+s} - \mu \right\| \ge \delta \right) < \frac{\eta}{2}, \ \forall u, v > w_0 \text{ and } \forall \alpha, \beta \in \mathbb{N}.$$

Then,

$$\begin{aligned}
&\mathrm{Cr}\left\{\left\|\frac{1}{u_1v_1}\sum_{r=0}^{u_1-1}\sum_{s=0}^{v_1-1}\mu_{\alpha_0+r,\beta_0+s}-\frac{1}{u_2v_2}\sum_{r=0}^{u_2-1}\sum_{s=0}^{v_2-1}\mu_{\alpha_0+r,\beta_0+s}\right\|\geq\delta\right\}\\
&\leq \mathrm{Cr}\left\{\left\|\frac{1}{u_1v_1}\sum_{r=0}^{u_1-1}\sum_{s=0}^{v_1-1}\mu_{\alpha_0+r,\beta_0+s}-\mu\right\|\geq\delta\right\}\\
&+\mathrm{Cr}\left\{\left\|\frac{1}{u_2v_2}\sum_{r=0}^{u_2-1}\sum_{s=0}^{v_2-1}\mu_{\alpha_0+r,\beta_0+s}-\mu\right\|\geq\delta\right\}<\frac{\eta}{2}+\frac{\eta}{2}=\eta,
\end{aligned}$$

for all $u_1, v_1, u_2, v_2 > w$ and for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$. Hence, $\{\mu_{\alpha,\beta}\}$ is almost Cauchy in credibility. □

Definition 3.10. *A fuzzy variable sequence $\{\mu_{\alpha,\beta}\}$ is known as Cesàro summable to μ if*

$$\lim_{u,v\to\infty}\left\|\frac{1}{uv}\sum_{\alpha=1}^{u}\sum_{\beta=1}^{v}\mu_{\alpha,\beta}(\phi)-\mu(\phi)\right\|=0,$$

for every $\phi \in A$, $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$.

Theorem 3.7. *Take $\{\mu_{\alpha,\beta}\}$ as a double fuzzy variable sequence. If the sequence is convergent, then it is also Cesàro summable to the same limit.*

Proof. Let fuzzy variable sequence $\{\mu_{\alpha,\beta}\}$ be convergent. Then there is a constant H such that $\|\mu_{\alpha,\beta}(\phi)\| < H$ for all $\alpha, \beta \in \mathbb{N}$, $\phi \in A$, and given $\eta > 0$, there is an integer w_0 such that $\|\mu_{\alpha,\beta}(\phi) - \mu(\phi)\| < \eta$ for all $\alpha, \beta > w_0$. Select an integer $P \geq w_0$ such that $P > \frac{2Hw_0}{\eta}$. Then, we acquire

$$\begin{aligned}
\left\|\frac{1}{uv}\sum_{\alpha=1}^{u}\sum_{\beta=1}^{v}\mu_{\alpha,\beta}(\phi)-\mu(\phi)\right\| &= \left\|\frac{(\mu_{1,1}(\phi)-\mu(\phi))+...+(\mu_{u,v}(\phi)-\mu(\phi))}{uv}\right\|\\
&\leq \frac{\|(\mu_{1,1}(\phi)-\mu(\phi))\|+...+\|(\mu_{w_0,w_0}(\phi)-\mu(\phi))\|}{uv}\\
&+\frac{\|(\mu_{w_0+1,w_0+1}(\phi)-\mu(\phi))\|+...+\|(\mu_{u,v}(\phi)-\mu(\phi))\|}{uv}\\
&\leq \frac{2Hw_0}{uv}+\frac{(u-w_0)(v-w_0)\eta}{uv}<\eta+\eta=2\eta.
\end{aligned}$$

This denotes that $\{\mu_{\alpha,\beta}\}$ is Cesàro summable. □

4. Conclusions

As we all know, the idea of convergence is critical in credibility theory. We examined how the study contributes to the summability theory and credibility theory research domains in the following ways in the paper: First, some critical and valuable definitions were presented. Then, some forms of almost convergence concept of the fuzzy variable double sequence were defined, and some mathematical aspects of those new convergence concepts were supplied. Specifically, we looked at several types of almost convergence, such as almost convergence almost surely, almost convergence in credibility, and almost convergence in the mean of the fuzzy variable double sequences. The results in the study produced by us did not previously exist. Future research will provide intriguing outcomes.

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