

## PERMUTATIONS WITH NON-DECREASING TRANSPOSITION ARRAYS AND PATTERN AVOIDANCE

FUFA BEYENE<sup>1\*</sup>, ROBERTO MANTACI<sup>2</sup>, §

ABSTRACT. We give some results about a bijection associating each permutation with a subexcedant function (*i.e.*, a function  $f : [n] \rightarrow [n]$  such that  $f(i) \leq i, 1 \leq i \leq n$ ). This function is related to a particular decomposition of the permutation as a product of transpositions and therefore it has been called transposition array in the literature. In particular, we identify anti-excedance positions of the permutation through its transposition array and we give an expression of the bijection in terms of the cycle structure of the permutation. We give a characterization of a family of permutations having a non-decreasing transposition array and study length 3 pattern avoidance therein.

Keywords: Permutation; Subexcedant function; Transposition array; Pattern avoidance; Integer partition.

AMS Subject Classification: 05A05; 05A17; 05A19.

### 1. INTRODUCTION

Let  $n$  be a fixed positive integer and  $[n] := \{1, 2, \dots, n\}$ . At least since the works of Lehmer ([9], 1960) or perhaps even much earlier (Laisant [8], 1888), permutations over  $[n]$  have been coded with integer  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  such that  $1 \leq a_i \leq i, \forall i \in [n]$ , or equivalently with functions  $f : [n] \rightarrow [n]$  such that  $1 \leq f(i) \leq i, \forall i \in [n]$ , called *subexcedant functions*. A subexcedant function  $f$  is written as the word  $f_1 f_2 \cdots f_n$ , where  $f_i = f(i), \forall i \in [n]$ .

Permutation codes are interesting because certain algorithms perform better over the codes than they do over the permutations themselves. Codes allow for instance to implement efficient algorithms for the exhaustive generation of some specific classes of permutations [4, 5, 6, 10, 11, 12]. To do so, one often has to “read” the properties of the permutation in its code and this gives birth to interesting combinatorial problems.

---

<sup>1</sup> Addis Ababa University- Department of Mathematics, 1176 Addis Ababa, Ethiopia.  
e-mail: fufa.beyene@aau.edu.et; ORCID: <https://orcid.org/0000-0002-4578-5652>.

\* Corresponding author.

<sup>2</sup> Université de Paris, IRIF, 8 Pl. Aurélie Numours, 75013 Paris, France.  
e-mail: mantaci@irif.fr; ORCID: <https://orcid.org/0000-0002-5906-3394>.

§ Manuscript received: June 19, 2022; accepted: October 18, 2022.

TWMS Journal of Applied and Engineering Mathematics, Vol.14, No.2 © Işık University, Department of Mathematics, 2024; all rights reserved.

Any permutation over  $[n]$  can be decomposed in several ways as a product of *transpositions*, cycles of length 2, but it can be decomposed only in one way as a product of transpositions of the kind  $(n, f_n)(n-1, f_{n-1}) \cdots (1, f_1)$  and in one way as  $(1, f'_1)(2, f'_2) \cdots (n, f'_n)$ , where  $f_1 f_2 \cdots f_n = f$  and  $f'_1 f'_2 \cdots f'_n = f'$  are subexcedant functions over  $[n]$ . In the product of these transpositions, if  $f(i) = i$  for some  $i$  correspondingly for  $f'$ , then the corresponding transposition degenerates into the identity. The first version, associating the code  $f$  to the permutation  $\sigma = \phi(f) = (n, f_n)(n-1, f_{n-1}) \cdots (1, f_1)$  was studied for instance by Rakotondrajao and the second author [10], who related excedances to appropriate statistics on the corresponding codes. The second version, associating the code  $f'$  to the permutation  $\sigma = \tilde{\phi}(f) = (1, f'_1)(2, f'_2) \cdots (n, f'_n)$  was independently introduced by Baril [2], and who also studied in [3] in particular the positions of weak excedances in a permutation using the corresponding code and who designates this permutation code with the term *transposition array*, a name that can be sensibly attributed to the bijection of the first version as well.

In this article we study transposition arrays obtained by the bijection  $\phi$  described in [10]. The organization of the paper is as follows: In Section 2, we recall some definitions and some basic results about this permutation code. In Section 3, we provide a new definition of the bijection  $\phi$  based on the decomposition of the permutation as a product of disjoint cycles. In Section 4, we study permutations whose transposition array is non-decreasing. We conclude the article with a section studying the avoidance of length 3 patterns in the set of permutations with non-decreasing transposition arrays.

## 2. NOTATION AND PRELIMINARIES

Let  $\mathfrak{S}_n$  be the symmetric group of permutations over the set  $[n]$ . A permutation  $\sigma \in \mathfrak{S}_n$  can be given as a word  $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$  or in cycle notation, where as usual a *cycle* in  $\sigma$  can be written as  $(j, \sigma(j), \sigma^2(j), \dots, \sigma^{t-1}(j))$ , where  $t$ , the *length* of the cycle, is the first positive integer such that  $\sigma^t(j) = j$ . Cycles of length one are *fixed points*. The cycle notation is  $\sigma = C_1 C_2 \cdots C_k$ , where the  $C_i$ 's are the different and thus disjoint cycles of  $\sigma$ .

A permutation  $\sigma$  is said to have an excedance at  $i \in [n]$  if  $\sigma(i) > i$ , a weak excedance if  $\sigma(i) \geq i$ , and an anti-excedance if  $\sigma(i) \leq i$ . If  $\sigma$  has an excedance (respectively, weak excedance or anti-excedance) at  $i$ ,  $\sigma(i)$  is called an excedance (respectively, weak excedance or anti-excedance) letter. If  $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \in \mathfrak{S}_n$ , then we use the notation:

$$\begin{aligned} \text{Exc}(\sigma) &:= \{i \in [n] : \sigma(i) > i\}, \\ \text{Wex}(\sigma) &:= \{i \in [n] : \sigma(i) \geq i\}, \\ \text{Ax}(\sigma) &:= \{i \in [n] : \sigma(i) \leq i\}, \text{ and} \\ \text{AxL}(\sigma) &:= \{\sigma(i) : i \in \text{Ax}(\sigma)\}. \end{aligned}$$

Furthermore,  $\text{exc}(\sigma) := |\text{Exc}(\sigma)|$ ,  $\text{wex}(\sigma) := |\text{Wex}(\sigma)|$ , and  $\text{ax}(\sigma) := |\text{Ax}(\sigma)|$ .

Recall that a subexcedant function is a function  $f : [n] \rightarrow [n]$  such that  $1 \leq f_i \leq i, \forall i \in [n]$ . We let  $F_n$  denote the set of all subexcedant functions over  $[n]$ . Let  $f = f_1 f_2 \cdots f_n \in F_n$ . Then the image set  $\text{Im}(f)$  denotes the set of elements in  $f([n])$  and  $\text{im}(f)$  is its cardinality. Further we define the *statistics*  $\text{Rmo}$ ,  $\text{Plat}$  and  $\text{Fix}$  as follows:

$$\begin{aligned} \text{Rmo}(f) &:= \{i : 1 \leq i \leq n, f_i \notin \{f_{i+1}, \dots, f_n\}\}, \\ \text{Plat}(f) &:= \{i : 1 \leq i < n, f_i = f_{i+1}\}, \text{ and} \\ \text{Fix}(f) &:= \{i : 1 \leq i \leq n, f_i = i\}. \end{aligned}$$

Furthermore,  $\text{plat}(f) := |\text{Plat}(f)|$  and  $\text{fix}(f) := |\text{Fix}(f)|$ .

**2.1. The inverse function of  $\phi$ .** As shown in [10], the procedure to obtain the subexcedant function  $f = f_1 f_2 \cdots f_n$  associated with a permutation  $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$  such that  $\phi(f) = \sigma$  is described by an  $n$ -step loop.

- For  $i = n, n - 1, \dots, 1$ :
  - set  $f_i = \sigma(i)$
  - set  $\sigma = (i, \sigma(i)) \circ \sigma$  (this swaps  $i$  and  $\sigma(i)$  in the permutation)

For  $i = n, n - 1, \dots, 1$  we let  $\sigma_i$  be the permutation at the iteration of the “for” loop corresponding to a given  $i$ , then the procedure sets  $f_i = \sigma_i(i)$ , where  $\sigma_n = \sigma$  and  $\sigma_i = (i + 1, \sigma_{i+1}(i + 1)) \circ \sigma_{i+1}$ .

**Example 2.1.** Take  $\sigma = 23154$ . Then  $\sigma_5 = 2315\bar{4}$  and  $f_5 = \sigma_5(5) = \sigma(5) = 4$  and  $\sigma_4 = (5, 4) \circ \sigma_5 = 231\bar{4}5$ . Now  $f_4 = \sigma_4(4) = 4$  and  $\sigma_3 = (4, 4) \circ \sigma_4 = 231\bar{4}5$ . From  $\sigma_3$  we have  $f_3 = \sigma_3(3) = 1$  and  $\sigma_2 = (3, 1) \circ \sigma_3 = 2\bar{1}3\bar{4}5$ . So  $f_2 = \sigma_2(2) = 1$  and  $\sigma_1 = (2, 1) \circ \sigma_2 = \bar{1}2\bar{3}4\bar{5}$ . Finally from  $\sigma_1$  we have  $f_1 = 1$ . Therefore,  $f = f_1 f_2 f_3 f_4 f_5 = 11144$ .

**Remark 2.1.** We note that at the end of the iteration for a given  $i$ , the permutation  $\sigma_i$  fixes all points greater than  $i$  (whence it may be regarded as a permutation on  $[i]$ ). Therefore, the algorithm sorts the permutation  $\sigma$  (by selecting the maximum and swapping).

**2.2. Some properties of  $\phi$ .**

**2.2.1. The insertion method.** By observing the sorting process of the permutation  $\sigma$  when we compute its corresponding transposition array  $f = \phi^{-1}(\sigma)$ , we have another way to express and compute  $\sigma = \phi(f)$  other than as a product of transpositions. Indeed, a given subexcedant function  $f = f_1 f_2 \cdots f_n$  contains the information to easily construct  $\sigma$  by “undoing” that sorting process. The algorithm follows. Start with the identity  $\sigma_1 = 1$ , suppose that the integers  $1, 2, \dots, i - 1$  have already been inserted. Then replace the integer  $f_i$  by  $i$  and append the integer  $f_i$  at the end. If  $f_i = i$ , then simply append  $f_i$  at the end.

**Example 2.2.** Take  $f = 11144 \in F_5$ , then we construct  $\sigma = \phi(f)$  as follows.

$$\begin{aligned} \sigma_1 &= 1 \\ \sigma_2 &= 21 \\ \sigma_3 &= 231 \\ \sigma_4 &= 2314 \\ \sigma_5 &= 23154 = \sigma \end{aligned}$$

As shown in [10, Proposition 3.5], the set  $\text{Im}(f)$  coincides with the set of anti-excedance letters of  $\sigma = \phi(f)$ , i.e.,  $i \in \text{Im}(f)$  if and only if  $\sigma^{-1}(i) \in \text{Ax}(\sigma)$ . Independently, in [3, Lemma 4], Baril has proved that the set  $\text{Im}(f)$  coincides with the set of weak excedances of  $\sigma^{-1} = \tilde{\phi}(f)$ . Actually these two statements are equivalent, since obviously we have

**Lemma 2.1.** *The integer  $i \in [n]$  is an anti-excedance of  $\sigma$  if and only if  $\sigma(i)$  is a weak excedance of  $\sigma^{-1}$ . □*

In a similar manner we have the following.

**Proposition 2.1.** *Let  $f = f_1 f_2 \cdots f_n \in F_n, \phi(f) = \sigma$ , and  $i \in [n]$ .*

- (1)  $\text{Rmo}(f) = \text{Ax}(\sigma)$ .
- (2)  $f_i = \sigma(i)$  iff  $i \in \text{Ax}(\sigma)$ .

*Proof. Point 1.* In the product  $(n, f_n) \cdots (2, f_2)(1, f_1)$  the transpositions  $(j, f_j)$  with  $j < i$  fix the integer  $i$ ; the transposition  $(i, f_i)$  transforms  $i$  into an integer  $f_i \leq i$ . If  $i \notin \text{Ax}(\sigma)$ , i.e.,  $\sigma(i) > i$ , then the integer  $f_i$  must appear in one of the terms  $(j, f_j)$  for  $j > i$ . Therefore, there exists  $j > i$  such that  $f_i = f_j$ , i.e.,  $i \notin \text{Rmo}(f)$ . Conversely, if  $i \notin \text{Rmo}(f)$ , i.e., there exists  $j > i$  such that  $f_i = f_j$ , then the transposition  $(i, f_i)$  transforms  $i$  into an integer  $f_i$  and the transposition  $(j, f_i)$  transforms  $f_i$  into a  $j$ . Now  $j$  can appear in the transpositions  $(\ell, f_\ell)$  with  $\ell > j$  if and only if  $j = f_\ell \in \text{Im}(f)$ . In this case  $j$  is transformed into a larger integer  $\ell$ , and so on. Thus,  $\sigma(i) > i$  and  $i \notin \text{Ax}(\sigma)$ .

**Point 2.** Suppose  $\sigma$  has an anti-excedance at  $i$ . Then  $\sigma(i) \leq i$ . Observe that the procedure to compute  $\phi^{-1}(\sigma)$  does not move  $\sigma(i)$  during its first  $n - i$  iterations. Indeed each of the steps for  $k = n, n - 1, \dots, i + 1$  swaps the integers  $k$  and  $\sigma(k)$ . None of the integers involved in the swaps can be  $\sigma(i)$ , because on one hand  $k$  only takes values larger than  $i$  and  $\sigma(i) \leq i$  and on the other hand none of the  $\sigma(k)$  can be equal to  $\sigma(i)$  because  $\sigma$  is a bijection. Thus  $\sigma_i(i) = \sigma(i)$ , and hence  $f_i = \sigma(i)$ .  $\square$

### 3. TRANSPOSITION ARRAYS AND CYCLES

**Definition 3.1.** Let  $\sigma \in \mathfrak{S}_n$ . For any integer  $i \in [n]$  we define the nearest orbital minorant of  $i$  (under  $\sigma$ ) as the integer  $j \leq i$  such that  $j = \sigma^t(i)$  with  $t \geq 1$  chosen as small as possible. We denote this integer by  $\text{nom}_\sigma(i)$  or simply  $\text{nom}(i)$  when there is no ambiguity regarding the permutation  $\sigma$ .

We now provide an alternative definition of the transposition array  $\phi^{-1}(\sigma)$  that is related to the cycle structure of the permutation  $\sigma$ . It suffices to observe the effect of multiplying the initial permutation by the transpositions  $(i, \sigma(i))$ .

**Theorem 3.1.** For every  $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \in \mathfrak{S}_n$  we have  $\phi^{-1}(\sigma) = f = f_1 f_2 \cdots f_n$ , where  $f_i = \text{nom}(i)$ ,  $\forall i \in [n]$ .

*Proof.* We use descending recursion. In other terms, to prove that the claim is true for all  $i \in [n]$ , we show that:

- (1) it is true for  $n$  and
- (2) if it is true for all integers  $j$  with  $i < j \leq n$ , then it is true for  $i$ .

The claim is trivially true for  $i = n$  since  $f_n = \sigma(n)$  and  $\sigma(n) \leq n$ . Then let  $i < n$ . By Proposition 2.1 (2), the claim is trivially true if  $i$  is an anti-excedance (with  $t = 1$ , where  $t$  is as above). Then let us suppose that  $\sigma$  has an excedance at  $i$ , say  $\sigma(i) = j_1$  with  $j_1 > i$ . Observe that during the iteration step (of the procedure to construct  $f$  from  $\sigma$ ) corresponding to  $j_1$ , the multiplication of the permutation by the transposition  $(j_1, \sigma_{j_1}(j_1))$  changes the image of  $i$  from  $j_1$  to  $\sigma_{j_1}(j_1) = f_{j_1}$ . But by recursion hypotheses,  $f_{j_1} = \sigma^{t_1}(j_1)$ , where  $t_1$  is the smallest integer such that  $j_2 := \sigma^{t_1}(j_1) < j_1$ . We can then define a sequence of integers  $j_1 > j_2 > \cdots > j_p > j_{p+1}$  such that :

- $j_p > i \geq j_{p+1}$ ;
- $j_1 = \sigma(i)$  and  $j_{p+1} = \sigma_i(i) = f_i$ ;
- $j_{k+1} = \sigma_{j_k}(j_k) = \sigma^{t_k}(j_k)$  for  $1 \leq k \leq p$  and  $t_k$  is the smallest positive integer such that  $\sigma^{t_k}(j_k) < j_k$ . (Thus  $j_{k+1} = \sigma^{1+t_1+\cdots+t_k}(i)$ .)

Therefore, if  $t = 1 + \sum_{k=1}^p t_k$  we have  $f_i = j_{p+1} = \sigma^t(i)$ , we only have to prove that  $t$  is the smallest integer such that  $\sigma^t(i) \leq i$ . Suppose there exists an integer  $s < t$  such that  $\sigma^s(i) \leq i$ . Hence there exists an integer  $p_0 \leq p$  such that

$$1 + \sum_{k=1}^{p_0-1} t_k < s < 1 + \sum_{k=1}^{p_0} t_k.$$

Let  $r = s - \left(1 + \sum_{k=1}^{p_0-1} t_k\right)$ , then

$$\sigma^s(i) = \sigma^{\left(1 + \sum_{k=1}^{p_0-1} t_k\right) + r}(i) = \sigma^r(\sigma^{1 + \sum_{k=1}^{p_0-1} t_k}(i)) = \sigma^r(j_{p_0}),$$

this integer is smaller than  $i$ , therefore it is smaller than  $j_{p_0}$ , but  $r$  is smaller than  $t_{p_0}$  and this is a contradiction because, by recurrence hypothesis  $t_{p_0}$  is the smallest integer such that  $\sigma^{t_{p_0}}(j_{p_0}) \leq j_{p_0}$ .  $\square$

When  $\sigma$  is represented graphically as the reunion of cyclic graphs (each corresponding to a cycle), to find  $\text{nom}(i)$  it suffices to start from  $i$  and follow the arcs of the cycle (connecting  $i$  and  $\sigma(i)$ ) until one meets an integer  $j \leq i$ . Obviously, when (and only when)  $i$  is the minimum of its own cycle, one has  $\text{nom}(i) = i$ .

Theorem 3.1 suggests also a new interpretation of the integer  $f_i$  and a new algorithm to go from  $f$  to  $\sigma$  (as a product of disjoint cycles and not as a word).

**Proposition 3.1.** *If  $f \in F_n$ , then  $\sigma = \phi(f)$  can be constructed as follows. For  $i = 1, 2, \dots, n$ :*

- if  $f_i = i$ , then add a new singleton cycle:  $(i)$
- if  $f_i < i$ , then insert  $i$  before  $f_i$  in its cycle.

**Example 3.1.** *Take  $f = 1132532 \in F_7$ . Then  $\phi(f) = \sigma$  can be obtained as follows:*

$$\begin{aligned} &(1) \\ &(2, 1) \\ &(2, 1)(3) \\ &(4, 2, 1)(3) \\ &(4, 2, 1)(3)(5) \\ &(4, 2, 1)(6, 3)(5) \\ &(4, 7, 2, 1)(6, 3)(5) = \phi(f) = \sigma. \end{aligned}$$

The above construction shows that the number of fixed points of  $f$  and the number of cycles of  $\sigma = \phi(f)$  are equal.

Theorem 3.1 implies in particular that the integers  $i$  and  $f_i$  are always in the same cycle of the permutation for all  $i, 1 \leq i \leq n$ .

**Corollary 3.1.** *Let  $f = f_1 f_2 \cdots f_n \in F_n$  and  $\phi(f) = \sigma$ .*

- (1)  $i \in \text{Fix}(f)$  iff  $i$  is a minimum of a cycle of  $\sigma$ .
- (2) If  $f_i = f_j$ , then  $i$  and  $j$  are in the same cycle of  $\sigma$ .

#### 4. NON-DECREASING TRANSPOSITION ARRAYS

We say that a subexcedant function  $f = f_1 f_2 \cdots f_n$  is *non-decreasing* if  $1 = f_1 \leq f_2 \leq \cdots \leq f_n$ . Let  $F_n^{\nearrow}$  denote the set of all non-decreasing subexcedant functions over  $[n]$ . For  $n \geq 1$ , we have

$$|F_n^{\nearrow}| = c_n = \frac{1}{n+1} \binom{2n}{n}, \tag{1}$$

the Catalan number (the set  $F_n^{\nearrow}$  and the set of *N-E lattice paths* from the point  $(0, 0)$ , the origin, to the point  $(n, n)$  and never going above the diagonal are in bijection [5, Lemma 1.32]).

We let  $\mathfrak{S}_n^{\nearrow} := \{\sigma \in \mathfrak{S}_n : \phi^{-1}(\sigma) \in F_n^{\nearrow}\}$ , i.e., the set of permutations whose transposition array is non-decreasing. For instance,  $F_3^{\nearrow} = \{111, 112, 113, 122, 123\}$  and  $\mathfrak{S}_3^{\nearrow} = \{231, 312, 213, 132, 123\}$ .

Here we present a characterization for the permutations having non-decreasing transposition arrays under the bijection  $\phi^{-1}$  and describe a certain number of their properties.

If  $f = f_1 f_2 \cdots f_n$  is a non-decreasing subexcedant function, then we can encode  $f$  by the vector  $r = (r_1, r_2, \dots, r_n)$ , where  $r_i$  is the number of occurrences of the letter  $i$  in  $f$ . These vectors have the following characterization.

$$0 \leq r_i \leq n - i + 1, \forall i \in [n], r_1 \neq 0 \text{ and } \sum_{i=1}^n r_i = n.$$

**Proposition 4.1.** *If  $\sigma = \sigma(1) \cdots \sigma(n-1) \in \mathfrak{S}_{n-1}$  is associated to a non-decreasing transposition array and  $\sigma' = (n, j) \circ \sigma$ , where  $\sigma(n-1) \leq j \leq n$ , then so is  $\sigma'$ .*

*Proof.* If  $f = f_1 f_2 \cdots f_{n-1}$  is the transposition array of  $\sigma$ , then the transposition array of  $\sigma'$  is  $f' = f_1 f_2 \cdots f_{n-1} j$ . Since  $f_{n-1} = \sigma(n-1) \leq j$ ,  $f'$  is non-decreasing.  $\square$

**Proposition 4.2.** *Let  $\sigma \in \mathfrak{S}_n$  and  $f$  be its transposition array. Then,  $f$  is non-decreasing if and only if*

- (1) *the subword of anti-excedance letters of  $\sigma$  is increasing. In other terms, for any two anti-excedances  $i$  and  $j$  of  $\sigma$  with  $i < j$ , one has  $\sigma(i) < \sigma(j)$ , and*
- (2) *elements of  $[n]$  having the same nearest orbital minorant under  $\sigma$  form an integer interval, i.e., if  $j \in \text{Im}(f)$ , then  $f^{-1}(j) = [a, b] = \{a, a+1, \dots, b\}$  for some integers  $a \leq b$ .*

*Proof.* Let  $r = (r_1, r_2, \dots, r_n)$  be the vector code of  $f$ . By [10, Proposition 3.5], the elements of  $\text{Im}(f)$  are the anti-excedances of  $\sigma$ , and these are the integers  $k$  such that  $r_k \neq 0$ . Furthermore, by Proposition 2.1, only the rightmost occurrences of each element of  $\text{Im}(f)$  in  $f$  correspond to the (positions of the) anti-excedances. If  $f$  is non-decreasing, then for each  $k$  such that  $r_k \neq 0$ , that position is exactly  $\sum_{\ell=1}^k r_\ell$ . Further, observe that the nearest orbital minorants of the elements of  $[n]$  under  $\sigma$  are elements of  $\text{Im}(f)$ . If  $i < k < j$ , where  $i, j, k \in [n]$  with  $\text{nom}(i) = \text{nom}(j)$ , then  $\text{nom}(i) = \text{nom}(k)$  since  $f$  is non-decreasing.

Conversely, it is obvious that the subexcedant function  $f = \phi^{-1}(\sigma)$  with  $\sigma$  having the given conditions is non-decreasing.  $\square$

**Proposition 4.3.** *If  $\sigma \in \mathfrak{S}_n^\nearrow$ , then the anti-excedances of  $\sigma$  and their images entirely characterize  $\sigma$ .*

*Proof.* Let  $\{i_1 < i_2 < \cdots < i_k = n\}$  be the anti-excedances of  $\sigma$  and  $\{j_1 = 1 < j_2 < \cdots < j_k\}$  their respective images under  $\sigma$  (with  $i_t \geq j_t, \forall t \in [k]$ ). By Proposition 2.1,  $i_t$  is the rightmost occurrence such that  $f_{i_t} = j_t$  and the number of positions having  $j_t$  as a value is  $i_t - i_{t-1}$  (if one assumes  $i_0 = 0$ ). Therefore,  $f$  is entirely determined and consequently  $\sigma$  is too.  $\square$

**Example 4.1.** *Let  $\sigma \in \mathfrak{S}_9^\nearrow$  with  $\text{Ax}(\sigma) = \{3, 4, 6, 9\}$  and  $\text{AxL}(\sigma) = \{1, 2, 3, 5\}$ , respectively. If  $f = \phi^{-1}(\sigma)$ , then we have  $f = 111233555 \in \mathfrak{F}_9^\nearrow$ , and thus  $\sigma = 471263895$ .*

Let  $T_n$  be the set of tuples of integer pairs  $\langle (i_1, j_1), (i_2, j_2), \dots, (i_k, j_k) \rangle$  satisfying the following:

- (i)  $1 \leq k \leq n$ ,
- (ii)  $1 \leq i_1 < i_2 < \cdots < i_k = n$ ,
- (iii)  $1 = j_1 < j_2 < \cdots < j_k$ , and
- (iv)  $j_s \leq i_{s-1} + 1$  for  $2 \leq s \leq k$ .

We can deduce the following corollary as a new proof of a result already known as an incarnation of Catalan numbers [7, 11].

**Corollary 4.1.** *For  $n \geq 0$ , we have  $|T_n| = c_n$ .*

*Proof.* Two such sequences  $\langle i_1, i_2, \dots, i_k \rangle$  and  $\langle j_1, j_2, \dots, j_k \rangle$  constitute respectively the set of anti-excedances and their images of a permutation associated with a non-decreasing transposition array, namely  $1^{i_1} j_2^{i_2-i_1} \dots j_k^{n-i_{k-1}}$ , and by (1) we have the result.  $\square$

Note that Proposition 4.3 suggests the existence of a procedure to reconstruct directly the unique permutation associated with a non-decreasing transposition array while only knowing its anti-excedances and their images.

**Lemma 4.1.** *Let  $\langle (i_1, j_1), (i_2, j_2), \dots, (i_k, j_k) \rangle \in T_n$ . Then it is possible to construct a permutation  $\sigma$  over  $[n]$  as follows.*

- For all  $s \in \{1, 2, \dots, k\}$ , insert  $j_s$  at position  $i_s$  (i.e., set  $\sigma(i_s) = j_s$ );
- For all  $i \notin \{i_1, i_2, \dots, i_k\}$  taken in decreasing order, if  $i_{s-1} < i < i_s$  for a certain  $s$ , then insert at the position  $i$  the value  $\sigma^{-t}(j_s)$ , where  $t$  is the smallest positive integer such that  $\sigma^{-t}(j_s)$  has not yet been inserted as the image of another integer, i.e., set  $\sigma(i) = \sigma^{-t}(j_s)$ .

*Proof.* We prove that the procedure is well defined, namely that a  $t$  such that  $\sigma^{-t}(j_s)$  has not yet been inserted as the image of another integer can always be found. Note that for  $2 \leq s \leq k$ ,  $j_s \leq i_{s-1} + 1$ . If  $\sigma(j_s)$  is already assigned, then  $j_s = i_{s_1}$  for some  $s_1 < s$ , and  $\sigma(j_s) = j_{s_1}$ . If also  $\sigma^2(j_s)$  is already assigned,  $j_{s_1} = i_{s_2}$  for some  $s_2 < s_1$ , and  $\sigma^2(j_s) = j_{s_2}$  and so on. Then there exists a smallest  $m$  such that  $\sigma^m(j_s)$  has not yet been assigned, since as we have a descending sequence of positive integers, the number of steps is finite. However, if no  $t$  existed as claimed in the lemma, no such  $m$  could exist. If there were no such  $t$ , then there must be some non-negative integers  $t_1 < t_2$ , such that  $\sigma^{-t_1}(j_s) = \sigma^{-t_2}(j_s)$  and that their sum is smallest possible under these conditions. If then  $t_1 \geq 1$ , then  $\sigma(\sigma^{-t_1}(j_s)) = \sigma^{-(t_1-1)}(j_s) = \sigma^{-(t_2-1)}(j_s)$ , which is a contradiction. Thus  $t_1 = 0$ , whence  $j_s = \sigma^0(j_s) = \sigma^{-t_2}(j_s)$ . This implies that  $\sigma$  has a cycle of length  $t_2$  and hence  $m$  would not exist. Therefore, instead, there exists a largest  $t$  such that  $\sigma^{-t}(j_s)$  is known, whence  $\sigma^{-t}(j_s)$  is not yet assigned as a  $\sigma$  value.  $\square$

**Theorem 4.1.** *Let  $\langle (i_1, j_1), (i_2, j_2), \dots, (i_k, j_k) \rangle \in T_n$ . Then  $\sigma$  constructed as in Lemma 4.1 has anti-excedances exactly at  $i_1, i_2, \dots, i_k$  with anti-excedance letters  $j_1, j_2, \dots, j_k$ , respectively and its corresponding transposition array is*

$$f = 1^{i_1} j_2^{i_2-i_1} \dots j_k^{n-i_{k-1}}.$$

*Proof.* Trivially, if  $i \in \{i_1, i_2, \dots, i_k\}$ , say  $i = i_s$ , then  $\sigma$  has an anti-excedance at  $i$ , because  $\sigma(i_s) = j_s \leq i_{s-1} + 1 \leq i_s$ , therefore  $(\phi^{-1}(\sigma))(i_s) = \sigma(i_s) = j_s$ . Assume that  $i \notin \{i_1, i_2, \dots, i_k\}$ , say  $i_{s-1} < i < i_s$  for a certain  $s$ . Now we prove that  $\sigma$  has an excedance at  $i$  and that  $(\phi^{-1}(\sigma))(i) = j_s$ . Let  $i_{s-1} < i < i_s$  for a certain  $s$ , and let  $\sigma(i) = \sigma^{-t}(j_s)$  with  $t$  the integer such that  $\sigma^{-t}(j_s)$  has not yet been inserted as the image of another integer. We start by proving that for all  $p \leq t$  we have  $\sigma^{-p}(j_s) > i$ . This is trivial for  $p = 1$  since  $\sigma^{-1}(j_s) = i_s > i$ . Suppose that for some  $1 < p_0 \leq t$  one has  $\sigma^{-p_0}(j_s) \leq i$ , i.e.,  $\sigma^{-1}(\sigma^{-(p_0-1)}(j_s)) \leq i$  and let us choose  $p_0$  as small as possible with this property. This implies that there exists an integer  $\ell \leq i$  such that the image  $\sigma(\ell)$  is already defined as  $\sigma^{-(p_0-1)}(j_s)$ , which is larger than  $i$  for the minimality hypotheses on  $p_0$ . But the only integers smaller than or equal to  $i$  whose images are already defined are the anti-excedances

at positions on the left of  $i$ , while

$$\sigma(\ell) = \sigma^{-(p_0-1)}(j_s) > i \geq \ell,$$

so  $\ell$  would be an excedance, a contradiction. This implies in particular that  $\sigma(i) = \sigma^{-t}(j_s) > i$  and hence that  $i$  is an excedance for  $\sigma$ .

Let us prove now that  $(\phi^{-1}(\sigma))(i) = j_s$ . Since  $\sigma(i) = \sigma^{-t}(j_s)$ , we have  $\sigma^{t+1}(i) = j_s$ . By hypotheses  $j_s \leq i_{s-1} + 1 \leq i$ , according to Theorem 3.1,  $j_s$  is equal to  $(\phi^{-1}(\sigma))(i)$  if  $t + 1$  is the smallest integer such that  $\sigma^{t+1}(i) \leq i$ , so we only need to prove that for all  $q < t + 1$  one has  $\sigma^q(i) > i$ . Suppose there exists  $q_0 < t + 1$  such that  $\sigma^{q_0}(i) \leq i$ . We have  $q_0 > 1$ , because  $i$  is an excedance for  $\sigma$ . Now,

$$\sigma^{-(t+1-q_0)}(j_s) = \sigma^{-(t+1-q_0)}(\sigma^{t+1}(i)) = \sigma^{t+1-t-1+q_0}(i) = \sigma^{q_0}(i) \leq i.$$

This is a contradiction with the previous part of the proof because  $t + 1 - q_0 < t$ . Thus, for all  $p \leq t$ , we have  $\sigma^{-p}(j_s) > i$ . □

### 5. AVOIDANCE OF LENGTH 3 PATTERNS IN $\mathfrak{S}_n^\nearrow$

In this section, we study length 3 pattern avoidance in the set  $\mathfrak{S}_n^\nearrow$ . We denote the set of permutations in  $\mathfrak{S}_n^\nearrow$  avoiding a pattern  $\pi$  by  $\mathfrak{S}_n^\nearrow(\pi)$ .

**The case 123.** The first few terms of the sequence of the numbers of 123-avoiding permutations over  $[n], n \geq 1$ , corresponding to non-decreasing transposition arrays is  $1, 2, 4, 4, 3, 0, 0, \dots$

**Proposition 5.1.** *We have  $|\mathfrak{S}_n^\nearrow(123)| = 0$  for  $n \geq 6$ .*

*Proof.* If  $\sigma \in \mathfrak{S}_n^\nearrow$  such that  $\text{ax}(\sigma) \geq 3$ , then  $\sigma$  must contain the pattern 123 since the subword of anti-excedance letters of  $\sigma$  is increasing. Let  $\text{ax}(\sigma) \leq 2$ . If  $\text{ax}(\sigma)=1$ , then  $\sigma = 234 \dots n1$  and it contains the pattern 123 for all  $n \geq 4$ . Now suppose that  $\text{ax}(\sigma)=2$  and let  $\text{Ax}(\sigma) = \{i, n\}$  such that  $i < n$ . Then  $\sigma(i)=1 < \sigma(n)$ . Let  $\sigma(n) = j \geq 2$ . Then, the transposition array of  $\sigma$  has the form  $f = 11 \dots 1jj \dots j \in F_n^\nearrow$ . If  $j > i$ , then  $j = i+1$  and thus  $\sigma = \phi(f) = 23 \dots i1(i+2)(i+3) \dots n(i+1)$ . So removing 1 and  $i+1$  leaves an increasing sequence of length  $n-2$ . Similarly, if  $j \leq i$ , then  $\sigma = 23 \dots (j-1)(i+1)(j+1) \dots i1(i+2)(i+3) \dots nj$  and removing  $i+1, 1$  and  $j$  leaves an increasing sequence of length  $n-3$ , and the pattern 123 occurs in  $\sigma$  for  $n \geq 6$ . □

**The case 132.** In a subexcedant function  $f$ , a plateau  $i$  of  $f$  is said to be of height  $h$  if  $f_i = f_{i+1} = h$ . Further,  $f$  has a *stair* at  $i \in [n-1]$  if  $f_{i+1} = f_i + 1 \geq 3$ . A *staircase* in  $f$  is the letters of a maximal interval of integers in which all the integers except the last are stairs.

Let  $V_n := \{f \in F_n^\nearrow : f \text{ may have plateaus only of height } 1\}$ , i.e.,  $f \in V_n$  iff  $f_1 = f_2 = \dots = f_k = 1 < f_{k+1} < f_{k+2} < \dots < f_n$ , where  $1 \leq k \leq n$ . For instance,  $V_3 = \{111, 112, 113, 123\}$ .

**Proposition 5.2.** *If  $\sigma \in \mathfrak{S}_n^\nearrow(132)$ , then  $f = \phi^{-1}(\sigma) \in V_n$ .*

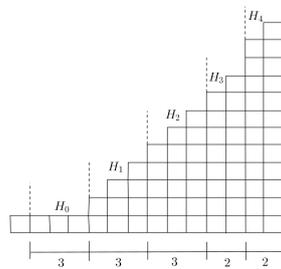
*Proof.* Let  $\sigma \in \mathfrak{S}_n^\nearrow(132)$  and suppose that  $f = \phi^{-1}(\sigma) \notin V_n$ , i.e.,  $f$  has plateau(s) of height  $h$  with  $h \geq 2$ . Let  $k$  be the rightmost position such that  $f_k = 1$  and  $p > k + 1$  be the rightmost position such that  $f_p = h$ . By Proposition 2.1,  $\sigma$  has an anti-excedance in  $k$  with image value 1 and an anti-excedance in  $p$  with image value  $h \leq p$ . In the position  $p - 1$ , we have  $f_{p-1} = h$  and an excedance for  $\sigma$ , i.e., a letter larger than  $p - 1$ , therefore  $\sigma$  would have a pattern 132 in the positions  $k, p-1$  and  $p$ , a contradiction. □

**Remark 5.1.** *Let  $f \in V_n$ , where  $\text{plat}(f) = k$  and  $\sigma = \phi(f)$ . Then*

- (1) for all  $i$  with  $k+1 \leq i \leq n$ , we have  $\sigma(i) = f_i$ , because all these are anti-excedance positions. Thus,  $\text{AxL}(\sigma)$  is an increasing sequence of integers, and  $\text{Exc}(\sigma) = [k]$ ;
- (2) if there exists  $i > 1$  such that  $f_i = i$ , then  $f_j = j, \forall j > i$ , in particular  $f_n = n$ .

**Definition 5.1.** Let  $f \in V_n$ , where  $\text{plat}(f) = k$ . We decompose  $f = 1|H_0|L_1|\dots|L_p$ , where  $H_0 = 1^{k_0}$ ,  $k_0 = k$ , and the  $L_i$ 's are staircases. If  $H_i$  is the last block of  $L_{j-1}$ , then we cut  $L_j$  into blocks of size  $k_i$  (with a possible remainder of size  $(\ell_j \bmod k_i)$  and let  $k_{i+1} = ((\ell_j - 1) \bmod k_i) + 1$ , where  $\ell_j = |L_j|$ . Then, let the resulting decomposition be  $f = 1|H_0|H_1|H_2|\dots|H_s$  with  $H_i$  having size  $k_i, 0 \leq i \leq s$ .

For instance, take  $f = 1\ 1\ 1\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 11\ 12$ . We have  $L_0 = 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9, L_2 = 11\ 12, k = k_0 = 3, \ell_1 = 8$ , and  $\ell_2 = 2$ . First, we cut  $L_1$  into two blocks of size  $k_0 = 3$  and the block of size the remainder  $(5 \bmod 3) = 2$ . So  $k_1 = k_2 = 3, k_3 = 2$ . Next, we cut  $L_2$  into block of size  $k_3 = 2$  since the remainder  $(2 \bmod 2) = 0$ . So  $k_4 = 2$ . Thus,  $f$  becomes  $f = 1|H_0|H_1|H_2|H_3|H_4 = 1|1\ 1\ 1|2\ 3\ 4|5\ 6\ 7|8\ 9|11\ 12$  (see the following figure).



Let  $\mathfrak{D}_n := \{f \in V_n : \sigma = \phi(f) \in \mathfrak{S}_n^{\nearrow}(132), \text{Fix}(f) = \{1\}\}$ , and  $\mathfrak{D}_{n,k} := \{f \in \mathfrak{D}_n : \text{plat}(f) = k\}$ . Further, let  $d_n := |\mathfrak{D}_n|$  and  $d_{n,k} := |\mathfrak{D}_{n,k}|$ . The following proposition characterizes the elements of  $\mathfrak{D}_n$ .

**Proposition 5.3.** Let  $f \in V_n$  be given in its decomposed form  $f = 1|H_0|H_1|H_2|\dots|H_s$  as in Definition 5.1. Then,  $f \in \mathfrak{D}_n$  if and only if

$$f_{k_0+k_1+\dots+k_i+2} = k_0 + k_1 + \dots + k_{i-1} + 2, \quad 1 \leq i \leq s, \tag{2}$$

i.e., each block  $H_i$  with  $i > 0$  begins with the integer equal to the initial position of the block  $H_{i-1}$ .

*Proof.* Suppose  $f \in \mathfrak{D}_n$ , where  $\text{plat}(f) = k_0$ . We shall use induction to prove that  $f$  satisfies (2) and that

$$\sigma_{1+k_0+\dots+k_i}(j) = 1 + k_0 + \dots + k_{i-1} + j, \tag{3}$$

where  $j = 1, 2, \dots, k_i$ , and  $(\sigma_a \in \mathfrak{S}_a, 1 \leq a \leq n$ , as given in Section 2.2.1).

Observe that  $\sigma_{1+k_0} = \phi(1H_0) = 23 \dots (k_0+1)1$ , and thus (3) holds for  $i = 0$ . We show that  $f_{k_0+2} = 2$ . If  $f_{k_0+2} > 2$ , then the integer  $\sigma_{1+k_0}(1)=2$  remains unmoved in the successive steps of the construction of  $\sigma$  because  $2 \notin \text{Im}(f)$ . So  $\sigma(1) = 2$ . Since  $f_n < n$  and  $\text{Exc}(\sigma) = [k_0]$ , we have some  $j \in [2, k_0]$  such that  $\sigma(j) = n$ . This implies that  $\sigma$  contains the pattern 132 in the positions  $1, j, k_0 + 2$  and this is a contradiction. Thus, instead,  $f_{k_0+2} = 2$  and (2) holds for  $i = 0$ .

Now assume that (2) and (3) hold for  $i = q-1$ . We prove it for  $i = q$ . If we let  $t := k_0+k_1+\dots+k_q+2$  (the inial position of the block  $H_{q+1}$ ), then  $k_0+k_1+\dots+k_{q-1}+2 = t-k_q$ . By (3) (for  $i = q-1$ ) we have the largest  $k_{q-1}$  integers already inserted into the first  $k_{q-1}$  positions of  $\sigma_{t-k_{q-1}}$  in increasing order. Since  $H_q = [t-k_q-k_{q-1}, t-k_{q-1}-1]$  (by (2)) and

$k_{q-1} \geq k_q$ , the next  $k_q$  steps insert the largest  $k_q$  integers in  $\sigma_{t-1}$  in increasing order. Therefore, (3) holds.

Now we prove (2), *i.e.*,  $f_t = t - k_q$ . Since  $f \in V_n$ , we have  $f_t \geq f_{t-1} + 1$ . If  $f_t = f_{t-1} + 1$ , then  $t-1$  is a stair, whence  $k_{q-1} = k_q$ , and hence  $f_t = f_{t-k_q} + k_q = (t - k_q - k_{q-1}) + k_{q-1} = t - k_q$  indeed. Thus, instead assume that  $f_t > f_{t-1} + 1$ . By (3) we have  $\sigma_{t-k_q-1}(j) = t - k_q - k_{q-1} - 1 + j$ ,  $1 \leq j \leq k_{q-1}$ . If  $f_t < t - k_q$ , then there is some  $j \in [k_q + 2, k_{q-1}]$  such that  $\sigma_{t-1}(j) = \sigma_{t-k_q-1}(j) = f_t$ . Since  $\sigma_t$  is obtained from  $\sigma_{t-1}$  by applying the transposition  $(t, f_t)$ , we have  $\sigma_t(j) = t$  and  $\sigma_t(t) = f_t$ . Thus,  $\sigma_t(k_q + 1) = \sigma_{t-k_q-1}(k_q + 1) = t - k_{q-1} = f_{t-1} + 1 < f_t = \sigma_t(t) < \sigma_t(j)$  and  $k_q + 1 < j < t$ . Since in the successive steps  $\sigma_t(k_q + 1)$  and  $\sigma_t(t)$  remain unmoved and  $\sigma_t(j)$  can only be replaced by a larger integer, we have the pattern 132 in  $\sigma$  and this is a contradiction. If  $f_t > t - k_q$ , then  $f_t$  is one of the largest  $k_q - 1$  integers already inserted in positions  $[2, k_q]$  and the positions  $1, \sigma_{t-1}^{-1}(f_t)$  and  $t$  would create the pattern 132. Therefore,  $f_t = t - k_q$ .

Conversely, suppose that  $f$  satisfies the given condition. We can decompose the block of excedance letters of  $\sigma$  (which are the letters in the first  $k_0$  positions) into blocks  $P_1|P_2|\cdots|P_u$  made of increasing numbers and such that for every  $a \in P_j, b \in P_{j+1}$  we have  $a > b$ , where  $j = 1, 2, \dots, u - 1$ . Indeed for all  $r$  with  $1 \leq r \leq s$ , it can be verified that if  $\tau = \phi(1|H_0|H_1|H_2|\cdots|H_r)$ , then  $\tau$  is obtained from  $\tau' = \phi(1|H_0|H_1|H_2|\cdots|H_{r-1})$  by replacing the integers  $\tau'(1), \dots, \tau'(k_r)$  with the  $k_r$  largest integers in increasing order and appending  $\tau'(1), \dots, \tau'(k_r)$  at the end. Thus, if  $\sigma$  had an occurrence 132, then the 3 and the 2 could not both be among the anti-excedance letters because they form an increasing sequence (not necessarily an interval). Hence at least the 1 and the 3 must be in the excedance block, and thus in the same block  $P_j$  for some  $j$  because the  $P_j$ 's are intervals arranged in decreasing order of their first elements. So there can be no integer playing the role of 2 outside  $P_j$ . Thus there is no such occurrence.  $\square$

**Theorem 5.1.** *The number  $a_n := |\mathfrak{S}_n^{\nearrow}(132)|$  satisfies the recurrence relation:*

$$a_n = a_{n-1} + p_{n-1}, n \geq 1, a_0 = 0, p_0 = 1, \quad (4)$$

where  $p_n$  is the number of integer partitions of  $n$ .

*Proof.* Note that  $a_n$  is the number of non-decreasing transposition arrays  $f$  over  $[n]$  such that  $\sigma = \phi(f) \in \mathfrak{S}_n^{\nearrow}(132)$ . The term  $a_{n-1}$  in the right-hand side of the recurrence relation can be explained easily. Let  $f = f_1 f_2 \cdots f_{n-1}$  such that  $\sigma = \phi(f) \in \mathfrak{S}_{n-1}^{\nearrow}(132)$ . Then the function  $f'$  obtained from  $f$  by appending  $n$  at its end corresponds to a permutation  $\sigma' = \phi(f') \in \mathfrak{S}_n^{\nearrow}(132)$ . This term counts the non-decreasing subexcedant functions with fixed points and such that  $\phi(f') \in \mathfrak{S}_n^{\nearrow}(132)$  and it contributes  $a_{n-1}$  to  $a_n$ . For the term  $p_{n-1}$ , we show in the following lemmas and propositions that the number of functions  $f$  with no fixed points greater than 1, with  $\sigma = \phi(f) \in \mathfrak{S}_n^{\nearrow}(132)$ , equals the number  $p_{n-1}$  of all integer partitions of  $n-1$  and more precisely that the number  $d_{n,k}$  equals the number of integer partitions over  $[n-1]$  with the largest part equal to  $k$  [11]. Thus, this contributes  $p_{n-1}$  to the number  $a_n$ .  $\square$

**Remark 5.2.** *The sequence of the numbers  $a_n$  is the same as the sequence given in OEIS number A000070, but shifted by one step. In particular, it has an ordinary generating function*

$$\frac{1}{1-x} \prod_{j \geq 1} \frac{x}{1-x^j}.$$

**Lemma 5.1.** *Let  $f \in \mathfrak{D}_{n,k}$  and  $f'$  be obtained from  $f$  by increasing each integer greater than or equal to  $k+2$  by 1 and inserting a 1 at the beginning, then  $f' \in \mathfrak{D}_{n+1, k+1}$ .*

*Proof.* If  $f$  is decomposed as in Definition 5.1, then we have  $f_{k_0+k_1+2} = k_0 + 2$ . If  $f' = f'_1 f'_2 \cdots f'_{n+1}$ , then

$$f'_j = \begin{cases} 1, & \text{if } 1 \leq j \leq k_0 + 2; \\ f_{j-1}, & \text{if } k_0 + 3 \leq j \leq k_0 + k_1 + 2; \\ f_{j-1} + 1, & \text{if } k_0 + k_1 + 3 \leq j \leq n + 1. \end{cases}$$

Let us show that  $f' \in \mathfrak{D}_{n+1, k+1}$ , i.e.,  $f'$  satisfies the condition in Proposition 5.3. The number of blocks of  $f$  and  $f'$  are the same but the size of  $H'_0$  of  $f'$  is increased by 1, where  $f' = 1|H'_0|H'_1|H'_2| \cdots |H'_s$ . So  $f'_{k'_0+2} = f'_{k'_0+3} = f_{k_0+2} = 2$ , and  $f'_{k'_0+k'_1+\cdots+k'_i+2} = f_{k_0+1+k_1+\cdots+k_i+2} + 1 = k_0 + k_1 + \cdots + k_{i-1} + 3 = k'_0 + k'_1 + \cdots + k'_i + 2, i \geq 1$ . Thus,  $f' \in \mathfrak{D}_{n+1, k+1}$ .  $\square$

**Lemma 5.2.** *Let  $f \in \mathfrak{D}_{n, k}$  and  $f'$  be obtained from  $f$  by increasing each integer greater than 1 by  $k$  and then inserting the subword  $23 \cdots k + 1$  after the rightmost occurrence of 1. Then  $f' \in \mathfrak{D}_{n+k, k}$ .*

*Proof.* Since  $((\ell_1 - 1) \bmod k) + 1 = ((\ell_1 + k - 1) \bmod k) + 1$  the insertion of the subword  $23 \cdots k + 1$  in the operation creates a new block of size  $k$  in  $f'$  after  $H_0$ . That is,  $f' = 1|H_0|H'_1|H'_2| \cdots |H'_{s+1}$ , where  $H'_1 = 2\ 3\ 4 \cdots k + 1$  and  $a + k \in H'_i$  if  $a \in H_{i-1}, i \geq 2$ . So,  $f'_{k_0+2} = 2$  and

$$\begin{aligned} f'_{k'_0+k'_1+\cdots+k'_i+2} &= f'_{k_0+k_0+k_1+\cdots+k_{i-1}+2} \\ &= f_{k_0+k_0+k_1+\cdots+k_{i-1}+2-k_0} + k_0 \\ &= (k_0 + k_1 + \cdots + k_{i-2} + 2) + k_0 \\ &= k'_0 + k'_1 + \cdots + k'_{i-1} + 2, \quad i \geq 1. \end{aligned}$$

Thus,  $f' \in \mathfrak{D}_{n+k, k}$ .  $\square$

From Lemma 5.1 and Lemma 5.2, we deduce that each function  $f$  in  $\mathfrak{D}_{n, k}$ , can uniquely be obtained either from a function in  $\mathfrak{D}_{n-1, k-1}$  by the construction in Lemma 5.1 or from a function in  $\mathfrak{D}_{n-k, k}$  by the construction in Lemma 5.2. Furthermore, the two sets obtained this way are disjoint because those obtained by the construction in Lemma 5.1 do not have  $k + 2$  in their image, while those obtained by the construction in Lemma 5.2 do. As a result, we have the following proposition.

**Proposition 5.4.** *The number  $d_{n, k}$  satisfies the recurrence relation:*

$$d_{n, k} = d_{n-1, k-1} + d_{n-k, k}, \tag{5}$$

where  $d_{0, 0} = 1$  and  $d_{n, k} = 0$  if  $n \leq 0$  or  $k \leq 0$  and  $n$  and  $k$  are not both zero.  $\square$

This recurrence is the same as the one satisfied by the number of integer partitions of  $n$  whose largest part is  $k$  [11, Section 1.7], we then deduce the following.

**Corollary 5.1.** *For  $n \geq 1$ , we have  $d_n = p_{n-1}$ .*  $\square$

**Remark 5.3.** *The number  $d_{n, k}$  equals the number of permutations  $\sigma \in \mathfrak{S}_n^{\nearrow}(132)$ , where  $\sigma(n) < n$ , having  $k$  excedances.*

It is possible to define a direct bijection between the set  $\mathfrak{D}_n$  and the set  $P_{n-1}$  of all integer partitions of  $n-1$ . Let  $f \in \mathfrak{D}_n$  and  $\rho : \mathfrak{D}_n \mapsto P_{n-1}$  be the map which associates  $f$  with an integer partition  $\rho(f) = \lambda \vdash n-1$  defined as follows. Decompose  $f$  as indicated in Definition 5.1:  $f = 1|H_0|H_1| \cdots |H_s$ , and call  $k_i$  the size of  $H_i$ , then we set  $\rho(f) = \lambda = k_0 k_1 \cdots k_s$ , this is clearly a partition of  $n-1$  with the largest part equal to  $k_0$ .

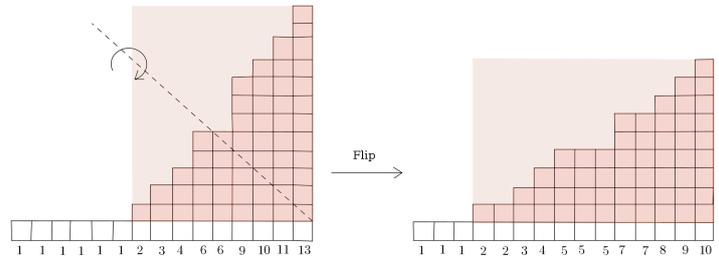
**Proposition 5.5.** *The map  $\rho : \mathfrak{D}_n \mapsto P_{n-1}$  is a bijection.*

*Proof.* We will show that  $\rho$  can be inverted. Let  $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$  be a partition of  $n-1$ . Let us define  $\rho^{-1}(\lambda) = f$  over  $[n]$  as follows. First start with  $k = \lambda_1$  plateaus of height 1, i.e., set  $f_2 = \cdots = f_{k+1} = 1$ . Then for  $i \geq 2$  insert  $\lambda_i$  consecutive integers starting from the integer  $2 + \sum_{j=1}^{i-2} \lambda_j$  (where, as usual  $\sum_{j=1}^0 \lambda_j = 0$ ).  $\square$

Let  $\lambda = 55533221$ . We have  $\lambda \vdash 26$ . Starting from  $k = \lambda_1 = 5$  we obtain

$$f = 1|1\ 1\ 1\ 1\ 1|2\ 3\ 4\ 5\ 6|7\ 8\ 9\ 10\ 11|12\ 13\ 14|17\ 18\ 19|20\ 21|23\ 24|25 \in \mathfrak{D}_{27}.$$

**The case 213.** We will prove that  $|\mathfrak{S}_n^{\nearrow}(213)| = |\mathfrak{S}_n^{\nearrow}(132)|, n \geq 1$ . In order to do so we introduce an involution in the set  $F_n^{\nearrow}$ , called “Flip”. This involution associates each  $f = f_1 f_2 \cdots f_n \in F_n^{\nearrow}$  with the function  $\text{Flip}(f) = f'_1 f'_2 \cdots f'_n$  given as follows:  $f'_i = n+1 - \sum_{k=1}^{n-i+1} r_k$ , where  $(r_1, r_2, \dots, r_n)$  is the  $r$ -vector of  $f$ . For instance, let  $f = 1\ 1\ 1\ 1\ 1\ 1\ 2\ 3\ 4\ 6\ 6\ 9\ 10\ 11\ 13$ . The  $r$ -vector of  $f$  is  $(6, 1, 1, 1, 0, 2, 0, 0, 1, 1, 1, 0, 1, 0, 0)$ , and hence  $\text{Flip}(f) = 1\ 1\ 1\ 1\ 2\ 2\ 3\ 4\ 5\ 5\ 5\ 7\ 7\ 8\ 9\ 10$ .



Although non-decreasing subexcedant functions can be seen as partitions of the integer  $\sum_{i=1}^n f_i$  and although this operation is similar to the conjugation in the set of integer partitions, the two operations are different.

**Remark 5.4.** *If  $f \in F_n^{\nearrow}$ , then  $\text{Flip}(f)$  starts with  $n - f_n$  plateaus of height 1.*

**Lemma 5.3.** *If  $f' = f_1 f_2 \cdots f_{n-1} \in F_{n-1}^{\nearrow}$  and  $f = f_1 f_2 \cdots f_{n-1} j$ , where  $j \geq f_{n-1}$ , then  $\text{Flip}(f)$  is obtained from  $\text{Flip}(f')$  by adding 1 to the integers in the  $j-1$  largest positions and inserting a 1 at the beginning.*

*Proof.* Suppose that  $f'$  has  $r$ -vector  $(r_1, r_2, \dots, r_{n-1})$  and  $\text{Flip}(f') = g' = g'_1 \cdots g'_{n-1}$ . Then the  $r$ -vector of  $f = f' \cdot j$  is  $(r_1^*, \dots, r_n^*)$ , where  $r_i^* = r_i + \delta_{i,j}$ ,  $r_n^* = \delta_{n,j}$ . So, we have  $\text{Flip}(f) = g_1 g_2 \cdots g_n$ , where  $g_i = n+1 - \sum_{k=1}^{n-i+1} r_k^*$ . It is easy to see that  $g_1 = 1, g_i = g'_{i-1}$  for  $2 \leq i < n-j+2$ , and  $g_i = g'_{i-1} + 1$  for  $i \geq n-j+2$ , i.e.,  $\text{Flip}(f)$  is exactly the function obtained from  $\text{Flip}(f')$  by adding 1 to the integers in the  $j-1$  largest positions and inserting a 1 at the beginning.  $\square$

**Theorem 5.2.** *If  $f \in F_n^{\nearrow}$  is the transposition array of the permutation  $\sigma$ , then  $\text{Flip}(f)$  is the transposition array of the permutation  $\tau = ((\sigma^{-1})^r)^c$ , where  $r$  and  $c$  denote the operations of reverse and complement of a permutation respectively.*

*Proof.* For any permutation  $\pi \in \mathfrak{S}_n$  one has  $(\pi^r)^c = (\pi^c)^r = \psi_n \circ \pi \circ \psi_n$ , where  $\psi_n$  is the permutation  $\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$ , therefore the equality to prove becomes:  $\tau = \psi_n \circ \sigma^{-1} \circ \psi_n$ . We prove the result by induction on  $n$ . The result is trivially true for  $n = 1$ . Suppose  $n > 1$  and that the result is true for  $n-1$ . Let  $f = f' \cdot j$ , where  $f'$  is the prefix of length  $n-1$  of  $f$  and  $j = f_n \geq f_{n-1} = f'_{n-1}$ . Let  $\sigma'$  be the permutation

whose transposition array is  $f'$  and let  $\tau'$  be the permutation whose transposition array is  $\text{Flip}(f')$ . By induction hypotheses  $\tau' = \psi_{n-1} \circ (\sigma')^{-1} \circ \psi_{n-1}$  or equivalently (since  $\psi_{n-1}$  is an involution)  $\psi_{n-1} \circ \tau' \circ \psi_{n-1} = (\sigma')^{-1}$ . On the other hand, by the insertion method,  $\sigma = (n, j) \circ \sigma'$  and hence  $\sigma^{-1} = (\sigma')^{-1} \circ (n, j)$ , therefore the relation to be proven becomes:

$$\tau = \psi_n \circ \psi_{n-1} \circ \tau' \circ \psi_{n-1} \circ (n, j) \circ \psi_n. \tag{6}$$

By the definition of the bijection  $\phi$  we have  $\tau' = \prod_{i=1}^{n-1} (n-i, f'_{n-i})$ , then by Lemma 5.3,  $\tau = \prod_{i=1}^{j-1} (n-i+1, f'_{n-i} + 1) \circ \prod_{i=j}^{n-1} (n-i+1, f'_{n-i})$ . By plugging these values in Equation (6) we obtain

$$\prod_{i=1}^{j-1} (n-i+1, f'_{n-i} + 1) \circ \prod_{i=j}^{n-1} (n-i+1, f'_{n-i}) = \psi_n \circ \psi_{n-1} \circ \left( \prod_{i=1}^{n-1} (n-i, f'_{n-i}) \right) \circ \psi_{n-1} \circ (n, j) \circ \psi_n.$$

In the symmetric group, if a permutation  $\alpha$  is written as product of cycles and  $\beta$  is another permutation, the conjugate  $\beta\alpha\beta^{-1}$  can be computed by replacing every integer  $i$  with  $\beta(i)$  in the product of cycles giving  $\alpha$ .

$$\begin{aligned} \prod_{i=1}^{j-1} (n-i+1, f'_{n-i} + 1) \circ \prod_{i=j}^{n-1} (n-i+1, f'_{n-i}) &= \psi_n \circ \left( \prod_{i=1}^{n-1} (i, n-f'_{n-i}) \right) \circ (n, j) \circ \psi_n \\ &= \prod_{i=1}^{n-1} (n-i+1, f'_{n-i} + 1) \circ (1, n-j+1) \end{aligned}$$

by simplifying we get

$$\prod_{i=j}^{n-1} (n-i+1, f'_{n-i}) = \prod_{i=j}^{n-1} (n-i+1, f'_{n-i} + 1) \circ (1, n-j+1)$$

because of Remark 5.4, all the  $f'_i$  for  $i = 1, \dots, n-j+1$  are equal to 1.

$$(n-j+1, 1) \cdots (3, 1)(2, 1) = (n-j+1, 2) \cdots (3, 2)(2, 2)(1, n-j+1).$$

It is straightforward to check that these two permutations are equal (and equal to the cycle  $(1, 2, \dots, n-j+1)$ ). □

**Corollary 5.2.** *For  $n \geq 1$ , we have  $|\mathfrak{S}_n^\nearrow(213)| = |\mathfrak{S}_n^\nearrow(132)|$ .*

*Proof.* A permutation  $\sigma$  contains the pattern 132 if and only if  $((\sigma^{-1})^r)^c$  contains the pattern  $((132)^{-1})^r)^c = 213$ , therefore  $(\phi \circ \text{Flip} \circ \phi^{-1})$  is a bijection between  $\mathfrak{S}_n^\nearrow(213)$  and  $\mathfrak{S}_n^\nearrow(132)$ . □

**The case 231.** We do not have an explicit expression for the number of permutations of this class, but we can provide a lower bound showing that it grows exponentially with  $n$ . The first few terms of this class are 1, 2, 4, 9, 20, 45, 103, 235, 538, 1233, ...

Let us recursively define the set  $X_n$  of non-decreasing subexcedant functions over  $[n]$ , where  $f \in X_n$  is obtained from  $f' \in X_{n-1}$  by appending  $n-1$  or  $n$  at its end, or from  $f \in X_{n-3}$  by appending the subword  $(n-3)(n-3)(n-2)$  at its end. Thus, by construction we have the recurrence relation

$$|X_n| = 2|X_{n-1}| + |X_{n-3}|, n \geq 3, |X_0| = 0, |X_1| = 1, |X_2| = 2.$$

**Proposition 5.6.** *If  $f \in X_n$ , then  $\sigma = \phi(f) \in \mathfrak{S}_n^\nearrow(231)$ .*

*Proof.* Let  $f \in X_n$  and  $\sigma = \phi(f)$ . Then  $f$  is obtained from  $f' \in X_{n-1}$  or  $f' \in X_{n-3}$  by the operations defined above. Let  $f \in X_n$  be obtained from  $f' \in X_{n-1}$  by appending  $n - 1$  or  $n$  at its end. Then  $\sigma = (n, n - 1) \circ \sigma'$  or  $\sigma = (n, n) \circ \sigma'$ , respectively, where  $\sigma' = \phi(f') \in \mathfrak{S}_{n-1}^\nearrow(231)$ . We show that  $\sigma \in \mathfrak{S}_n^\nearrow(231)$ . The case where  $\sigma = (n, n) \circ \sigma'$  is obvious. Let  $\sigma = (n, n - 1) \circ \sigma'$  and assume that the pattern 231 occurs in  $\sigma$ , *i.e.*, there are integers  $i < j < k$  such that  $\sigma(j) > \sigma(i) > \sigma(k)$ . If  $\sigma(j) \neq n$ , then since  $\sigma(n) = n - 1$  we have that  $\sigma(i) = \sigma'(i), \sigma(j) = \sigma'(j), \sigma(k) = \sigma'(k)$ , and thus the subword  $\sigma'(i)\sigma'(j)\sigma'(k)$  would create the pattern 231 in  $\sigma'$ . But this is a contradiction. If  $\sigma(j) = n$ , then since  $\sigma'$  can be obtained from  $\sigma$  by replacing the integer  $n$  by  $n - 1$ ,  $\sigma'(j) = n - 1$  which is the largest integer in  $\sigma'$ . This implies that  $\sigma'$  contains the pattern 231, but this is also a contradiction. Therefore,  $\sigma$  avoids the pattern 231.

Now consider that  $f$  is obtained from  $f' \in X_{n-3}$  by appending the subword  $(n - 3)(n - 3)(n - 2)$  at its end. Then we have  $\sigma = (n, n - 2)(n - 1, n - 3)(n - 2, n - 3)\sigma'$  and  $\sigma(n - 2) = n - 1, \sigma(n - 1) = n - 3$  and  $\sigma(n) = n - 2$ . If  $\sigma$  contains the pattern 231, then there exist integers  $i < j < k \leq n - 3$  with  $\sigma(j) > \sigma(i) > \sigma(k)$ . The operation of obtaining  $\sigma'$  from  $\sigma$  affects the pattern 231 only if  $\sigma(j) = n$ . In this case we have  $\sigma'(j) = n - 3$  which is the largest integer in  $\sigma'$ . Thus,  $\sigma'$  contains the pattern 231 and this is a contradiction.  $\square$

So  $|\mathfrak{S}_n^\nearrow(231)|$  grows at least as fast as  $|X_n|$ .

**The case 312.** In this case we show that  $|\mathfrak{S}_n^\nearrow(312)| = 2^{n-1}, n \geq 1$ . Let  $Y_n := \{f = f_1 f_2 \cdots f_n \in F_n^\nearrow : f_i = i, \forall i \in \text{Im}(f)\}$ . For instance,  $Y_3 = \{111, 113, 122, 123\}$ .

**Proposition 5.7.** *A permutation  $\sigma$  belongs to  $\mathfrak{S}_n^\nearrow(312)$  if and only if its transposition array  $f$  belongs to  $Y_n$ .*

*Proof.* Assume that  $f = f_1 f_2 \cdots f_n \in Y_n$  and  $\sigma = \phi(f)$ . Let  $\text{Im}(f) = \{\ell_1, \ell_2, \dots, \ell_k\}$ . Then  $f_{\ell_i} = \ell_i, i \in \{1, 2, \dots, k\}$ . We know that  $\ell_i$  is a fixed point of  $f$  if and only if  $\ell_i$  is the minimum of its cycle in  $\sigma$ . So all integers in the cycle of  $\ell_i$  have nom equal to  $\ell_i$ , hence they have to appear in increasing order in the cycle (when we write the cycle with its minimum  $\ell_i$  at the beginning). Furthermore, since  $f$  is a non-decreasing, all the integers in the cycle of  $\ell_i$  form an integer interval, therefore this cycle is  $(\ell_i, \ell_i + 1, \dots, \ell_{i+1} - 1)$ . Thus, if  $\sigma$  contains a 312-pattern, *i.e.*, there are  $i < j < k$  such that  $\sigma(i) > \sigma(k) > \sigma(j)$ , then  $i, j, k, \sigma(i), \sigma(j), \sigma(k)$  must be in the same cycle, say  $C_r = (\ell_r, \sigma(\ell_r), \dots)$ , where  $\ell_r < \sigma(\ell_r) < \sigma^2(\ell_r) < \dots$ . But this a contradiction.

Conversely, let  $\sigma \in \mathfrak{S}_n^\nearrow(312)$  and  $f = \phi^{-1}(\sigma) = f_1 f_2 \cdots f_n$ . Suppose that  $\text{Im}(f) = \{\ell_1, \ell_2, \dots, \ell_k\}$  and  $\ell_j$  is the smallest image value of  $f$  such that  $f_{\ell_j} < \ell_j$ . Let  $t$  and  $s$  be the positions of the leftmost and the rightmost occurrences of the value  $\ell_j$  in  $f$ . Since  $\ell_{j-1} \in \text{Im}(f)$  is a fixed point it begins a cycle in  $\sigma$ , say  $C$ . Note that  $\text{nom}(\ell_j) = \ell_{j-1}$ . Thus,  $\ell_j, t, s \in C$ . That is,  $C = (\ell_{j-1}, \ell_{j-1} + 1, \dots, \ell_j - 1, \dots, t, \dots, s, \ell_j, \dots, t - 1)$ , where the integers between  $\ell_{j-1}$  and  $t$  (if any) are greater than  $t$ . Thus, there is a 312 pattern in the positions  $\ell_{j-1}, t - 1$  and  $s$ , but this is a contradiction.  $\square$

**Corollary 5.3.** *The number of 312-avoiding permutations in  $\mathfrak{S}_n^\nearrow$  having exactly  $k$  cycles is equal to*

$$|\{f \in Y_n : \text{im}(f) = k\}| = \binom{n - 1}{k - 1}, \quad 0 \leq k - 1 \leq n - 1.$$

*Proof.* For every subexcedant function  $f = f_1 f_2 \cdots f_n$  we have  $f_1 = 1$ . Thus there are  $n - 1$  remaining positions from which we can choose  $k - 1$  positions, this can be possible in  $\binom{n-1}{k-1}$ , so that we have  $f_i = i$ . Then we fill in the remaining positions in one way to get

the required function  $f$ . The number of cycles of the permutation coincides with  $\text{im}(f)$  because  $f_i = i$  if and only if  $i = \min(C)$ , where  $C$  is a cycle of  $\sigma = \phi(f)$ .  $\square$

**The case 321.** In this case we can provide a lower bound that shows that the size of this class grows exponentially with respect to  $n$ . The first few terms of this class are 1, 2, 5, 13, 35, 95, 261, 719, 1990, ...

We note that a permutation  $\sigma \in \mathfrak{S}_n$  is 321-avoiding if and only if both the subword of excedance letters and the subword of anti-excedance letters are increasing. Therefore, a permutation  $\sigma \in \mathfrak{S}_n^\nearrow$  is 321-avoiding if and only if the subword of excedance letters is increasing, *i.e.*, if  $i_1 < i_2 < \dots < i_k$  is the increasing sequence of excedances of  $\sigma$ , then  $\sigma(i_1) < \sigma(i_2) < \dots < \sigma(i_k)$ . Obviously, we obtain transposition arrays of permutations satisfying this property when we append  $n$  or  $n-1$  to the transposition array of a permutation  $\sigma$  in  $\mathfrak{S}_{n-1}^\nearrow(321)$ , therefore we have the following.

**Proposition 5.8.** *For  $n \geq 1$ , we have*

$$|\mathfrak{S}_n^\nearrow(321)| \geq 2^{n-1}.$$

**Acknowledgement.** We thank the editor and the anonymous referees for their comments and suggestions. The first author is grateful for the financial support extended by the cooperation agreement between International Science Program (ISP) at Uppsala University and Addis Ababa University. We appreciate the hospitality we got from IRIF during the research visits of the first author. We also thank Jörgen Backelin of Stockholm University for the crucial discussions we had with him and his very useful suggestions.

#### REFERENCES

- [1] Alexandersson, P. and Getachew, F., (2021), An involution on derangements preserving excedances and right-to-left minima, Preprint, <https://arXiv.org/2105.08455v2>.
- [2] Baril, J.-L., (2007), Gray code for permutations with fixed number of cycles, *Discrete Math.*, 307, pp. 1557-1571.
- [3] Baril, J.-L., (2013), Statistics-preserving bijections between classical and cyclic permutations, *Inform. Process. Lett.*, 113, pp. 17-22.
- [4] Baril, J.-L., and Vajnovszki, V., (2017), A permutation code preserving a double Eulerian bivariate, *Discrete Appl. Math.*, 224, pp. 9-15.
- [5] Bona, M., (2007), *Introduction to Enumerative Combinatorics*, McGraw-Hill.
- [6] Foata, D., and Zeilberger, D., (1990), Denert's Permutation statistic is indeed Euler-Mahonian, *Stud. Appl. Math.*, 83, pp. 31-59.
- [7] Hu, Y., (2014), On the number of fixed-length semiorders, *J. Integer Seq.* 17, Article 14.1.6.
- [8] Laisant, Ch. A., (1888), Sur la numération factorielle, application aux permutations, *Bulletin de la Société Mathématique de France*, 16, pp. 176-183.
- [9] Lehmer, D. H., (1960), Teaching Combinatorial tricks to a computer, *Proc. Sympos. Appl. Math. Combinatorial Analysis*, Amer. Math. Soc., 10, pp. 179-193.
- [10] Mantaci, R., and Rakotondrajao, F., (2001), A permutation representation that knows what "Eulerian" means, *Discrete Math. Theor. Comput. Sci.*, 4, pp. 101-108.
- [11] Stanley, R. P., (2011), *Enumerative Combinatorics*, Cambridge University Press, Vol. 1, Sec-ed.
- [12] Tenner, B. E., (2017), Reduced word manipulation: patterns and enumeration, *Journal of Algebraic Combinatorics*, 46, pp. 189-217.



**Fufa Beyene** received his B.Ed. degree in Mathematics from Mekelle University, Ethiopia, in 2007, M.Sc. degree in Mathematics (Combinatorics) from Addis Ababa University, Ethiopia, in 2011, and he is a Ph.D. candidate in Mathematics (Enumerative Combinatorics) at Addis Ababa University, Ethiopia. His area of research is Enumerative and Algebraic Combinatorics, and Algorithms.



**Roberto Mantaci** received his M.Sc. degree in Mathematics from University of Palermo, Italy, in 1987, Ph.D. degree in Theoretical Computer Science from University Paris 7, France, in 1991, and Ph.D. degree in Mathematics, University Roma 1, Italy, in 1993. His area of research is Algorithms, Automata, Programming, Enumerative and Algebraic Combinatorics.

---

---