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STARLIKENESS AND CONVEXITY OF INTEGRAL OPERATORS INVOLVING MITTAG-LEFFLER FUNCTIONS

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ABSTRACT. In this paper, we shall find the order of starlikeness and convexity for integral operators

$$\mathbb{F}_{\alpha_j,\beta_j,\lambda_j,\zeta}(z)=\left\{\zeta\int_0^z t^{\zeta-1}\prod_{j=1}^{n}\left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(t)}{t}\right)^{1/\lambda_j}dt\right\}^{1/\zeta},$$

where the functions $\mathbb{E}_{\alpha_j,\beta_j}$ are the normalized Mittag-Leffler functions.

Keywords: Analytic functions; Starlike and convex functions; Integral operators; Mittag-Leffler function.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z:|z|<1\}$. A function $f(z)\in\mathcal{A}$ is said to be starlike of order δ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\delta \qquad (z\in\mathbb{U}) \tag{2}$$

for some $\delta(0\leq\delta<1)$. We denote by $\mathcal{S}^*(\delta)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order δ in $\mathbb{U}$. Clearly $\mathcal{S}^*(\delta)\subseteq\mathcal{S}^*(0)=\mathcal{S}^*$, where $\mathcal{S}^*$is the class of functions that are starlike in $\mathbb{U}$. Also, a function $f(z)\in\mathcal{A}$ is said to be convex of order δ if it satisfies

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\delta \qquad (z\in\mathbb{U}) \tag{3}$$

for some $\delta(0\leq\delta<1)$. We denote by $\mathcal{C}(\delta)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathbb{U}$. Clearly $\mathcal{C}(\delta)\subseteq\mathcal{C}(0)=\mathcal{C}$, the class of functions that are convex in $\mathbb{U}$.

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Let $E_\alpha(z)$ be the function defined by

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad (z\in\mathbb{C},\operatorname{Re}(\alpha)>0\ ).$$

The function $E_\alpha(z)$ was introduced by Mittag-Leffler [16] and is, therefore, known as the Mittag-Leffler function. A more general function $E_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced

by Wiman [20, 21] and defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad (z,\alpha,\beta\in\mathbb{C},\operatorname{Re}(\alpha)>0\ ). \tag{4}$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in ([2, 3, 4, 8, 10], [11]-[18]).

Observe that Mittag-Leffler function $E_{\alpha,\beta}$ does not belong to the family $\mathcal{A}$. Therefore, we consider the following normalization of the Mittag-Leffler function:

$$\begin{aligned}\mathbb{E}_{\alpha,\beta}(z)&=\Gamma(\beta)zE_{\alpha,\beta}(z)\\&=z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}z^n,\end{aligned} \tag{5}$$

where $z,\alpha,\beta\in\mathbb{C};\beta\neq 0,-1,-2,\cdots$ and $\operatorname{Re}(\alpha)>0$.

Whilst formula (5) holds for complex-valued α,β and $z\in\mathbb{C}$, however in this paper, we shall restrict our attention to the case of real-valued α,β and $z\in\mathbb{U}$. Observe that the function $\mathbb{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example, $\mathbb{E}_{2,1}(z)=z\cosh\sqrt{z}$, $\mathbb{E}_{2,2}(z)=\sqrt{z}\sinh\sqrt{z}$, $\mathbb{E}_{2,3}(z)=2[\cosh\sqrt{z}-1]$ and $\mathbb{E}_{2,4}(z)=6[\sinh\sqrt{z}-\sqrt{z}]/\sqrt{z}$.

Geometric properties including starlikeness, convexity and close-to-convexity for the Mittag-Leffler function $E_{\alpha,\beta}$ were recently investigated by Bansal and Prajapat in [5].

Recently, Srivastava *et al.*[19] introduced a new integral operator $\mathbb{F}_{\alpha_j,\beta_j,\lambda_j,\zeta}$ involving Mittag-Leffler functions given by

$$\mathbb{F}(z)=\mathbb{F}_{\alpha_j,\beta_j,\lambda_j,\zeta}(z)=\left\{\zeta\int_0^z t^{\zeta-1}\prod_{j=1}^{n}\left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(t)}{t}\right)^{1/\lambda_j}dt\right\}^{1/\zeta}, \tag{6}$$

where the functions $\mathbb{E}_{\alpha_j,\beta_j}$ are the normalized Mittag-Leffler functions defined by

$$\mathbb{E}_{\alpha_j,\beta_j}(z)=\Gamma(\beta_j)zE_{\alpha_j,\beta_j}(z).$$

and the parameters $\lambda_1,\lambda_1,\ldots,\lambda_n$ and ζ are are positive real numbers such that the integrals in (6) exist. Here and throughout in the sequel every many-valued function is taken with the principal branch.

Several authors studied univalency, starlikeness and convexity of certain integral operators, see [1, 6, 7, 9, 17, 22]. In the present paper, we will find the order of starlikeness and convexity for the above integral operator involving Mittag-Leffler functions and defined by (6).

In order to prove our main results, we recall the following lemmas.

Lemma 1.1. *([15]). Let $\Phi(u,v)$ be a complex valued function,*

$$\Phi : \mathbb{D} \to \mathbb{C}, \qquad (\mathbb{D} \subset \mathbb{C}^2)$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $\Phi(u,v)$ satisfies

(i) $\Phi(u,v)$ is continuous in $\mathbb{D}$;
(ii) $(1,0) \in \mathbb{D}$ and $\mathrm{Re}(\Phi(1,0)) > 0$;
(iii) $\mathrm{Re}(\Phi(iu_2, v_1)) \leq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ and such that $v_1 \leq -(1+u_2^2)/2$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in $\mathbb{U}$ such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{U}$. If $\mathrm{Re}(\Phi(p(z), zp'(z))) > 0$ $(z \in \mathbb{U})$, then $\mathrm{Re}(p(z)) > 0$ $(z \in \mathbb{U})$.

Lemma 1.2. *([5]) Let $\alpha \geq 1$ and $0 \leq \eta < 1$. Suppose also that*

$$\Psi(\eta) = \frac{(3-\eta) + \sqrt{5\eta^2 - 18\eta + 17}}{2(1-\eta)}.$$

If $\beta \geq \Psi(\eta)$, then $\mathbb{E}_{\alpha,\beta}$ is starlike function of order η.

Lemma 1.3. *([19]) Let $\alpha \geq 1$ and $\beta \geq 1$. Then*

$$\left| \frac{z\mathbb{E}'_{\alpha,\beta}(z)}{\mathbb{E}_{\alpha,\beta}(z)} - 1 \right| \leq \frac{2\beta+1}{\beta^2 - \beta - 1}, \qquad (z \in \mathbb{U}). \tag{7}$$

2. Main Results

Our first result provides the order of starlikeness for integral operator of the type (6).

Theorem 2.1. *Let $\alpha_j \geq 1, 0 \leq \eta_j < 1$, and*

$$\beta_j \geq \frac{(3-\eta_j) + \sqrt{5\eta_j^2 - 18\eta_j + 17}}{2(1-\eta_j)},$$

for all $j = 1, 2, 3, \ldots, n$. Suppose also that $\lambda_1, \lambda_2, \ldots, \lambda_n, \zeta$ are positive real numbers such that

$$\sum_{j=1}^{n} \frac{1-\eta_j}{\lambda_j} \leq \zeta,$$

then $\mathbb{F}(z) \in \mathcal{S}^*(\delta)$, where

$$\delta = \frac{-\left(\sum_{j=1}^{n} \frac{2(1-\eta_j)}{\lambda_j} - 2\zeta + 1\right) + \sqrt{\left(\sum_{j=1}^{n} \frac{2(1-\eta_j)}{\lambda_j} - 2\zeta + 1\right)^2 + 8\zeta}}{4\zeta}, \quad 0 \leq \delta < 1. \tag{8}$$

Proof. Define the function $p(z)$ by

$$\frac{z\mathbb{F}'(z)}{\mathbb{F}(z)} := \delta + (1-\delta)p(z), \tag{9}$$

where δ as given in (8).

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is analytic in $\mathbb{U}$. It follows from (6) and (9) that

$$\frac{z^\zeta \prod_{j=1}^{n} \left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(z)}{z}\right)^{1/\lambda_i}}{\mathbb{F}^\zeta(z)} = \delta + (1-\delta)p(z). \tag{10}$$

Differentiating (10) logarithmically, we obtain

$$\sum_{j=1}^{n}\frac{1}{\lambda_j}\left(\frac{z\mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)}\right)=\zeta(1-\delta)p(z)+\frac{(1-\delta)zp'(z)}{\delta+(1-\delta)p(z)}+\sum_{j=1}^{n}\frac{1}{\lambda_j}-\zeta(1-\delta). \tag{11}$$

From Lemma 1.1, $\mathbb{E}_{\alpha_j,\beta_j}$ is starlike function of order η_j for all $j=1,2,3,\ldots,n$, therefore we have

$$\begin{aligned}&\sum_{j=1}^{n}\frac{1}{\lambda_j}\mathrm{Re}\left(\frac{z\mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)}\right)\\ =\ &\mathrm{Re}\left\{\zeta(1-\delta)p(z)+\frac{(1-\delta)zp'(z)}{\delta+(1-\delta)p(z)}+\sum_{j=1}^{n}\frac{1-\eta_j}{\lambda_j}-\zeta(1-\delta)\right\}>0.\end{aligned} \tag{12}$$

If we define the function $\Phi(u,v)$ by

$$\Phi(u,v)=\zeta(1-\delta)u+\frac{(1-\delta)v}{\delta+(1-\delta)u}+\sum_{j=1}^{n}\frac{1-\eta_j}{\lambda_j}-\zeta(1-\delta) \tag{13}$$

with $u=u_1+iu_2$ and $v=v_1+iv_2$, then

(i) $\Phi(u,v)$ is continuous in $\mathbb{D}=\mathbb{C}^2$;

(ii) $(1,0)\in\mathbb{D}$ and $\mathrm{Re}(\Phi(1,0))=\sum\limits_{j=1}^{n}\frac{1-\eta_j}{\lambda_j}>0$;

(iii) For all $(iu_2,v_1)\in\mathbb{D}$ and such that $v_1\le-(1+u_2^2)/2$,

$$\begin{aligned}\mathrm{Re}(\Phi(iu_2,v_1)) &= \frac{\delta(1-\delta)v_1}{\delta^2+(1-\delta)^2u_2^2}+\sum_{j=1}^{n}\frac{1-\eta_j}{\lambda_j}-\zeta(1-\delta)\\ &\le \frac{A+Bu_2^2}{C}\end{aligned} \tag{14}$$

where

$$A=\delta\left(2\zeta\delta^2+\left(\sum_{j=1}^{n}\frac{2(1-\eta_j)}{\lambda_j}-2\zeta+1\right)\delta-1\right),$$

$$B=(1-\delta)^2\left(\sum_{j=1}^{n}\frac{2(1-\eta_j)}{\lambda_j}-2\zeta(1-\delta)\right)-\delta(1-\delta),$$

and

$$C=2\delta^2+2(1-\delta)^2u_2^2.$$

The right hand side of (14) is negative if $A\le0$ and $B\le0$. From $A\le0$, we have the value of δ given by (8) and from $B\le0$, we have $0\le\delta<1$. Therefore, the function $\Phi(u,v)$ satisfies the conditions in Lemma 1.1. Thus we have $\mathrm{Re}(p(z))>0\,(z\in\mathbb{U})$, that is $\mathbb{F}(z)\in\mathcal{S}^*(\delta)$. □

Let $n=1$, $\alpha_1=\alpha$, $\beta_1=\beta$, $\lambda_1=\lambda$ and $\eta_1=0$ in Theorem 2.1, we have the following result.

Corollary 2.1. *Let $\alpha \geq 1$ and $\beta \geq \frac{3+\sqrt{17}}{2}$. Then*

$$\mathbb{F}_{\alpha,\beta,\lambda,\zeta}(z) = \left\{ \zeta \int_0^z t^{\zeta-1} \left(\frac{\mathbb{E}_{\alpha,\beta}(t)}{t} \right)^{1/\lambda} dt \right\}^{1/\zeta} \in \mathcal{S}^*(\delta)$$

where λ and ζ are positive real numbers such that $\frac{1}{\lambda} \leq \zeta$, and

$$\delta = \frac{-\left(\frac{2}{\lambda} - 2\zeta + 1\right) + \sqrt{\left(\frac{2}{\lambda} - 2\zeta + 1\right)^2 + 8\zeta}}{4\zeta}, \quad 0 \leq \delta < 1.$$

Putting $\lambda = 1$ and $\zeta = 1$ in Corollary 2.1, we immediately have

Corollary 2.2. *Let $\alpha \geq 1$ and $\beta \geq \frac{3+\sqrt{17}}{2}$. Then $\mathbb{F}_{\alpha,\beta,1,1}(z) = \int_0^z \left(\frac{\mathbb{E}_{\alpha,\beta}(t)}{t} \right) dt$ is starlike of order $1/2$ in $\mathbb{U}$.*

Example 2.1. *Let $\mathbb{E}_{2,4}(z) = 6[\sinh\sqrt{z} - \sqrt{z}]/\sqrt{z}$, then $\int_0^z \frac{6[\sinh\sqrt{t} - \sqrt{t}]}{t^{3/2}} dt$ is starlike of order $1/2$ in $\mathbb{U}$.*

Making use Lemma 1.3, we determine the order of convexity for integral operator of the type (6).

Theorem 2.2. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 1$, $\beta_1, \beta_2, \ldots, \beta_n \geq \frac{1}{2}(1+\sqrt{5})$ and consider the normalized Mittag-Leffler functions $\mathbb{E}_{\alpha_j,\beta_j}$ defined by*

$$\mathbb{E}_{\alpha_j,\beta_j}(z) = \Gamma(\beta_j) z E_{\alpha_j,\beta_j}(z). \tag{15}$$

Let $\beta = \min\{\beta_1, \beta_2, \ldots, \beta_n\}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be nonzero positive real numbers. Moreover, suppose that these numbers satisfy the following inequality

$$0 \leq 1 - \frac{2\beta + 1}{\beta^2 - \beta - 1} \sum_{j=1}^{n} \frac{1}{\lambda_j} < 1.$$

Then the function $\mathbb{F}_{\alpha_j,\beta_j,\lambda_j}$ defined by

$$\mathbb{F}_{\alpha_j,\beta_j,\lambda_j}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(t)}{t} \right)^{1/\lambda_j} dt, \tag{16}$$

is in $\mathcal{C}(\delta)$, where

$$\delta = 1 - \frac{2\beta + 1}{\beta^2 - \beta - 1} \sum_{j=1}^{n} \frac{1}{\lambda_j}.$$

Proof. We observe that $\mathbb{E}_{\alpha_j,\beta_j} \in \mathcal{A}$, i.e. $\mathbb{E}_{\alpha_j,\beta_j}(0) = \mathbb{E}'_{\alpha_j,\beta_j}(0) - 1 = 0$, for all $j \in \{1, 2, \ldots, n\}$. On the other hand, it is easy to see that

$$\mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(z) = \prod_{j=1}^{n} \left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(z)}{z} \right)^{1/\lambda_j}$$

and

$$\frac{z\mathbb{F}''_{\alpha_j,\beta_j,\lambda_j}(z)}{\mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(z)} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \left(\frac{z\mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)} - 1 \right),$$

or, equivalently,

$$1+\frac{z\mathbb{F}''_{\alpha_j,\beta_j,\lambda_j}(z)}{\mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(z)}=\sum_{j=1}^{n}\frac{1}{\lambda_j}\left(\frac{z\mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)}\right)+1-\sum_{j=1}^{n}\frac{1}{\lambda_j}. \tag{17}$$

Taking the real part of both terms of (17), we have

$$\operatorname{Re}\left\{1+\frac{z\mathbb{F}''_{\alpha_j,\beta_j,\lambda_j}(z)}{\mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(z)}\right\}=\sum_{j=1}^{n}\frac{1}{\lambda_j}\operatorname{Re}\left(\frac{z\mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)}\right)+\left(1-\sum_{j=1}^{n}\frac{1}{\lambda_j}\right). \tag{18}$$

Now, by using the inequality (7) for each β_j, where $j\in\{1,2,\dots,n\}$, we obtain

$$\begin{aligned}\operatorname{Re}\left\{1+\frac{z\mathbb{F}''_{\alpha_j,\beta_j,\lambda_j}(z)}{\mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(z)}\right\} &= \sum_{j=1}^{n}\frac{1}{\lambda_j}\operatorname{Re}\left(\frac{z\mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)}\right)+\left(1-\sum_{j=1}^{n}\frac{1}{\lambda_j}\right)\\ &> \sum_{j=1}^{n}\frac{1}{\lambda_j}\left(1-\frac{2\beta_j+1}{\beta_j^2-\beta_j-1}\right)+\left(1-\sum_{j=1}^{n}\frac{1}{\lambda_j}\right)\\ &= 1-\frac{2\beta+1}{\beta^2-\beta-1}\sum_{j=1}^{n}\frac{1}{\lambda_j}\end{aligned}$$

for all $z\in\mathbb{D}$ and $\beta_1,\beta_2,\dots,\beta_n\geq\frac{1}{2}(1+\sqrt{5})$. Here we used that the function $\varphi:(\frac{1}{2}(1+\sqrt{5}),\infty)\to\mathbb{R}$, defined by

$$\varphi(x)=\frac{2x+1}{x^2-x-1},$$

is decreasing. Therefore, for all $j\in\{1,2,\dots,n\}$ we have

$$\frac{2\beta_j+1}{\beta_j^2-\beta_j-1}\leq\frac{2\beta+1}{\beta^2-\beta-1}. \tag{19}$$

Because $0\leq 1-\frac{2\beta+1}{\beta^2-\beta-1}\sum_{j=1}^{n}\frac{1}{\lambda_j}<1$,we get $\mathbb{F}_{\alpha_j,\beta_j,\lambda_j}(z)\in\mathcal{C}(\delta)$, where $\delta=1-\frac{2\beta+1}{\beta^2-\beta-1}\sum_{j=1}^{n}\frac{1}{\lambda_j}$. This completes the proof. □

Let $n=1$, $\alpha_1=\alpha$, $\beta_1=\beta$ and $\lambda_1=\lambda$ in Theorem 2.1, we have the following result.

Corollary 2.3. *Let $\alpha\geq 1$, $\beta\geq\frac{1}{2}(1+\sqrt{5})$ and $\lambda>0$. Moreover, suppose that these numbers satisfy the following inequality*

$$0\leq 1-\frac{2\beta+1}{\lambda(\beta^2-\beta-1)}<1.$$

Then the function $\mathbb{F}_{\alpha,\beta,\lambda}$ defined by

$$\mathbb{F}_{\alpha,\beta,\lambda}(z)=\int_0^z\left(\frac{\mathbb{E}_{\alpha,\beta}(t)}{t}\right)^{1/\lambda}dt,$$

is in $\mathcal{C}(\delta)$, where

$$\delta=1-\frac{2\beta+1}{\lambda(\beta^2-\beta-1)}.$$

Example 2.2. (*i*) *If* $0\leq 1-\frac{5}{\lambda}<1$, *then* $\int_0^z\left(\frac{\sinh\sqrt{t}}{\sqrt{t}}\right)^{1/\lambda}dt\in\mathcal{C}(\delta);\delta=1-\frac{5}{\lambda};\lambda\geq 5.$

(*ii*) *If* $0 \leq 1 - \frac{7}{5\lambda} < 1$, *then* $\int\limits_0^z \left(\frac{2[\cosh \sqrt{t} - 1]}{t} \right)^{1/\lambda} dt \in \mathcal{C}(\delta); \delta = 1 - \frac{7}{5\lambda}; \lambda \geq 7/5.$

(*iii*) *If* $0 \leq 1 - \frac{9}{11\lambda} < 1$, *then* $\int\limits_0^z \left(\frac{6[\sinh \sqrt{t} - \sqrt{t}]}{t^{3/2}} \right)^{1/\lambda} dt \in \mathcal{C}(\delta); \delta = 1 - \frac{9}{11\lambda}; \lambda \geq 9/11.$

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