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ON $\mathcal{I}$-CONVERGENCE ALMOST SURELY OF COMPLEX UNCERTAIN SEQUENCES

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Abstract. In this paper, we explore the concepts of $\mathcal{I}$-convergence almost surely and $\mathcal{I}^*$-convergence almost surely in complex uncertain theory and study some of their properties and identify the relationships between them. Also, we introduced the notions of $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequence almost surely of complex uncertain sequences and investigate their relationships.

Keywords: Uncertainty theory, complex uncertain variable, $\mathcal{I}$-convergence, $\mathcal{I}^*$- convergence.

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1. Introduction

Characteristics of different types of convergences of a sequence are making a huge impact on mathematical analysis. The concept of statistical convergence, which is an extension of the usual idea of convergence, was introduced by Fast[9] and Steinhaus[27], individually in the year 1951. But the research on this concept got flourish soon after the works of $\check{S}al\acute{a}t$[23] and Fridy[10] came into literature. As an extension of statistical convergence, the idea of $\mathcal{I}$ and $\mathcal{I}^*$-convergence was introduced by Kostyrko et al.[15]. Thereafter, lots of interesting developments have occurred in $\mathcal{I}$-convergence and related topics like Savaş and Das[24, 25], Mursaleen et al.[18], Debnath and Rakshit[6], Savaş et al.[26], Choudhury and Debnath[2], Hazarika[11], and many more.

There are various types of problems in the real world that have uncertainty. Dealing with these important problems along with uncertainty is challenging work. In 2007, Liu[16] proposed a theory, named uncertainty theory to overcome these specific types of

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problems. Later, he[17] also improved this theory in 2009. Afterward, uncertainty theory made an entry into the area of mathematics, and it has become an active area of research. In [16], Liu defined different types of convergence of a sequence of real uncertain variables namely convergence almost surely (a.s.), convergence in measure, convergence in mean, and also convergence in distribution. Then You[29] defined convergence uniformly almost surely (u.a.s.) and gave the relationships among all these different convergence of a sequence of real uncertain variables. Chen et al.[1] extended this work by introducing these various convergences for the sequence of complex uncertain variables. To know more about complex uncertain variables, we may refer to[21, 30, 31]. Recently, Tripathy and Nath[28] defined statistical convergence for complex uncertain sequences. After that, many researchers investigated the nature of convergence of sequences in an uncertain environment like Dowari and Tripathy[7, 8], Das et al.[3], Debnath and Das[4, 5], Debnath and Rakshit[6], Khan et al.[12], Kişi[13, 14], Nath and Tripathy[20], Saha et al.[22] and many more.

Inspired by the above works, in this article, we introduce the notion of $\mathcal{I}^*$-convergence almost surely in complex uncertain theory, examined several properties, and identify the relationships between $\mathcal{I}$ and $\mathcal{I}^*$-convergence almost surely of complex uncertain sequence. Moreover, we define $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequence almost surely of complex uncertain sequence. It can be observe that if $\mathcal{I} = \mathcal{I}_d$, then $\mathcal{I}$-convergence almost surely coincide with statistical convergence almost surely of complex uncertain sequence and if $\mathcal{I}$ be an admissible ideal, then $\mathcal{I}^*$-convergence almost surely coincide with $\mathcal{I}$-convergence almost surely but the converse is not true.

2. Definitions and Preliminaries

In this section, we provide some basic definitions and results on generalized convergence concepts and the theory of uncertainty which will be used throughout the article.

Definition 2.1. [15] *A non-void class $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if $\mathcal{I}$ is additive (i.e., $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$) and hereditary (i.e., $A \in \mathcal{I}$ and $B \subseteq A \implies B \in \mathcal{I}$).*
An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq 2^{\mathbb{N}}$. A non-trivial ideal $\mathcal{I}$ is said to be admissible if $\mathcal{I}$ contains every finite subset of $\mathbb{N}$.

Example 2.2. *(i) $\mathcal{I}_f :=$ The set of all finite subsets of $\mathbb{N}$ forms a non-trivial admissible ideal.*
(ii) $\mathcal{I}_d :=$ The set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal.

Definition 2.3. [10] *A sequence (x_n) is said to be statistically convergent to ℓ provided that for each $\varepsilon > 0$ such that $\lim\limits_{n\to\infty} \frac{1}{n}\big|\{k \leq n : |x_k - \ell| \geq \varepsilon\}\big| = 0$, $n \in \mathbb{N}$.*

Definition 2.4. [15] *A sequence (x_n) is said to be $\mathcal{I}$-convergent to ℓ, if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\} \in \mathcal{I}$. The usual convergence of sequences is a special case of $\mathcal{I}$-convergence ($\mathcal{I}$=$\mathcal{I}_f$-the ideal of all finite subsets of $\mathbb{N}$). The statistical convergence of sequences is also a special case of $\mathcal{I}$-convergence. In this case, $\mathcal{I}$=$\mathcal{I}_d = \{A \subseteq \mathbb{N} : \lim\limits_{n\to\infty} \frac{|A\cap\{1,2,\ldots,n\}|}{n} = 0\}$, where $|A|$ is the cardinality of the set A.*

Definition 2.5. [15] *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence (x_n) is said to be $\mathcal{I}^*$-convergent to ℓ, if there exists a set $A = \{m_1 < m_2 < \cdots < m_n < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that*

$$\lim_{n\to\infty} |x_{m_n} - \ell| = 0.$$

Definition 2.6. [19] *A sequence (x_n) is said to be $\mathcal{I}$-Cauchy, if for every $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that $\{n \in \mathbb{N} : |x_n - x_{n_0}| \geq \varepsilon\} \in \mathcal{I}$.*

Definition 2.7. [19] *A sequence (x_n) is said to be $\mathcal{I}^*$-Cauchy, if there exists a set $A = \{m_1 < m_2 < \cdots < m_n < \cdots\} \subset \mathbb{N}$, $A \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_n}) is a Cauchy sequence i.e., $\lim\limits_{i,j\to\infty} \left|x_{m_i} - x_{m_j}\right| = 0$.*

Definition 2.8. [15] *An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is said to satisfy the condition AP, if for every countable family of mutually disjoint sets $\{C_n\}_{n\in\mathbb{N}}$ from $\mathcal{I}$, there exists a countable family of sets $\{B_n\}_{n\in\mathbb{N}}$ such that the symmetric difference $C_j \triangle B_j$ is finite for every $j \in \mathbb{N}$ and $\bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$.*

Definition 2.9. [16] *Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1 (Normality): $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (Duality): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axiom 3 (Subadditivity): For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$,

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event. To obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu[17] *as:*

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

Definition 2.10. [21] *A variable $\zeta = \xi + i\eta$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers is a complex uncertain variable if and only if ξ and η are uncertain variables, where ξ and η are the real and imaginary parts of ζ, respectively.*

Definition 2.11. [1] *A complex uncertain sequence (ζ_n) is said to be convergent almost surely (a.s) to ζ if for every $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} \|\zeta_n(\gamma) - \zeta(\gamma)\| = 0, \text{ for every } \gamma \in \Lambda.$$

Symbolically we write $\zeta_n \xrightarrow{A_s} \zeta$.

Definition 2.12. [28] *A complex uncertain sequence (ζ_n) is said to be statistically convergent almost surely to ζ if for every $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \leq n : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right\}\right| = 0, \text{ for every } \gamma \in \Lambda.$$

Symbolically we write $\zeta_n \xrightarrow{S^{A_s}} \zeta$.

Definition 2.13. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent almost surely to ζ if for every $\varepsilon > 0$, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\} \in \mathcal{I}, \ \textit{for every } \gamma \in \Lambda.$$

Symbolically we write $\zeta_n \xrightarrow{A_s(\mathcal{I})} \zeta$.

Throughout the paper, we consider $\mathcal{I}$ is a non-trivial admissible ideal of $\mathbb{N}$.

3. Main Results

Theorem 3.1. *If a complex uncertain sequence (ζ_n) is convergent almost surely to ζ, then (ζ_n) is $\mathcal{I}$-convergent almost surely to ζ.*

Proof. It follows directly from the fact that $\mathcal{I}_f \subset \mathcal{I}$. □

Remark 3.2. *But the converse of Theorem 3.1 is not true in general.*

Example 3.3. *Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)}, & if \ \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)}, & if \ \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)} < \frac{1}{2} \\ \frac{1}{2}, \ otherwise \end{cases} \quad for \ n = 1, 2, 3, \cdots.$$

Also, the complex uncertain variables defined by

$$\zeta_n(\gamma) = \begin{cases} i\beta_n, \ if \ \gamma \in \{\gamma_1, \gamma_4, \gamma_9, \cdots\} \\ 0, \ otherwise \end{cases} \quad for \ n = 1, 2, 3, \cdots,$$

where $\beta_n = \begin{cases} n, \ if \ n = k^2, k \in \mathbb{N} \\ 0, \quad otherwise \end{cases} \quad for \ n = 1, 2, 3, \cdots$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$.
For any $\varepsilon > 0$ and there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$, we have
$\Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\} = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma)\| \geq \varepsilon\Big\}$
$= \Big\{1, 4, 9, \cdots\Big\} \in \mathcal{I}$ *for every* $\gamma \in \Lambda$.
Thus the sequence (ζ_n) is $\mathcal{I}$-convergent almost surely to $\zeta \equiv 0$ but it is not convergent almost surely to $\zeta \equiv 0$.

Theorem 3.4. *Let $\zeta, \zeta_1, \zeta_2, \cdots$ be complex uncertain variables defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. If $\zeta_n \xrightarrow{A_s(\mathcal{I})} \zeta$, then ζ is uniquely determined.*

Proof. If possible let $\zeta_n \xrightarrow{A_s(\mathcal{I})} \zeta$, and $\zeta_n \xrightarrow{A_s(\mathcal{I})} \zeta^*$ for some $\zeta(\gamma) \neq \zeta^*(\gamma)$. Let $\varepsilon > 0$ be arbitary. Then for any $\varepsilon > 0$, we have
$A = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| < \frac{\varepsilon}{2}\Big\} \in \mathcal{F}(\mathcal{I})$ and
$B = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta^*(\gamma)\| < \frac{\varepsilon}{2}\Big\} \in \mathcal{F}(\mathcal{I})$.

Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$ this implies $A \cap B \neq \phi$. Let $m \in A \cap B$.

Then $\|\zeta_m(\gamma) - \zeta(\gamma)\| < \frac{\varepsilon}{2}$ and $\|\zeta_m(\gamma) - \zeta^*(\gamma)\| < \frac{\varepsilon}{2}$

Therefore
$$\begin{aligned}\|\zeta(\gamma) - \zeta^*(\gamma)\| &= \|\zeta_m(\gamma) - \zeta^*(\gamma) + \zeta(\gamma) - \zeta_m(\gamma)\| \\ &\leq \|\zeta_m(\gamma) - \zeta^*(\gamma)\| + \|\zeta(\gamma) - \zeta_m(\gamma)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Hence the results.

□

Theorem 3.5. *If the complex uncertain sequence (ζ_n) and (ζ_n^*) are $\mathcal{I}$-convergent almost surely to ζ and ζ^*, respectively, then*

- *(i) $(\zeta_n + \zeta_n^*)$ is $\mathcal{I}$-convergent almost surely to $\zeta + \zeta^*$.*
- *(ii) $\zeta_n - \zeta_n^*$ is $\mathcal{I}$-convergent almost surely to $\zeta - \zeta^*$.*
- *(iii) $(c\zeta_n)$ is $\mathcal{I}$-convergent almost surely to $c\zeta$, where $c \in \mathbb{C}$.*

Proof. (i) Let $\varepsilon > 0$, then $A = \Big\{n \in \mathbb{N} : \big(\|\zeta_n(\gamma) - \zeta(\gamma)\| < \frac{\varepsilon}{2}\big)\Big\} \in \mathcal{F}(\mathcal{I})$
and $B = \Big\{n \in \mathbb{N} : \big(\|\zeta_n^*(\gamma) - \zeta^*(\gamma)\| < \frac{\varepsilon}{2}\big)\Big\} \in \mathcal{F}(\mathcal{I})$.
Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$ this implies $A \cap B \neq \phi$. Therefore for all $n \in A \cap B$ we have,

$$\|(\zeta_n(\gamma) + \zeta_n^*(\gamma)) - (\zeta(\gamma) + \zeta^*(\gamma))\| \leq \|\zeta_n(\gamma) - \zeta(\gamma)\| + \|\zeta_n^*(\gamma) - \zeta^*(\gamma)\| < \varepsilon.$$

i.e, $\Big\{n \in \mathbb{N} : \|(\zeta_k + \zeta_k^*) - (\zeta + \zeta^*)\| < \varepsilon\Big\} \in \mathcal{F}(\mathcal{I})$.
Hence $(\zeta_n + \zeta_n^*)$ is $\mathcal{I}$-convergent almost surely to $\zeta + \zeta^*$.

(ii) It is similar to the proof of (i) above and therefore omitted.
(iii) The proof is easy so omitted. □

Theorem 3.6. *If the complex uncertain sequences (ζ_n) and (ζ_n^*) are $\mathcal{I}$-convergent almost surely to ζ and ζ^*, respectively, and there exist positive numbers $p_1, p, q_1,$ and q such that $p_1 \leq \|\zeta_n\|, \|\zeta\| \leq p$ and $q_1 \leq \|\zeta_n^*\|, \|\zeta^*\| \leq q$ for any n, then*

- *(i) $(\zeta_n \zeta_n^*)$ is $\mathcal{I}$-convergent almost surely to $\zeta\zeta^*$.*
- *(ii) $(\frac{\zeta_n}{\zeta_n^*})$ is $\mathcal{I}$-convergent almost surely to $\frac{\zeta}{\zeta^*}$.*

Proof. (i) Let $\varepsilon > 0$, and $p, q > 0$ then

$$A = \Big\{n \in \mathbb{N} : \|\zeta_n - \zeta\| < \frac{\varepsilon}{2q}\Big\} \in \mathcal{F}(\mathcal{I})$$

$$\text{and } B = \Big\{n \in \mathbb{N} : \|\zeta_n^* - \zeta^*\| < \tfrac{\varepsilon}{2p}\Big\} \in \mathcal{F}(\mathcal{I}).$$

Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$ this implies $A \cap B \neq \phi$. Therefore for all $n \in A \cap B$ we have,

$$\begin{aligned}\|\zeta_n \zeta_n^* - \zeta\zeta^*\| &= \|\zeta_n \zeta_n^* - \zeta_n \zeta^* + \zeta_n \zeta^* - \zeta\zeta^*\| \\ &\leq \|\zeta_n \zeta_n^* - \zeta_n \zeta^*\| + \|\zeta_n \zeta^* - \zeta\zeta^*\| \\ &\leq p\,\|\zeta_n^* - \zeta^*\| + q\,\|\zeta_n - \zeta\|. \\ &< \varepsilon\end{aligned}$$

i.e, $\Big\{n \in \mathbb{N} : \|\zeta_n\zeta_n^* - \zeta\zeta^*\| < \varepsilon\Big\} \in \mathcal{F}(\mathcal{I})$.
Hence $(\zeta_n\zeta_n^*)$ is $\mathcal{I}$-convergent almost surely to $\zeta\zeta^*$.

(ii) It is similar to the proof of (i) above and therefore omitted. □

Theorem 3.7. *If every subsequence of a complex uncertain sequence (ζ_n) is $\mathcal{I}$-convergent almost surely to ζ, then (ζ_n) is $\mathcal{I}$-convergent almost surely to ζ.*

Proof. If possible let, every subsequence of a complex uncertain sequence (ζ_n) is $\mathcal{I}$-convergent almost surely to ζ, but (ζ_n) is not $\mathcal{I}$-convergent almost surely to ζ. Then there exists some $\varepsilon > 0$ such that

$$A = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\} \notin \mathcal{I}.$$

So A must be an infinite set. Let $A = \{n_1 < n_2 < \cdots < n_j < \cdots\}$. Now we define a sequence (ζ_j^*) as $\zeta_j^* = \zeta_{n_j}$ for $j \in \mathbb{N}$. Then (ζ_j^*) is a subsequence of (ζ_n) which is not $\mathcal{I}$-convergent almost surely to ζ, a contradiction. □

Remark 3.8. *But the converse of Theorem 3.7 is not true in general.*

Example 3.9. *From Example 3.3, we see that the complex uncertain sequence (ζ_n) is $\mathcal{I}$-convergent almost surely to $\zeta \equiv 0$.*
Now we define a subsequence (ζ_m^*) of (ζ_n) by
$(\zeta_m^*) = (\zeta_{m_n})$, *where $m_n = n^2, n \in \mathbb{N}$ which is not $\mathcal{I}$-convergent almost surely to $\zeta \equiv 0$.*

Theorem 3.10. *Let (ζ_n) and (ζ_n^*) be two complex uncertain sequences such that (ζ_n^*) be convergent almost surely to ζ and $\{n \in \mathbb{N} : \zeta_n \neq \zeta_n^*\} \in \mathcal{I}$. Then (ζ_n) is $\mathcal{I}$-convergent almost surely to ζ.*

Proof. Let the complex uncertain sequences (ζ_n^*) be convergent almost surely to ζ and $\{n \in \mathbb{N} : \zeta_n \neq \zeta_n^*\} \in \mathcal{I}$.
Then for every $\varepsilon > 0$, $\Big\{n \in \mathbb{N} : \|\zeta_n^*(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\}$ is a finite set and therefore

$$\Big\{n \in \mathbb{N} : \|\zeta_n^*(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\} \in \mathcal{I}.$$

Now $\Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\}$

$$\subseteq \Big\{n \in \mathbb{N} : \|\zeta_n^*(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\} \cup \{n \in \mathbb{N} : \zeta_n \neq \zeta_n^*\} \in \mathcal{I}.$$

Hence the sequence (ζ_n) is $\mathcal{I}$-convergent almost surely to ζ and the proof is complete. □

Definition 3.11. *Let $\zeta, \zeta_1, \zeta_2, \cdots$ be complex uncertain variables defined on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}^*$-convergent almost surely to ζ if there exists a set $A = \{m_1 < m_2 < \cdots < m_n < \cdots\} \subset \mathbb{N}$, $A \in \mathcal{F}(\mathcal{I})$ and an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} \|\zeta_{m_n}(\gamma) - \zeta(\gamma)\| = 0, \ for\ every\ \gamma \in \Lambda.$$

Symbolically we write $\zeta_n \xrightarrow{A_s(\mathcal{I}^*)} \zeta$.

Example 3.12. *Consider the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2, \cdots\}$ *with power set and* $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ *and*

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_n \in \Lambda} \frac{1}{2^n}$$

Also, the complex uncertain variables defined by

$$\zeta_n(\gamma) = \begin{cases} i\beta_n, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases} \quad for\ n = 1, 2, 3, \cdots,$$

where $\beta_n = \begin{cases} n, & if\ n = k^2, k \in \mathbb{N} \\ 1, & otherwise \end{cases} \quad for\ n = 1, 2, 3, \cdots$

and $\zeta \equiv 0$. *Take* $\mathcal{I}$=$\mathcal{I}_d$.
Then there exists a set $A = (\mathbb{N} \setminus B) \in \mathcal{F}(\mathcal{I})$, *where* $B = \{1, 4, 9, \cdots\} \in \mathcal{I}$ *for which*

$$\lim_{n\to\infty} \|\zeta_{m_n}(\gamma) - \zeta(\gamma)\| = 0$$

for every $\gamma \in \Lambda$ *with* $\mathcal{M}\{\Lambda\} = 1$.
Thus the sequence (ζ_n) *is* $\mathcal{I}^*$*-convergent almost surely to* $\zeta \equiv 0$.

Theorem 3.13. *If* $\zeta_n \xrightarrow{A_s(\mathcal{I}^*)} \zeta$, *then* $\zeta_n \xrightarrow{A_s(\mathcal{I})} \zeta$.

Proof. Let us assume that $\zeta_n \xrightarrow{A_s(\mathcal{I}^*)} \zeta$. Then there exists a set $A = \{m_1 < m_2 < \cdots < m_n < \cdots\} \in \mathcal{F}(\mathcal{I})$, and for every $\varepsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ and there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\|\zeta_{m_n}(\gamma) - \zeta(\gamma)\| < \varepsilon \quad \forall n \geq n_0$$

and for every $\gamma \in \Lambda$.
Let $Y = \mathbb{N} \setminus A$. It is clear that $Y \in \mathcal{I}$. Then for any $\varepsilon > 0$,

$$U(\gamma, \varepsilon) = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\} \subseteq Y \cup \{m_1 < m_2 < \cdots < m_{n_0}\} \in \mathcal{I}.$$

Hence $\zeta_n \xrightarrow{A_s(\mathcal{I})} \zeta$ and the proof is complete. □

Remark 3.14. *But the converse of Theorem 3.13 is not true in general.*

Example 3.15. *Let* $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$, *where* $D_j = \{2^{j-1}k : 2$ *does not divide* $k, k \in \mathbb{N}\}$ *be the decomposition of* $\mathbb{N}$ *such that each* D_j *is infinite and* $D_j \cap D_k = \phi$, *for* $j \neq k$. *Let* $\mathcal{I}$ *be the class of all subsets of* $\mathbb{N}$ *that can intersect only finite number of* $D_j{}'s$. *Then* $\mathcal{I}$ *is a nontrivial admissible ideal of* $\mathbb{N}$ *(Kostyrko et al.*[15]*).*
Now we consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2, \cdots\}$ *with power set and* $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ *and*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)}, & if\ \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)}, & if\ \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases} \quad for\ n = 1, 2, 3, \cdots.$$

Also, the complex uncertain variables are defined by

$$\zeta_n(\gamma) = i\beta_n \text{ if } \gamma \in \{\gamma_1, \gamma_2, \cdots\},$$

where $\beta_n = \frac{1}{j}$, *if* $n \in D_j$ *for* $n = 1, 2, 3, \cdots$ *and* $\zeta \equiv 0$.
It is clear that the sequence (ζ_n) *is* $\mathcal{I}$*-convergent almost surely to* $\zeta \equiv 0$. *But this sequence is not* $\mathcal{I}^*$*-convergent almost surely to* $\zeta \equiv 0$. *Because for any set* $H \in \mathcal{I}$ *there exists* $p \in \mathbb{N}$ *such that* $H \subseteq \bigcup_{j=1}^{p} D_j$ *and as a consequence* $D_{p+1} \subseteq \mathbb{N} \setminus H$. *Let* $A = \mathbb{N} \setminus H$, *then* $A \in \mathcal{F}(\mathcal{I})$ *for which we can define a subsequence* ζ_{m_n} *which is not convergent almost surely to* $\zeta \equiv 0$.
Hence the sequence ζ_n *is not* $\mathcal{I}^*$*-convergent almost surely to* $\zeta \equiv 0$.

Theorem 3.16. *Let* (ζ_n) *be a complex uncertain sequence in an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *such that* $\zeta_n \xrightarrow{A_s(\mathcal{I})} \zeta$, *then* $\zeta_n \xrightarrow{A_s(\mathcal{I}^*)} \zeta$ *if* $\mathcal{I}$ *satisfies the condition (AP).*

Proof. Let us assume that $\zeta_n \xrightarrow{A_s(\mathcal{I})} \zeta$. Then there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ and for any $\varepsilon > 0$, the set $U(\gamma, \varepsilon) = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\} \in \mathcal{I}$ for every $\gamma \in \Lambda$. Now we construct a countable family of mutually disjoint sets $\{U_k(\gamma)\}_{k \in \mathbb{N}}$ in $\mathcal{I}$ by considering

$$U_1(\gamma) = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq 1\Big\}$$

and

$$U_k(\gamma) = \Big\{n \in \mathbb{N} : \frac{1}{k} \leq \|\zeta_n(\gamma) - \zeta(\gamma)\| < \frac{1}{k-1}\Big\} = U(\gamma, \frac{1}{k}) \setminus U(\gamma, \frac{1}{k-1})$$

Since $\mathcal{I}$ satisfies the condition (AP), so for the above countable collection $\{U_k(\gamma)\}_{k \in \mathbb{N}}$ there exists another countable family of subsets $\{V_k(\gamma)\}_{k \in \mathbb{N}}$ of $\mathbb{N}$ satisfying $U_j(\gamma) \triangle V_j(\gamma)$ is finite for all $j \in \mathbb{N}$ and $V(\gamma) = \bigcup_{j=1}^{\infty} V_j(\gamma) \in \mathcal{I}$.

Let $\delta > 0$ be arbitary. By Archimedean property we can choose $k \in \mathbb{N}$ such that $\frac{1}{k+1} < \delta$. Then

$$\Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \delta\Big\} \subseteq \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \frac{1}{k+1}\Big\} = \bigcup_{j=1}^{k+1} U_j(\gamma) \in \mathcal{I}.$$

Since $U_j(\gamma) \triangle V_j(\gamma)$ is finite $(j = 1, 2, \cdots, k+1)$ there exists an $n_0 \in \mathbb{N}$, such that

$$\bigcup_{j=1}^{k+1} V_j(\gamma) \cap (n_0, \infty) = \bigcup_{j=1}^{k+1} U_j(\gamma) \cap (n_0, \infty).$$

Choose $n \in \mathbb{N} \setminus V(\gamma) \in \mathcal{F}(\mathcal{I})$ such that $n > n_0$. Therefore we must have $n \notin \bigcup_{j=1}^{k+1} V_j(\gamma)$ and so $n \notin \bigcup_{j=1}^{k+1} U_j(\gamma)$. Then there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ $\|\zeta_n(\gamma) - \zeta(\gamma)\| < \frac{1}{k+1} < \varepsilon$ for every $\gamma \in \Lambda$. Hence $\zeta_n \xrightarrow{A_s(\mathcal{I}^*)} \zeta$. □

Definition 3.17. *Suppose that* $\zeta, \zeta_1, \zeta_2, \cdots$ *are complex uncertain variables defined on the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$. *A complex uncertain sequence* (ζ_n) *is said to be* $\mathcal{I}$*-Cauchy* *sequence almost surely if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta_{n_0}(\gamma)\| \geq \varepsilon\Big\} \in \mathcal{I},\ for\ every\ \gamma \in \Lambda.$$

Theorem 3.18. *If a complex uncertain sequence (ζ_n) is $\mathcal{I}$-convergent almost surely to ζ then it is $\mathcal{I}$-Cauchy sequence almost surely.*

Proof. Let the complex uncertain sequence (ζ_n) be $\mathcal{I}$-convergent almost surely to ζ. Then for every $\varepsilon > 0$ and there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$, we have

$$U(\gamma, \varepsilon) = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\} \in \mathcal{I}\ for\ every\ \gamma \in \Lambda.$$

Clearly, $\mathbb{N} \setminus U(\gamma, \varepsilon) \in \mathcal{F}(\mathcal{I})$ and therefore it is non-empty. Choose $n_0 \in \mathbb{N} \setminus U(\gamma, \varepsilon)$. Then we have

$$\|\zeta_{n_0}(\gamma) - \zeta(\gamma)\| < \varepsilon\ for\ every\ \gamma \in \Lambda.$$

Let $V(\gamma, \varepsilon) = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta_{n_0}(\gamma)\| \geq 2\varepsilon\Big\} \in \mathcal{I}$ for every $\gamma \in \Lambda$. Now we prove that the following inclusion is true $V(\gamma, \varepsilon) \subseteq U(\gamma, \varepsilon)$.
For if $r \in V(\gamma, \varepsilon)$ we have

$$2\varepsilon \leq \|\zeta_r(\gamma) - \zeta_{n_0}(\gamma)\| \leq \|\zeta_r(\gamma) - \zeta(\gamma)\| + \|\zeta_{n_0}(\gamma) - \zeta(\gamma)\| < \|\zeta_r(\gamma) - \zeta(\gamma)\| + \varepsilon,$$

which imples $r \in U(\gamma, \varepsilon)$. Thus we conclude that $V(\gamma, \varepsilon) \in \mathcal{I}$, i.e., (ζ_n) is $\mathcal{I}$-Cauchy sequence almost surely.

□

Definition 3.19. *Let $\zeta, \zeta_1, \zeta_2, \cdots$ be complex uncertain variables defined on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}^*$-Cauchy sequence almost surely if there exists a set $A = \{m_1 < m_2 < \cdots < m_n < \cdots\} \subset \mathbb{N}$, $A \in \mathcal{F}(\mathcal{I})$ and an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{i,j \to \infty} \left\|\zeta_{m_i}(\gamma) - \zeta_{n_j}(\gamma)\right\| = 0,\ for\ every\ \gamma \in \Lambda.$$

Theorem 3.20. *If a complex uncertain sequence (ζ_n) is $\mathcal{I}^*$-Cauchy sequence almost surely then it is $\mathcal{I}$-Cauchy sequence almost surely.*

Proof. Let the complex uncertain sequence (ζ_n) is $\mathcal{I}^*$-Cauchy sequence almost surely. Then there exists a set $A = \{m_1 < m_2 < \cdots < m_n < \cdots\} \subset \mathbb{N}$, $A \in \mathcal{F}(\mathcal{I})$ and there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\|\zeta_{m_n}(\gamma) - \zeta_{m_k}(\gamma)\| < \varepsilon\ \ \forall n, k \geq n_0$$

and for every $\gamma \in \Lambda$.
Let $Y = \mathbb{N} \setminus A$. It is clear that $Y \in \mathcal{I}$. Then for any $\varepsilon > 0$,

$$U(\gamma, \varepsilon) = \Big\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta_{n_0}(\gamma)\| \geq \varepsilon\Big\} \subseteq Y \cup \{m_1 < m_2 < \cdots < m_{n_0}\} \in \mathcal{I}.$$

Hence the sequence (ζ_n) is $\mathcal{I}$-Cauchy sequence almost surely and the proof is complete. □

Remark 3.21. *But the converse of Theorem 3.20 is not true in general.*

Example 3.22. *Let* $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$*, where* $D_j = \{2^{j-1}k : 2 \text{ does not divide } k, k \in \mathbb{N}\}$ *be the decomposition of* $\mathbb{N}$ *such that each* D_j *is infinite and* $D_j \cap D_k = \phi$*, for* $j \neq k$*. Let* $\mathcal{I}$ *be the class of all subsets of* $\mathbb{N}$ *that can intersect only finite number of* $D_j{'}s$*. Then* $\mathcal{I}$ *is a nontrivial admissible ideal of* $\mathbb{N}$ *(Kostyrko et al.*[15]*).*
Now we consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2, \cdots\}$ *with power set and* $\mathcal{M}\{\Gamma\} = 1$*,* $\mathcal{M}\{\phi\} = 0$ *and*

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_n \in \Lambda} \frac{1}{2^n} \; for \; n = 1, 2, 3, \cdots .$$

Also, the complex uncertain variables are defined by

$$\zeta_n(\gamma) = i\beta_n \; if \; \gamma \in \{\gamma_1, \gamma_2, \cdots\},$$

where $\beta_n = \frac{1}{j+1}$*,* $if \; n \in D_j \; for \; n = 1, 2, 3, \cdots$ *and* $\zeta \equiv 0$*.*

It is clear that the sequence (ζ_n) *is* $\mathcal{I}$*-convergent almost surely to* $\zeta \equiv 0$*.*
By the theorem 3.18 the sequence (ζ_n) *is* $\mathcal{I}$*-Cauchy sequence almost surely.*

Next, we shall show that the complex uncertain sequence (ζ_n) *is not* $\mathcal{I}^*$*-Cauchy sequence almost surely. For this, if possible assume that the sequence* (ζ_n) *is* $\mathcal{I}^*$*-Cauchy sequence almost surely. Then* $\exists$ *a set* $A = \{m_1 < m_2 < \cdots < m_n < \cdots\} \in \mathcal{F}(\mathcal{I})$ *and for every* $\varepsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ *and there exists an event* Λ *with* $\mathcal{M}\{\Lambda\} = 1$ *such that*

$$\|\zeta_{m_n}(\gamma) - \zeta_{m_k}(\gamma)\| < \varepsilon \quad \forall n, k \geq n_0 \tag{1}$$

and for every $\gamma \in \Lambda$*.*
Since $\mathbb{N} \setminus A \in \mathcal{I}$ *so there exists a* $r \in \mathbb{N}$ *such that* $\mathbb{N} \setminus A \subset D_1 \cup D_2 \cup \cdots \cup D_r$*. But* $D_i \subset A$ $\forall i > r$*. In particular* D_{r+1}*,* $D_{r+2} \subset A$*. We see that from the construction of* $D_j's$*, for given any* $n_0 \in \mathbb{N}$ *there are* $m_n \in D_{r+1}$ *and* $m_k \in D_{r+2}$ *such that* $m_n, m_k \geq n_0$*. Therefore*

$$\|\zeta_{m_n}(\gamma) - \zeta_{m_k}(\gamma)\| = \left\| \frac{i}{r+1} - \frac{i}{r+2} \right\| = \frac{1}{(r+1)(r+2)}$$

If we take $\varepsilon = \frac{1}{3(r+1)(r+2)}$*, then there is no* $n_0 \in \mathbb{N}$ *whenever* $m_n, m_k \in M$ *with* $m_n, m_k \geq n_0$ *such that the equation* (1) *is holds. This is a contradiction so our assumption was wrong and hence* (ζ_n) *is not* $\mathcal{I}^*$*-Cauchy sequence almost surely.*

Theorem 3.23. *Let* (ζ_n) *be a complex uncertain sequence in an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *such that* (ζ_n) *is* $\mathcal{I}$*-Cauchy sequence almost surely, then* (ζ_n) *is* $\mathcal{I}^*$*-Cauchy sequence almost surely if* $\mathcal{I}$ *satisfies the condition (AP).*

Proof. Let (ζ_n) be an $\mathcal{I}$-Cauchy sequence almost surely. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ and there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$U(\varepsilon, \gamma) = \Big\{ n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta_{n_0}(\gamma)\| \geq \varepsilon \Big\} \in \mathcal{I} \; for \; every \; \gamma \in \Lambda.$$

Inperticular, for $\varepsilon = \frac{1}{i}$, $i \in \mathbb{N}$ we have $V_i(\gamma) = \Big\{ n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta_{n_0}(\gamma)\| < \frac{1}{i} \Big\}$ for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$.
Since $\mathcal{I}$ satisfies the condition (AP), then by Lemma (4) of [19], there exists a set $A =$

$\{m_1 < m_2 < \cdots < m_n < \cdots\} \subset \mathbb{N}$ such that $A \in \mathcal{F}(\mathcal{I})$ and $A \setminus V_i$ is finite for all $i \in \mathbb{N}$. By Archimedean property, we choose $j_0 \in \mathbb{N}$ such that $\frac{2}{j_0} < \varepsilon$. Then $A \setminus V_{j_0}$ is a finite set, so there exists $n_0 \in \mathbb{N}$ such that $m_n, m_k \in V_{j_0}$ for all $n, k \geq n_0$, i.e., $\|\zeta_{m_n}(\gamma) - \zeta_{n_0}(\gamma)\| < \frac{1}{j_0}$ and $\|\zeta_{m_k}(\gamma) - \zeta_{n_0}(\gamma)\| < \frac{1}{j_0}$ for all $n, k \geq n_0$ and for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$.

$$\begin{aligned}\text{Now } \|\zeta_{m_n}(\gamma) - \zeta_{m_k}(\gamma)\| &= \|\zeta_{m_n}(\gamma) - \zeta_{n_0}(\gamma) - \zeta_{m_k}(\gamma) + \zeta_{n_0}(\gamma)\| \\ &\leq \|\zeta_{m_n}(\gamma) - \zeta_{n_0}(\gamma)\| + \|\zeta_{m_k}(\gamma) - \zeta_{n_0}(\gamma)\| \\ &< \frac{1}{j_0} + \frac{1}{j_0} = \frac{2}{j_0} < \varepsilon \ \forall n, k \geq n_0\end{aligned}$$

and for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$.
Hence, the complex uncertain sequence is $\mathcal{I}^*$-Cauchy sequence almost surely. □

4. Conclusion

The main contribution of this paper is to provide the notion of $\mathcal{I}^*$-convergent almost surely of complex uncertain sequence and study some of its properties and identify the relationships between $\mathcal{I}$ and $\mathcal{I}^*$-convergent almost surely of complex uncertain sequence. Also, we define $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequence almost surely and study the relationship between them. These ideas and results are expected to be a source for researchers in the area of convergence of complex uncertain sequences. Also, these concepts can be generalized and applied for further studies.

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