FRICTIONAL CONTACT PROBLEMS INVOLVING P(X)-LAPLACIAN-LIKE OPERATORS

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ABSTRACT. This article is dedicated to studying a class of frictional contact problems involving the p(x)-Laplacian-like operator, on a bounded domain $\Omega \subseteq \mathbb{R}^2$. Using an abstract Lagrange multiplier technique and the Schauder fixed point theorem we establish the existence of a weak solution. Furthermore, we also obtain the uniqueness of the solution assuming that the datum f_1 satisfies a suitable monotonicity condition. The results here extend earlier theorems due to Cojocaru- Matei to the quasilinear case, with semilinearity f_1.

Keywords: $p(x)$- Kirchhoff type equation; weak solutions; existence and uniqueness of solutions; variable exponents; frictional contact condition; Schauder fixed point theorem.

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1. Introduction

In this paper we discuss the existence of weak solutions for the following nonlinear elliptic problem for the $p(x)$-Laplacian-like operator originated from a capillary phenomena.
Problem 1. Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$- M\Big(L(u)\Big)\Big[\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}})\Big] = f_1(x,u) \quad \text{in} \quad \Omega, \tag{1}$$

$$u = 0 \qquad\qquad \text{on} \quad \Gamma_1, \tag{2}$$

$$M(L(u))\left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\frac{\partial u}{\partial \nu} = f_2(x) \quad \text{on} \quad \Gamma_2, \tag{3}$$

$$\Big|M(L(u))\left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\frac{\partial u}{\partial \nu}\Big| \le g(x), \tag{4}$$

$$M((L(u))\left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\frac{\partial u}{\partial \nu} = -g\frac{u}{|u|}, \quad \text{if} \quad u \neq 0 \quad \text{on} \quad \Gamma_3 \tag{5}$$

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where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with smooth enough boundary Γ, partitioned in three parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that meas $(\Gamma_i) > 0$, $(i = 1, 2, 3)$; $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$, $f_2 : \Gamma_2 \to \mathbb{R}$, $g : \Gamma_3 \to \mathbb{R}$ and $M : [0, +\infty[\to [m_0, +\infty[$ are given functions, $p \in C(\overline{\Omega})$ and $L(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}}}{p(x)}\, dx$.

The study of the $p(x)$- Kirchhoff type equations with nonlinear boundary conditions of different class have been a very interesting topic in the recent years. Let us just quote[2, 5, 6, 12, 20, 28] and references therein. One reason of such interest is due to their frequent appearance in applications such as the modeling of electrorheological fluids [25], image restoration [13], elastic mechanics [29] and continuum mechanics [4]. The other reason is that the nonlocal problems with variable exponent, in addition to their contributions to the modelization of many physical and biological phenomena, are very interesting from a purely mathematical point of view as well; we refer the reader to [1, 3, 11, 22, 23, 27]. Cojocaru-Matei [9] studied the unique solvability of problem (4) in the case $M(s) = 1, f_1(x, u) \equiv f_1(x)$, without the term $\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}$, $p =$ constant ≥ 2, which models the antiplane shear deformation of a nonlinearly elastic cylindrical body in frictional contact on Γ_3 with a rigid foundation; see, e.g. [26]. They used a technique involving dual Lagrange multipliers, which allows to write efficient algorithms to approximate the weak solutions; see [21]. For this situation, the behavior of the material is described by the Hencky-type constitutive law:

$$\sigma(x) = ktr\varepsilon(u(x))I_3 + \mu(x)\|\varepsilon^D(u(x))\|^{\frac{p(x)-2}{2}}\varepsilon^D(u(x))$$

where σ is the Cauchy stress tensor, tr is the trace of a Cartesian tensor of second order, ε is the infinitesimal strain tensor, u is the displacement vector, I_3 is the identity tensor, k, μ are material parameters, p is a given function; ε^D is the *deviator* of the tensor ε defined by $\varepsilon^D = \varepsilon - \frac{1}{3}(tr\varepsilon)I_3$ where $tr\varepsilon = \sum_{i=1}^{3} \varepsilon_{ii}$; see for instance [19].

Inspired by the above works, we study the existence of weak solutions for Problem 1, under appropriate assumptions on M and f_1, via Lagrange multipliers and the Schauder fixed point theorem. In this sense, we extend and generalize the result the main result in [9]. Also, we state a simple uniqueness result under suitable monotonicity condition on f_1.

The paper is designed as follows. In Section 2, we introduce the mathematical preliminaries and give several important properties of $p(x)$-Laplacian-like operator. We deliver a weak variational formulation with Lagrange multipliers in a dual space. Section 3, is devoted to the proofs of main results.

2. Preliminaries

For the reader's convenience, we point out some basic results on the theory of Lebesgue-Sobolev spaces with variable exponent. In this context we refer the reader to [15, 25] for details. Firstly we state some basic properties of spaces $W^{1,p(x)}(\Omega)$ which will be used later. Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1 ([18], Theorem 1.3)**.** *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces.*

Proposition 2.2 ([18], Theorem 1.4)**.** *Set $\rho(u) = \int_\Omega \Big|u(x)\Big|^{p(x)}\,dx$. For any $u \in L^{p(x)}(\Omega)$, then*

(1) *for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
(2) *$|u|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$;*
(3) *if $|u|_{p(x)} > 1$, then $|u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$;*
(4) *if $|u|_{p(x)} < 1$, then $|u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$;*
(5) *$\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ if and only if $\lim_{k\to+\infty} \rho(u_k) = 0$;*
(6) *$\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(u_k) = +\infty$.*

Proposition 2.3 ([16, 18], Theorem 1.2, Theorem 2.3)**.** *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.4 ([18], Theorem 1.15)**.** *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ holds a.e. in Ω. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\Big|\int_\Omega uv\,dx\Big| \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

We introduce the following closed space of $W^{1,p(x)}(\Omega)$

$$X = \{v \in W^{1,p(x)}(\Omega) : \gamma u = 0 \quad \text{a. e. on} \quad \Gamma_1\} \tag{6}$$

where γ denotes the Sobolev trace operator and $\Gamma_1 \subseteq \Gamma$, meas $(\Gamma_1) > 0$, therefore X is a separable reflexive Banach space. Now, we denote

$$\|u\|_X = |\nabla u|_{p(x)}, \qquad u \in X.$$

This functional represents a norm on X.

Proposition 2.5 ([7], Theorem 2.5)**.** *There exists $c > 0$ such that*

$$\|u\|_{1,p(x)} \le C\|u\|_X \quad \text{for all } u \in X.$$

Then, the norms $\|.\|_X$ and $\|.\|_{1,p(x)}$ are equivalent on X.
The derivative operator of L in weak sense $L' : X \to X'$ is

$$\langle L'u, v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}} \right).\nabla v \, dx, \ \forall u, v \in X. \tag{7}$$

Proposition 2.6. *The functional $L : X \to \mathbb{R}$ is convex. The mapping $L' : X \to X'$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type, namely*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty} L'(u_n)(u_n - u) \le 0 \text{ implies } u_n \to u,$$

where X' is the dual space of X.

Proof. This result is obtained in a similar manner as the one given in [24], Proposition 3.1. For the reader's convenience we sketch briefly the proof that L' is of (S_+) type. Let (u_ν) be a sequence of X such that $u_\nu \rightharpoonup u$ in X. By the strict monotonicity of L' we get

$$0 = \limsup_{\nu\to\infty} \langle L'u_\nu - L'u, u_\nu - u\rangle = \lim_{\nu\to\infty} \langle L'u_\nu - L'u, u_\nu - u\rangle,$$

thus $\lim_{\nu\to\infty}\langle L'u_\nu, u_\nu - u\rangle = 0$. Hence

$$\lim_{\nu\to\infty} \int_\Omega \Big(|\nabla u_\nu|^{p(x)-2}\nabla u_\nu + \frac{|\nabla u_\nu|^{2p(x)-2}\nabla u_\nu}{\sqrt{1+|\nabla u_\nu|^{2p(x)}}} \Big)(\nabla u_\nu - \nabla u)\, dx = 0. \tag{8}$$

But, using estimation

$$\int_\Omega \Big(|\nabla u_\nu|^{p(x)-2}\nabla u_\nu + \frac{|\nabla u_\nu|^{2p(x)-2}\nabla u_\nu}{\sqrt{1+|\nabla u_\nu|^{2p(x)}}} \Big)(\nabla u_\nu - \nabla u)\, dx \ge \int_\Omega \frac{1}{p(x)} |\nabla u_\nu|^{p(x)}\, dx$$
$$- \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + C\Big(\int_\Omega \frac{1}{p(x)} \sqrt{1+|\nabla u_\nu|^{2p(x)}}\, dx - \int_\Omega \frac{1}{p(x)} \sqrt{1+|\nabla u|^{2p(x)}}\, dx \Big)$$

we obtain, by (8), that

$$\lim_{\nu\to\infty} \int_\Omega \left(\frac{1}{p(x)} |\nabla u_\nu|^{p(x)} + C \int_\Omega \frac{1}{p(x)} \sqrt{1+|\nabla u_\nu|^{2p(x)}}\, dx \right)$$
$$= \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + C \int_\Omega \frac{1}{p(x)} \sqrt{1+|\nabla u|^{2p(x)}}\, dx \right).$$

So, the integrals of the family

$$\left\{ \frac{1}{p(x)} |\nabla u_\nu - \nabla u|^{p(x)} + C \frac{1}{p(x)} \left| \sqrt{1+|\nabla u_\nu|} - \sqrt{1+|\nabla u|} \right|^{2p(x)} \right\}$$

are absolutely equicontinuous on Ω. Consequently

$$\lim_{\nu\to\infty} \int_\Omega \Big(\frac{1}{p(x)} |\nabla u_\nu - \nabla u|^{p(x)}$$
$$+ C \frac{1}{p(x)} |\sqrt{1+|\nabla u_\nu|} - \sqrt{1+|\nabla u|}|^{2p(x)} \Big)\, dx = 0.$$

Therefore

$$\lim_{\nu\to\infty} \int_\Omega \frac{1}{p(x)} |\nabla u_\nu - \nabla u|^{p(x)}\, dx = 0$$

and

$$\lim_{\nu\to\infty} \int_\Omega \frac{1}{p(x)} |\sqrt{1+|\nabla u_\nu|} - \sqrt{1+|\nabla u|}|^{2p(x)}\, dx = 0.$$

Then, we conclude, from the last two equalities and Proposition 2.5 , that $u_\nu \to u$ in X as $\nu \to +\infty$. □

Now, we define the spaces

$$S = \Big\{ u \in W^{\frac{1}{p'(x)}, p(x)}(\Gamma) : \exists v \in X \quad \text{such that} \quad u = \gamma v \quad \text{a.e on} \quad \Gamma \Big\} \tag{9}$$

which is a real reflexive Banach space, $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ for all $x \in \Omega$, and

$$Y = S', \text{ the dual of the space S.} \tag{10}$$

Let us introduce a bilinear form

$$b : X \times Y \longrightarrow \mathbb{R} \quad : b(v, \mu) = \langle\ \mu, \gamma v\ \rangle_{Y \times S}, \tag{11}$$

a Lagrange multiplier $\lambda \in Y$,

$$\langle\ \lambda, z\ \rangle = -\int_{\Gamma_3} M\left(L(u)\right) \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) \frac{\partial u}{\partial \nu} z d\Gamma \quad , \quad \forall z \in S$$

and the set of Lagrange multipliers

$$\Lambda = \Big\{ u \in Y : \langle\ \mu, z\ \rangle \leqslant \int_{\Gamma_3} g(x)|z(x)| \quad , \quad \forall z \in S \Big\}. \tag{12}$$

From (4) we deduce that $\lambda \in \Lambda$.

Let u be a regular enough function satisfying Problem 1. After some computations we get (by using density results)

$$M\left(L(u)\right) \int_\Omega \left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2} \nabla u \right) . \nabla v dx = \int_\Omega f_1(x, u) v dx$$

$$+ \int_{\Gamma_2} f_2(x) \gamma v d\Gamma + M\left(L(u)\right) \int_{\Gamma_3} \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) \frac{\partial u}{\partial \nu} \gamma v d\Gamma \tag{13}$$

for all $v \in X$, where u satisfies (5) on Γ_3.

Now, we write problem (13) as an abstract mixed variational problem (by means a Lagrange multipliers technique)

We define the following operators:

$i)$ $A : X \to X'$, given by

$$\langle\ Au, v\ \rangle = M\left(L(u)\right) \int_\Omega \left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2} \nabla u \right) . \nabla v\, dx, \ u, v \in X. \tag{14}$$

$ii)$ $F : X \to X'$, given by

$$\langle\ F(u), v\ \rangle = \int_\Omega f_1(x, u) v dx + \int_{\Gamma_2} f_2(x) \gamma v dx \quad , \quad u, v \in X.$$

So, we are led to the following variational formulation of Problem 1.

Problem 1'. Find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} \langle\ Au, v\ \rangle + b(v, \lambda) \quad &= \quad \langle\ F(u), v\ \rangle \quad , \quad \forall v \in X \\ b(u, \mu - \lambda) \quad &\leq \quad 0 \quad \forall \mu \in \Lambda \subseteq Y. \end{aligned} \tag{15}$$

To solve this problem, we will apply the Schauder fixed point theorem.

Firstly, we "freeze" the state variable u on the function F, that is we fix $w \in X$ such that $f = F(w) \in X'$.

Hence, we arrive at the following abstract mixed variational problem.

Problem 2. Given $f \in X'$ find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} \langle\, Au, v \,\rangle + b(v,\lambda) &= \langle\, f, v \,\rangle \quad , \quad \forall v \in X \\ b(u, \mu - \lambda) &\leq 0 \quad \forall \mu \in \Lambda \subseteq Y. \end{aligned} \tag{16}$$

The unique solvability of Problem 2 is given under the following generalized assumptions. Let $(X, \|\|_X)$ and $(Y, \|\|_Y)$ be two real reflexive Banach space.

(B_1): $A : X \to X'$ is hemicontinuous;
(B_2): $\exists h : X \to \mathbb{R}$ such that
 (a) $h(tw) = t^\gamma h(w)$ with $\gamma > 1$, $\forall t > 0, w \in X$;
 (b) $\langle\, Au - Av, u - v \,\rangle_{X \times X} \geq h(v - u),\ \forall u, v \in X$;
 (c) $\forall (x_\nu) \subseteq X : x_\nu \rightharpoonup x$ in$X \Longrightarrow h(x) \leq \lim_{\nu\to\infty} \sup h(x_\nu)$.
(B_3): A is coercive.
(B_4): The form $b : X \times Y$ is bilinear, and
 (i) $\forall (u_\nu) \subseteq X : u_\nu \rightharpoonup u$ in $X \Longrightarrow b(u_\nu, \mu) \to b(u, \mu)$, for all $\mu \in \Lambda$.
 (ii) $\forall (\lambda_\nu) \subseteq Y : \lambda_\nu \rightharpoonup y$ in $Y \Longrightarrow b(v, \lambda_\nu) \to b(v, \lambda)$, for all $v \in X$.
 (iii) $\exists\, \widehat{\alpha} > 0 : \inf_{\substack{\mu \in I \\ u \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \dfrac{b(v,\mu)}{|v|_X |\mu|_Y} \geq \widehat{\alpha}$.
(B_5): Λ is a bounded closed convex subset of Y such that $0_Y \in \Lambda$.
(B_6): $\exists C_1 > 0, q > 0 : h(v) \geq C_1 \|v\|_X^q \quad , \forall v \in X$.

Theorem 2.1. *Assume (B_1) - (B_6). Then there exists a unique solution $(u, \lambda) \in X \times \Lambda$ of Problem 2.*

Proof. See [9], Theorem 1. □

To solve Problem 1', we start by stating the following assumptions on M , f_1 , f_2 and g

(A_1) $M : [0, +\infty[\to [m_0, +\infty[$ is a locally Lipschitz-continuous and nondecreasing function; $m_0 > 0$.
(A_2) $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying

$$|f_1(x,t)| \leq c_1 + c_2 |t|^{\alpha(x)-1} \ ,\ \forall (x,t) \in \Omega \times \mathbb{R},$$

$\alpha \in C_+(\overline{\Omega})$with $\alpha(x) < p^*(x),\ \alpha^+ < p^-$.
(A_3) $f_2 \in L^{p'(x)}(\Gamma_2),\ g \in L^{p'(x)}(\Gamma_3),\ g(x) \geq 0$ a.e on Γ_3.

We have the following properties about the operator A.

Proposition 2.7. *If (A_1) holds, then*

(i) *A is locally Lipschitz continuous.*
(ii) *A is bounded, strictly monotone. Furthermore*

$$\langle Au - Av, u - v \rangle \geq k_p \|u - v\|_X^{\hat{p}}$$

where

$$\hat{p} = \begin{cases} p^- & \textit{if } \|u - v\|_X > 1, \\ p^+ & \textit{if } \|u - v\|_X \leq 1. \end{cases}$$

So, we can take $h(v) = k_p \|v\|_X^{\hat{p}}$.
(iii) $\frac{\langle Au, u \rangle}{\|u\|_X} \to +\infty \quad$ *as* $\|u\|_X \to +\infty$.

Proof. (i) Assume that M is Lipschitz in $[0, R_1]$ with Lipschitz constant L_M, $R_1 > 0$. We have, for $u, v, w \in B(0, R_1)$

$$\begin{aligned}\langle Au - Av, w\rangle =& [M(L(u)) - M(L(v))] \int_\Omega \left[\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}\right) |\nabla u|^{p(x)-2}\nabla u\right].\nabla w\, dx \\ &+ M(L(v)) \int_\Omega \left[\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}\right) |\nabla u|^{p(x)-2}\nabla u \right. \\ &\left. - \left(1 + \frac{|\nabla v|^{p(x)}}{\sqrt{1 + |\nabla v|^{2p(x)}}}\right) |\nabla v|^{p(x)-2}\nabla v\right].\nabla w\, dx.\end{aligned}$$

Using the Lipschitz continuity of M, the Holder inequality and observing that $k(t) = \left(1 + \frac{t^p}{\sqrt{1+t^{2p}}}\right) t^{p-2}$ satisfies conditions i)-iii) of Lemma 1 in [10] , so there exist constants $K_1 \geq 0, K_2 > 0$ and $\gamma > 0$ such that

$$|k(|z|)z - k(|y|)y| \leq \gamma|z - y| \left[K_1 + K_2(|z| + |y|)\right]^{p-2} \quad \text{if } 2 \leq p < \infty, \forall y, z \in \mathbb{R}^n,$$

we get

$$|\langle Au - Av, w\rangle| \leq C\|u - v\|_X \|w\|_X,$$

which implies $\|Au - Av\|_{X'} \leq C\|u - v\|_X$.

ii)The functional $S \equiv L' : X \to X'$ defined by

$$\langle Su, v\rangle = \int_\Omega \left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}\right) |\nabla u|^{p(x)-2}\nabla u\right).\nabla v\, dx \qquad \forall u, v \in X, \tag{17}$$

is bounded (see Proposition 2.6). Hence, since M is continuous and L is bounded, A is bounded.

To obtain that A is strictly monotone, we observe that L' is strictly monotone.Hence, L is strictly convex. Moreover, since M is nondecreasing, $\hat{M}(t) = \int_0^t M(\tau)\, d\tau$ is convex in $[0, +\infty[$. Consequently, for all s,t $\in]0, 1[$ with $s + t = 1$ one has

$$\hat{M}(L(su + tv)) < \hat{M}(sL(u) + tL(v)) \leq s\hat{M}(L(u)) + t\hat{M}(L(v)), \forall u, v \in X, u \neq v.$$

This shows $\Psi(u) = \hat{M}(L(u))$ is strictly convex, then $\Psi'(u) = M(L(u))L'(u)$ is strictly monotone, which means that A is strictly monotone.

To establish the inequality in ii), we apply Lemma 3 in [8] to obtain

$$\begin{aligned}\langle Au - Av, u - v\rangle \geq& \int_\Omega \left\{ M(L(u)) \left[\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}\right) |\nabla u|^{p(x)-2}\nabla u\right]\right. \\ &\left. - M(L(v)) \left[\left(1 + \frac{|\nabla v|^{p(x)}}{\sqrt{1 + |\nabla v|^{2p(x)}}}\right) |\nabla v|^{p(x)-2}\nabla v\right]\right\}.(\nabla v - \nabla u)\, dx \\ \geq& m_0 \int_\Omega \frac{1}{p(x)} (|\nabla u - \nabla u|^{p(x)})\, dx \geq \frac{m_0}{p^+} \int_\Omega |\nabla u - \nabla u|^{p(x)}\, dx \\ \geq& \frac{m_0}{p^+} \|u - v\|_X^{\hat{p}}.\end{aligned}$$

iii)For $u \in X$ with $\|u\|_X > 1$ we have

$$\frac{\langle Au, u\rangle}{\|u\|_X} = \frac{M(L(u)) \int_\Omega \left[\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)} \right] dx}{\|u\|}$$
$$\geq m_0 \|u\|_X^{p^- -1} \to +\infty \text{ as } \|u\|_X \to +\infty.$$

□

Proposition 2.8. *The form* $b : X \times Y \to \mathbb{R}$ *defined in* (11) *is bilinear and, it verifies i), ii) and iii) in assumption* (B_4). *Moreover*

$$b(u,\mu) \leq \int_{\Gamma_3} g(x)|u(x)|\, d\Gamma \text{ for all } \mu \in \Lambda, \tag{18}$$

$$b(u,\lambda) = \int_{\Gamma_3} g(x)|u(x)|\, d\Gamma, \tag{19}$$

$$b(u, \mu - \lambda) \leq 0 \quad \text{for all } \mu \in \Lambda. \tag{20}$$

Moreover, Λ *is bounded.*

Proof. The assertions i), ii), iii) and Λ bounded are similarly as [9], Theorem 3, pages 138-139.

It is obvious to check (18). To justify (19), we have to show that, a.e. $x \in \Omega$

$$-M((L(u)) \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}} \right) \frac{\partial u(x)}{\partial \nu} u(x) = g(x)|u(x)|$$

In fact, let $x \in \Omega$. If $|u(x)| = 0$, then

$$-M((L(u)) \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}} \right) \frac{\partial u(x)}{\partial \nu} u(x) = 0 = g(x)|u(x)| \text{ on } \Gamma_3.$$

Otherwise, if $|u(x)| \neq 0$,then

$$-M((L(u)) \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}} \right) \frac{\partial u(x)}{\partial \nu} u(x) = g(x) \frac{(u(x))^2}{|u(x)|}$$
$$= g(x)|u(x)| \text{ on } \Gamma_3.$$

Furthermore, for all $\mu \in \Lambda$:

$$b(u, \mu - \lambda) = b(u,\mu) - b(u,\lambda) = \langle\ \mu, \gamma u\ \rangle_{Y\times S} - \langle\ \lambda, \gamma u\ \rangle_{Y\times S}\,. \tag{21}$$

Hence, thanks to (18), (19) and (21), we obtain (20). □

3. Existence and uniqueness of solutions

We are ready to solve Problem 1' (then Problem 1). For this, we consider the Banach spaces X and Y given in (6) and (10) respectively, the bilinear form b in (11) and the set Λ in (12).

Theorem 3.1. *Suppose* $(A_1) - (A_3)$ *hold. Then Problem 1' admits a solution* $(u, \lambda) \in X \times \Lambda$.

Proof. We apply the Schauder fixed point theorem.

As has been said before, we "freeze" the state variable u on the function F , that is, we fix $w \in X$ and consider the problem:

Find $u \in X$ and $\lambda \in \Lambda$ such that

$$\langle\ Au, v\ \rangle + b(v, \lambda) \quad = \quad \langle\ f, v\ \rangle \quad , \quad \forall v \in X, \tag{22}$$

$$b(u, \mu - \lambda) \quad \leq \quad 0 \quad \forall \mu \in \Lambda \subseteq Y \tag{23}$$

with $f = F(w) \in X'$. Note that by the hypotheses on α and f_1, given in (A_2), we have $f_1(w) \in L^{\alpha'(x)}(\Omega) \hookrightarrow X'$.

By Theorem 2.1, problem (22)-(23) has a unique solution $(u_w, \lambda_w) \in X \times \Lambda$.

Here we drop the subscript w for simplicity. Setting $v = u$ in (22) and $\mu = 0_Y$ in (23), using proposition 2.7 ii), we get

$$k_p \|u\|_X^{\hat{p}} \leq (2C_1 C_\alpha \|w\|_X^\sigma + 2C_2 C_\alpha |\Omega| + c_p |f_2|_{p'(x), \Gamma_2}) \|u\|_X \tag{24}$$

where

$$\sigma = \begin{cases} \alpha^- & \text{if } \|w\|_X > 1, \\ \alpha^+ & \text{if } \|w\|_X \leq 1, \end{cases}$$

and C_χ is the embedding constant of $X \hookrightarrow L^{\chi(x)}(\Omega)$.

Then

$$\|u\|_X \leq [C(1 + \|w\|_X)]^{\frac{1}{\hat{p}-1}}.$$

Therefore, either $\|u\|_X \leq 1$ or

$$\|u\|_X \leq [C(1 + \|w\|_X)]^{\frac{1}{p^- - 1}}. \tag{25}$$

Since $p^- > \alpha^+ + 1$, we have

$$t^{p^- - 1} - Ct^\sigma - C \to +\infty \qquad \text{as } t \to +\infty$$

Hence, there is some $\bar{R}_1 > 0$ such that

$$\bar{R_1}^{p^- - 1} - C\bar{R_1}^\sigma - C \geq 0. \tag{26}$$

From (25) and (26) we infer that if $\|w\|_X \leq \bar{R}_1$ then $\|u\|_X \leq \bar{R}_1$.

Thus there exists $R_1 = \min\{1, \bar{R}_1\}$ such that

$$\|u\|_X \leq R_1 \quad \text{for all } u \in X. \tag{27}$$

For this constant, define K as

$$K = \{v : v \in L^{\alpha(x)}(\Omega), \|v\|_X \leq R_1\}$$

which is a nonempty, closed, convex subset of $L^{\alpha(x)}(\Omega)$. We can define the operator

$$T : K \to L^{\alpha(x)}(\Omega), \qquad Tw = u_w$$

where u_w is the first component of the unique pair solution of the problem (22)-(23), $(u_w, \lambda_w) \in X \times \Lambda$.

From (27) $\|Tw\|_X \leq R_1$, for every $w \in K$, so that $T(K) \subseteq K$.

Moreover, if $(u_\nu)_{\nu \geq 1}$ $(u_{w_\nu} \equiv u_\nu)$ is a bounded sequence in K, then from (27) is also bounded in X. Consequently, from the compact embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, $(Tw_\nu)_{\nu \geq 1}$ is relatively compact in $L^{\alpha(x)}(\Omega)$ and hence, in K.

To prove the continuity of T , let $(w_\nu)_{\nu \geq 1}$ be a sequence in K such that

$$w_\nu \to w \quad \text{strongly in } L^{\alpha(x)}(\Omega) \tag{28}$$

and suppose $u_\nu = Tw_\nu$. The sequence $\{(u_\nu, \lambda_\nu)\}_{\nu\geq 1}$ satisfies

$$\begin{aligned} \langle\ Au_\nu, v\ \rangle + b(v, \lambda_\nu) &= \langle\ F(w_\nu), v\ \rangle \quad , \quad \forall v \in X \\ b(u_\nu, \mu - \lambda_\nu) &\leq 0 \quad \forall \mu \in \Lambda. \end{aligned}$$

Using (27)-(28) we can extract a subsequence (u_{ν_k}) of (u_ν) and a subsequence (w_{ν_k}) of (w_ν) such that

$$\begin{gathered} u_{\nu_k} \to u^* \text{weakly in} X, \\ u_{\nu_k} \to u^* \text{ strongly in } L^{\alpha(x)}(\Omega) \text{ and a.e. in } \Omega, \\ w_{\nu_k} \to w \quad \text{a.e. in } \Omega, \\ L(u_{\nu_k}) \to t_0, \quad \text{for some } t_0 \geq 0, \end{gathered} \tag{29}$$

and in view of continuity of M

$$M(L(u_{\nu_k})) \to M(t_0). \tag{30}$$

We shall show that $u^* = Tw$. To this end, by choosing $u_{\nu_k} - u^*$ as a test function, we have

$$\begin{gathered} \langle\ Au_{\nu_k}, u_{\nu_k} - u^*\ \rangle + b(u_{\nu_k} - u^*, \lambda_\nu) = \langle\ F(w_{\nu_k}), u_{\nu_k} - u^*,\ \rangle \\ \langle\ Au^*, u_{\nu_k} - u^*\ \rangle + b(u_{\nu_k} - u^*, \lambda^*) = \langle\ F(w), u_{\nu_k} - u^*\ \rangle\,. \end{gathered} \tag{31}$$

Then

$$\begin{aligned} &[M(L(u^*) - M(L(u_{\nu_k})] \int_\Omega \left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) |\nabla u^*|^{p(x)-2} \nabla u^*.(\nabla u_{\nu_k} - \nabla u^*)\, dx + \\ &M(L(u_{\nu_k})) \int_\Omega \left[\left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) |\nabla u^*|^{p(x)-2} \nabla u^* - \left(1 + \frac{|\nabla u_{\nu_k}|^{p(x)}}{\sqrt{1+|\nabla u_{\nu_k}|^{2p(x)}}}\right) \right. \\ &\left. |\nabla u_{\nu_k}|^{p(x)-2} \nabla u_{\nu_k} \right] .(\nabla u_{\nu_k} - \nabla u^*)\, dx + b(u_{\nu_k} - u^*, \lambda^* - \lambda_{\nu_k}) \\ &\qquad = \langle\ F(w) - F(w_{\nu_k}), u_{\nu_k} - u^*\ \rangle\,. \end{aligned} \tag{32}$$

Since $b(u_{\nu_k} - u^*, \lambda^* - \lambda_{\nu_k}) \geq 0$, again by the inequality of Lemma 3 in [8], $p \geq 2$, we obtain

$$\begin{aligned} &m_0 C_p \int_\Omega |\nabla u_{\nu_k} - \nabla u^*|^{p(x)}\, dx + [M(L(u^*) - M(L(u_{\nu_k})] \int_\Omega \left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) \\ &|\nabla u^*|^{p(x)-2} \nabla u^*.(\nabla u_{\nu_k} - \nabla u^*)\, dx \leq |\,\langle\ F(w_{\nu_k}) - F(w), u_{\nu_k} - u^*\ \rangle\,|. \end{aligned} \tag{33}$$

But, using (29) we get

$$\begin{aligned} &|[M(L(u^*) - M(L(u_{\nu_k})] \int_\Omega \left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) |\nabla u^*|^{p(x)-2} \nabla u^*.(\nabla u_{\nu_k} - \nabla u^*)\, dx| \\ &\leq \frac{\vartheta_{\nu_k}}{p^-} | \int_\Omega \left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) |\nabla u^*|^{p(x)-2} \nabla u^*.(\nabla u_{\nu_k} - \nabla u^*)\, dx| \to 0 \quad \text{as } k \to \infty, \end{aligned} \tag{34}$$

where $\vartheta_{\nu_k} = \max\{\|u_{\nu_k}\|_X^{p^-}, \|u_{\nu_k}\|_X^{p^+}\} + \max\{\|u^*\|_X^{p^-}, \|u^*\|_X^{p^+}\}$ is bounded.

Also, by (A_2), (29) and the compact embedding of $X \hookrightarrow L^{\alpha(x)}(\Omega)$ we deduce, thanks to the Krasnoselki theorem, the continuity of the Nemytskii operator

$$\begin{aligned} N_{f_1} : L^{\alpha(x)}(\Omega) &\to L^{\alpha'(x)}(\Omega) \\ w &\longmapsto N_{f_1}(w), \end{aligned} \tag{35}$$

given by $(N_{f_1}(w))(x) = f_1(x, w(x)), \quad x \in \Omega.$

Hence

$$\|f_1(w_{\nu_k}) - f_1(w)\|_{\alpha'(x)} \to 0.$$

It follows from the definition of F and the above convergence that

$$|\langle\, F(w_{\nu_k}) - F(w), u_{\nu_k} - u^* \,\rangle| \to 0. \tag{36}$$

Thus, from (33)-(36) we conclude that

$$u_{\nu_k} \to u^* \quad \text{strongly in } X.$$

Since the possible limit of the sequence $(u_\nu)_{\nu\geq 1}$ is uniquely determined, the whole sequence converges toward $u^* \in X$

Therefore, from (28) and the continuous embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, we get $u^* = Tw \equiv u_w$.

On the other hand

$$\begin{aligned} \frac{b(v,\lambda)}{\|v\|_X} &= \frac{\langle F(w), v\rangle - \langle Au, v\rangle}{\|v\|_X} \leq \frac{\langle F(w), v\rangle}{\|v\|_X} + \|Au\|_{X'} \\ &\leq \frac{1}{\|v\|_X}\left[\int_\Omega f_1(x,w)v\,dx + \int_{\Gamma_2} f_2(x)\gamma v\,d\Gamma\right] + L_A\|u\|_X + \|A0\|_{X'} \\ &\leq C(\|f_1(w)\|_{\alpha'(x)} + \|f_2\|_{p'(x),\Gamma_2} + \|A0\|_{X'} + 1). \end{aligned} \tag{37}$$

Next, using the boundedness of the operator N_{f_1} and the sequence $(u_\nu)_{\nu\geq 1}$, and the inf-sup property of the form b, we get $\|\lambda_\nu\|_Y \leq C$. It follows that up to a subsequence

$$\lambda_\nu \to \lambda_0 \quad \text{weakly in } Y$$

for some $\lambda_0 \in Y$.

So (u^*, λ^*) and (u^*, λ_0) are solutions of problem (22)-(23).Then, by the uniqueness $\lambda_0 = \lambda^* \equiv \lambda_w$. This shows the continuity of T.

To prove that T is compact, let $(w_\nu)_{\nu\geq 1} \subseteq K$ be bounded in $L^{\alpha(x)}(\Omega)$ and $u_\nu = T(w_\nu)$. Since $(w_\nu)_{\nu\geq 1} \subseteq K$, $\|w_\nu\|_X \leq C$ and then, up to a subsequence again denoted by $(w_\nu)_{\nu\geq 1}$ we have

$$w_\nu \to w \quad \text{weakly in } X.$$

By the compact embedding Xinto $L^{\alpha(x)}(\Omega)$, it follows that

$$w_\nu \to w \text{ strongly in } L^{\alpha(x)}(\Omega).$$

Now, following the same arguments as in the proof of the continuity of T we obtain

$$u_\nu = T(w_\nu) \to T(w) = u \quad \text{strongly in } X.$$

Thus

$$T(w_\nu) \to T(w) \;\text{ strongly in }\; L^{\alpha(x)}(\Omega).$$

Hence, we can apply the Schauder fixed point theorem to obtain that T possesses a fixed point. This gives us a solution of $(u, \lambda_0) \in X \times \Lambda$ of Problem 1', then a solution of Problem 1, which concludes the proof. □

Next, we consider the uniqueness of solutions of (15). To this end, we also need the following hypothesis on the nonlinear term f_1.

(A_4) There exists $b_0 \geq 0$ such that

$$(f(x,t) - f(x,s))(t-s) \leq b_0 |t-s|^{p(x)} \quad \text{a.e.} \;\; x \in \Omega, \forall t, s \in \mathbb{R}.$$

Our uniqueness result reads as follows.

Theorem 3.2. *Assume that* $(A_1) - (A_4)$ *hold. If, in addition* $2 \leq p$ *for all* $x \in \bar{\Omega}$*, then* (15) *has a unique weak solution provided that*

$$\frac{k_p}{b_0 \lambda_*^{-1}} < 1,$$

where

$$\lambda_* = \inf_{u \in X \setminus \{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\, dx}{\int_\Omega |u|^{p(x)}\, dx} > 0.$$

Proof. Theorem 3.1 gives a weak solution $(u, \lambda) \in X \times \Lambda$. Let $(u_1, \lambda_1), (u_2, \lambda_2)$ be two solutions of (15). Considering the weak formulation of u_1 and u_2 we have

$$\begin{aligned} \langle\, Au_i, v \,\rangle + b(v, \lambda_i) \;\; &= \;\; \langle\, F(u_i), v \,\rangle \quad , \quad \forall v \in X, \qquad (38) \\ b(u_i, \mu - \lambda_i) \;\; &\leq \;\; 0 \quad \forall \mu \in \Lambda \subseteq Y \quad\; i = 1, 2. \end{aligned}$$

By choosing $v = u_1 - u_2$, $\mu = \lambda_2$ if $i = 1$ and $\mu = \lambda_1$ if $i = 2$, we have

$$\begin{aligned} \langle\, Au_1 - Au_2, u_1 - u_2 \,\rangle + b(u_1 - u_2, \lambda_1 - \lambda_2) \;\; &= \;\; \langle\, F(u_1) - F(u_2), u_1 - u_2 \,\rangle \;, \forall v \in X, \\ b(u_1 - u_2, \lambda_2 - \lambda_1) \;\; &\leq \;\; 0 \quad \forall \mu \in \Lambda \subseteq Y. \qquad (39) \end{aligned}$$

It gives

$$\langle\, Au_1 - Au_2, u_1 - u_2 \,\rangle \leq \langle\, F(u_1) - F(u_2), u_1 - u_2 \,\rangle \,.$$

Then, from (39) and repeating the argument used in the proof of Proposition 2.7, ii), we get

$$\begin{aligned} k_p \int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)}\, dx &\leq |\,\langle\, f_1(u_1) - f_1(u_2), u_1 - u_2 \,\rangle\,| \\ &\leq |\int_\Omega (f_1(x, u_1) - f_1(x, u_2))(u_1 - u_2)\, dx| \\ &\leq |\int_\Omega |u_1 - u_2|^{p(x)}\, dx \leq b_0 \lambda_*^{-1} \int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)}\, dx. \end{aligned}$$

Consequently when $\frac{k_p}{b_0 \lambda_*^{-1}} < 1$, it follows that $u_1 = u_2$. This completes the proof. □

4. Conclusions

In this paper, we considered a class of frictional contact problems involving the p(x)-Laplacian-like operator, on a bounded domain $\Omega \subseteq \mathbb{R}^2$. . First, we establish important operator properties of $p(x)$-Laplacian-like operator and bilinear form. Then, under suitable assumptions (A_1)–(A_4), using an abstract Lagrange multiplier technique and the Schauder fixed point theorem, we showed that problem (1)-(5) possesses a unique weak solution. This method allows to write efficient algorithms in order to approximate the weak solutions and can be applied for solving similar nonlinear elliptic inequalities.

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References

[1] Aberqi A., Bennouna J., Benslimane O., Ragusa, M. A., (2022), Existence results for double phase problem in Sobolev-Orlicz spaces with variable exponents in complete manifold, Mediterranean Journal of Mathematics, 19 (4), Article number: 158.

[2] Afrouzi G. A., Mirzapour M., (2000), Eigenvalue problems for $p(x)$-Kirchhoff type equations, Electron. J. Differential Equations, 2013 (2013), 1–10, J. Math. 1 ,pp. 1-11.

[3] Afrouzi G. A., Chung, N. T, Naghizadeh Z., (2022), Multiple solutions for $p(x)$-Kirchhoff type problems with Robin boundary conditions, Electron. J. Differential Equations, Vol. 2022 , No. 24, pp. 1-16.

[4] Antontsev S., Shmarev S., (2006), Handbook of Differential Equations, Stationary Partial Differential Equations, vol. 3, (Chapter 1).

[5] Bazighifan, O.; Kumam, P., (2020), Oscillation Theorems for Advanced Differential Equations with p-Laplacian Like operators, Mathematics, 8(5) , 821.

[6] Bazighifan O., Abdeljawad T., (2020), Improved Approach for Studying Oscillatory Properties of Fourth-Order Advanced Differential Equations with p-Laplacian Like operator, Mathematics, 8(5), 656.

[7] Boureanu M., Matei A., (2014), Nonlinear problems with $p()$-growth conditions and applications to antiplane contact models, Adv. Nonlinear Stud., 14, pp. 295–313.

[8] Cabanillas L. E., Huaringa L., (2021), A nonlocal $p(x)\&q(x)$ elliptic transmission problem with dependence on the gradient Int. J. Appl. Math, 34 (1), pp. 93-108.

[9] Cojocaru M., Matei A., (2019), Well - posedness for a class of frictional contact models via mixed variational formulations, Nonlinear Anal RWA, 47, pp. 127-141.

[10] Chow, S. S., (1989), Finite element error estimates for non-linear elliptic equations of monotone type, Numer. Math., 54 , pp. 373-393.

[11] Chu C. M., Xiao Y. X., (2021), The multiplicity of nontrivial solutions for a new $p(x)$-Kirchhoff-Type elliptic problem, Journal of Function Spaces, art.n.1569376. (2021).

[12] Chung, N. T., (2013), Multiple solutions for a class of $p(x)$-Kirchhoff type problems with Neumann boundary conditions, Adv. Pure Appl. Math., 4 (2), pp. 165-177.

[13] Chen Y., Levine S., Rao R., (2006), Variable Exponent, Linear Growth Functionals in Image Restoration, SIAM Journal of Applied Mathematics 66 , pp. 1383–1406.

[14] Dai G., Ma R., (2011), Solutions for a $p(x)$-Kirchhoff-type equation with Neumann boundary data, Nonlinear Anal. TMA, 12, pp. 2666-2680.

[15] Diening, L., Harjulehto, P., Hasto, P., Ruzicka, M., (2011), Lebesgue and Sobolev spaces with variable exponents, Lecture Notes, vol. 2017, Springer-Verlag, Berlin.

[16] X. L. Fan, J. S. Shen, D. Zhao, (2001), Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl., 262, pp. 749–760.

[17] X. L. Fan, Q. H. Zhang, (2003), Existence of solutions for $p(x)$-Laplacian Dirichlet problems, Nonlinear Anal., 52, pp. 1843–1852.

[18] X. L. Fan, D. Zhao, (2001), On the Spaces $L^{p(x)}$ and $W^{m,p(x)}$, J. Math. Anal. Appl., 263, pp. 424–446.

[19] Han W., Sofonea M., (2002), Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, Studies in Advanced Mathematics 30, American Mathematical Society, Providence, RI-International Press, Somerville, MA.

[20] Hamdani M.K., Chung N. T., Repovš D.D , (2021), New class of sixth-order nonhomogeneous $p(x)$-Kirchhoff problems with sign-changing weight functions, Adv. Nonlinear Anal., 10, pp. 1117–1131.

[21] Hüeber, S., Matei, A., Wohlmuth, B., (2007), Efficient algorithms for problems with friction, SIAM J. Sci. Comput., 29(1), pp. 70–92. https://doi.org/10.1137/050634141.

[22] Hurtado E., Miyagaki O., Rodrigues R., (2022), Multiplicity of solutions to class of nonlocal elliptic problems with critical exponents, Math. Methods Appl. Sci., 45 (7), pp. 3949-3973.

[23] Razani A., (2021), Two weak solutions for fully nonlinear Kirchhoff-type problem, Filomat, 35 (10), pp. 3267–3278.

[24] Rodrigues M., (2012), Multiplicity of solutions on a nonlinear eigenvalue problem for p(x)-Laplacian-like operators, Mediterr. J. Math., 9(1), pp. 211-223.

[25] Ruzicka, M., (2002), Electrorheological Fluids: Modeling and Mathematical Theory), Springer-Verlag, Berlin.
[26] Sofonea M, Matei A., (2009), Variational inequalities with applications. A study of antiplane frictional contact problems, Advances in mechanics and mathematics, vol. 18., New York: Springer.
[27] Tsouli N., Haddaoui M., Hssini E.M., (2020), Multiple Solutions for a Critical p(x)-Kirchhoff Type Equations, Bol. Soc. Paran. Mat., 38(4), pp. 197-211.
[28] Vetro C., (2022), Variable exponent $p(x)$-Kirchhoff type problem with convection, J. Math. Anal. Appl., 506, 125721.
[29] Zhikov V. V., (1986), Averaging of functionals of the calculus of variations and elasticity theory, Izv. Akad. Nauk SSSR Ser. Mat., 50, pp. 675–710.

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