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REVERSE SHARP AND LEFT-T RIGHT-T PARTIAL ORDERING ON INTUITIONISTIC FUZZY MATRICES

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ABSTRACT. In this paper, we introduce the concept of reverse sharp ordering on Intuitionistic Fuzzy matrix (IFM) as a special case of minus ordering. We also introduce the concept of reverse left-T and right-T orderings for IFM as an analogue of left-star and right-star partial orderings for complex matrices. Several properties of these ordering are derived. We show that these ordering preserve its Moore-penrose inverse property. Finally, we show that these ordering are identical for certain class of IFM.

Keywords: Intuitionistic fuzzy matrices, Reverse sharp ordering, Reverse left-T and right-T ordering, g-inverse, Moore-penrose inverses.

AMS Subject Classification: 03E72, 15B15, 15B99.

1. Introduction

The complexity of problems in Economics, Engineering, Environmental Sciences and Social Sciences which cannot be solved by the well-known methods of classical Mathematics pose a great difficulty in today's practical world. To handle this type of situation Zadeh [16] first introduced the notion of fuzzy set to investigate both theoretical and practical applications of our daily activities. This traditional fuzzy set is sometimes may be very difficult to assign the membership value for fuzzy sets. In the current scenario intuitionistic fuzzy set (IFS) initiated by Atanassov [1] is appropriate for such a situation.

It is well known that generalized inverses exist for a complex matrices. However, this is the failure for fuzzy matrices, that is for $P \in F_{mn}$ under the max-min fuzzy operations the matrix equation PXP = P need not have a solution X. If P has a generalized inverse (g-inverse) then P is said to be regular. The concept of generalized inverse presents a very

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interesting area of research in matrix theory in the same way a regular matrix as one of which g-inverse exists, lays the foundation for research in fuzzy matrix theory.

The idea of fuzzy matrix was first presented by Thomosan [15] in 1977 and it has further developments by various researchers. The partial orderings on fuzzy matrices, which are equivalent to the star orderings on complex matrices, were started by Jian Miao Chen [4]. After that, a lot of works have been done using this notion. Meenachi [5] has characterizes the minus ordering on matrices in terms of their generalized inverses. Another novelty is the way she defines space ordering [6] on fuzzy matrices as a partial order on the set of all idempotent matrices in Fn. Punithavalli and Anandhkumar [11] have studied partial ordering on K - Idempotent intuitionistic fuzzy matrices. Punithavalli [10] has studied the Partial Orderings of m-Symmetric Fuzzy Matrices. Sriram and Murugadas [14] have discussed the Moore-Penrose Inverse of Intuitionistic Fuzzy Matrices. Muthu Guru Packiam and Krishna Mohan [7] have studied Partial orderings on k-idempotent fuzzy matrices. Atanassov [2][3] has introduced Intuitionistic fuzzy implications and modus ponens and on some types of fuzzy negations. Padder and Murugadas have discussed Algorithm for controllable and nilpotent intuitionistic fuzzy matrices and determinant theory for intuitionistic fuzzy matrices. Pradhan and Pal [12] have studied the generalized inverse of Atanassov's intuitionistic fuzzy matrices. Shyamal and Pal [13] have characterized distance between intuitionistic fuzzy matrices. This work is to compute the undetermined equation by using generalized inverses and partial ordering.

Partial ordering is a reflexive, anti-symmetric, transitive crisp binary relation R(X,X). The properties of this class of relations are denoted by the common symbol $\leq$. Therefore, $< x, y >$ represents $< x, y > \in R$ and indicates that x comes before y. The symbol $\geq$ denotes the reverse partial ordering $R^{-1}(X, X)$. We say that y succeeds x if $y \leq x$ implying that $< x, y > \in R^{-1}$. The symbols $\leq^P$, $\leq^Q$ and $\leq^R$ are used to denote the various partial orderings P, Q, and R respectively.

In section I , we introduce the concept of reverse sharp ordering on IFM as a special case of minus ordering. We established that for commuting pairs of matrices, sharp ordering and minus ordering are identical. We prove that under certain conditions sharp ordering reduces to the $T-$ ordering on IFM. We establish a set of necessary condition for IFM with specified row and column spaces to be under sharp order. We derive some properties of IFM under sharp ordering .Let $(IF)_n^{\#}$ denote the set of all IFM $Q \in F_n$ for which is group inverse $Q^{\#}$.

In section II, we introduce the concept of reverse left-T and right-T orderings for IFM as an analogue of left-star and right-star partial orderings for complex matrices. Several properties of these ordering are derived. We discuss the relation between these ordering with the T-ordering and minus ordering .We show that these ordering preserve its Moore-penrose inverse property. By using various generalized inverses the new type of minus orderings are discussed. Finally, we show that these ordering are identical for certain class of IFM.

1.1. **Research gaps.** As mentioned in the above introduction section, Meenakshi introduced the concept of Left T Right T and minus ordering on fuzzy matrices and Jian Miao Chen introduced Fuzzy matrix partial orderings and generalized inverses. Here, we have applied the concept of Reverse Sharp and Left-T Right-T Partial ordering on Intuitionistic Fuzzy Matrices. Both these concepts plays a significant role in hybrid fuzzy structure and we have applied Reverse Sharp and Left-T Right-T Partial ordering on Intuitionistic Fuzzy Matrices and studied some of the results in detail. First we present equivalent characterizations of a Reverse Sharp and Left-T Right-T Partial ordering on Intuitionistic Fuzzy Matrices and then, derive equivalent conditions for an intuitionistic fuzzy matrices. Also, using the g- inverses, we discuss some Theorems and examples for the reverse Sharp and Left-T Right-T Partial ordering on IFM.

2. Notations:

For IFM of $P \in (IF)_n$, P^T : Transpose of P, $R(P)$: Row space of P, $C(P)$: Column space of P, P^+ : Moore-Penrose inverse of P, $(IF)_n$ = Square Intuitionistic Fuzzy matrices of order n, $(IF)_n^{\#}$ = Intuitionistic Fuzzy group inverse of order n : $P \leq^T Q = T-$ ordering: $P \overset{T}{\geq} Q$ = Reverse T-ordering:$P \overset{\#}{>} Q$ = Reverse Sharp ordering: $P, Q \in (IF)_{m\times n}^{-}$ = Minus ordering: $P \leq^* Q$ = Partial ordering.

3. Preliminary and Definitions

Here we recall some preliminary definitions regarding the topic. By a fuzzy matrix, we mean a matrix over a fuzzy algebra. A fuzzy algebra is a mathematical system (F,+,.) with two binary operations addition (+) and multiplication (.) defined on a set F satisfying the following properties:
(P1) Idempotence p+p= p, p. p = p
(P2) Commutativity p+q=q+ p, p.q=q.p
(P3) Associativity p+(q+r)=(p+q)+r, p.(q.r)=(p.q).r
(P4) Absorption (p+p).q=p, p.(p+q)=p
(P5) Distributivity p.(q+r)=(p.q)+(p.q), p+(q.r)=(p+q).(p+r)
(P6) Universalbounds p+0=p, p+1=1; p.0=0, p.1=p. A fuzzy matrix can be interpreted as a binary fuzzy relation, which is defined as follows.

Definition 3.1. [8] *Let an IFM A of order m rows and n columns is in the form of* $A = [y_{ij}, < a_{ij\alpha}, a_{ij\beta} >]$ *, where* $a_{ij\alpha}$ *and* $a_{ij\beta}$ *are called the degree of membership and also the non-membership of* y_{ij} *in A, it preserving the condition* $0 \leq a_{ij\alpha} + a_{ij\beta} \leq 1$.

Definition 3.2. [5] *For* $P \in (IF)^{\#}{}_n$ *and* $Q \in (IF)_n$ *the reverse order sharp ordering denoted as* $\overset{\#}{>}$ *is defined as* $P \overset{\#}{>} Q \Leftrightarrow Q^{\#}Q = Q^{\#}P$ *and* $QQ^{\#} = PQ^{\#}$. *Since* $Q^{\#} \in Q\{1\}, P \overset{\#}{>} Q \Leftrightarrow P \geq Q$ *with respect to* $Q^{\#}$ *. Thus sharp ordering is the special case of minus ordering . In general, minus order need not imply sharp order need not imply T-order.*

This is illustrated in the following examples

Example 3.1. *Let* $P = \begin{bmatrix} <1,0> & <0,1> \\ <0,1> & <1,0> \end{bmatrix}$,

$$Q = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,1> \end{bmatrix}$$

Now, $Q\{1\} = \left\{ X/X = \begin{bmatrix} <0,1> & <1,0> \\ <1,0> & <\alpha,0> \end{bmatrix}, \alpha \in (IF) \right\}$

Clearly, $P \geq Q$ *with respect to* $Q^{-} = \begin{bmatrix} <0,1> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$

Definition 3.3. [6] *For* P, Q *belongs to* $(IF)_{m\times n}$ *the* $T-$ *ordering* $P \overset{\#}{\leq} Q$ *is well-defined as* $P \overset{\#}{\leq} Q \iff P^tP = P^tQ$ *and* $PP^t = QP^t$.

Definition 3.4. [7] *For* P, Q *belongs to* $(IF)_{m\times n}$ *the* $T-$ *Reverse ordering* $P \overset{\#}{\geq} Q$ *is defined as* $P \overset{\#}{\geq} Q \iff Q^tQ = Q^tP$ *and* $QQ^t = PQ^t$.

Definition 3.5. [6] *Let* $P, Q \in (IF)^{-}_{m\times n}$ *the minus ordering denoted as* $\leq$ *is defined as* $P \leq Q \iff P^{-}P = P^{-}Q$ *and* $PP^{-} = QP^{-}$ *for some* $P^{-} \in P1$

Definition 3.6. [4] *For* P, Q *belongs to* $(IF)_{m\times n}$ *partial ordering* $P \overset{*}{\leq} Q$ *is well-defined as* $P \overset{*}{\leq} Q \iff P^{+}P = P^{+} = QP^{+}$.

4. Main Results

Theorem 4.1. *For* $P, Q \in (IF)^{\#}{}_{n}$ *, if* $P \overset{\#}{>} Q$ *then we have*
(i) $Q = QQ^{\#}P = PQ^{\#}Q = PQ^{\#}P$
(ii) $QP^{\#}Q = PP^{\#}Q = QP^{\#}P = Q$.

Proof. (i) $P \overset{\#}{>} Q \Rightarrow P \geq Q$ with respect to $Q^{\#}$.
$P \geq Q \Leftrightarrow Q^{\#}Q = Q^{\#}P$ and $QQ^{\#} = PQ^{\#}$. for some $Q^{\#} \in Q\{1\}$
Now, $Q = Q(Q^{\#}Q) = QQ^{\#}P$
$Q = (QQ^{\#})Q = PQ^{\#}Q$
$Q = P(Q^{\#}Q) = PQ^{\#}P$
(ii) $P \geq Q \Rightarrow Q = QQ^{\#}P = PQ^{\#}Q$
For, $P^{\#} \in P\{1\}$
$QP^{\#}Q = (QQ^{\#}P)P^{\#}(PQ^{\#}Q)$
$QP^{\#}Q = QQ^{\#}(PP^{\#}P)Q^{\#}Q$
$= (QQ^{\#}P)Q^{\#}Q = QQ^{\#}Q = Q$
Hence, $QP^{\#}Q = Q$ for each $P^{\#} \in P\{1\}$
Similarly, $PP^{\#}Q = QP^{\#}P = Q$. □

Theorem 4.2. *For* $P, Q \in (IF)^{\#}{}_{n}, P \overset{\#}{>} Q \Leftrightarrow P^{\#} \overset{\#}{>} Q^{\#}$

Proof. Let $P \overset{\#}{>} Q$
Now, $(Q^{\#})^{\#}Q^{\#} = QQ^{\#} = Q^{\#}Q$
$= Q^{\#}(QP^{\#}P)$
$= (Q^{\#}Q)(P^{\#}P)$
$= (Q^{\#}Q)(PP^{\#})$
$= (Q^{\#}QP)P^{\#}$
$= QP^{\#}$
$(Q^{\#})^{\#}Q^{\#} = (Q^{\#})^{\#}P^{\#}$
Similarly, $Q^{\#}(Q^{\#})^{\#} = P^{\#}(Q^{\#})^{\#}$
Thus, $P^{\#} \overset{\#}{>} Q^{\#}$ converse follows from the fact $(Q^{\#})^{\#} = Q$. □

Theorem 4.3. *For* $P \in (IF)^{\#}{}_{n}$, *and* $Q \in (IF)_n$ *the conditions are equivalent*

$(i) P \overset{\#}{>} Q$
$(ii) PQ = Q^2 = QP.$

Proof. Since $Q^{\#}$ exists, $Q^{\#}Q^2 = (Q^{\#}Q)Q = QQ^{\#}Q = Q^2Q^{\#} = Q$
(i) ⇒ (ii) $QP = Q^2Q^{\#}P = Q(QQ^{\#}P) = QQ = Q^2$
Similarly,$PQ = Q^2$
(ii) ⇒ (i) $QQ^{\#} = (Q^2Q^{\#})Q^{\#}$
$= (PQQ^{\#})Q^{\#}$
$= P(Q^{\#}QQ^{\#})$
$= PQ^{\#}$
Similarly,$Q^{\#}Q = Q^{\#}P.$
Hence $P \overset{\#}{>} Q$ □

Theorem 4.4. *For $Q \in {(IF)^{\#}}_n$, and $P \in (IF)_n$ then $P \overset{\#}{>} Q \Leftrightarrow QP = PQ$ and $P \geq Q$.*

Proof. $P \overset{\#}{>} Q \Rightarrow P \geq Q, QP = PQ = Q^2$ (By Theorem 4.3)
Conversely ,$P \geq Q \Rightarrow Q = QP^-P = PQ^-P$ (By Theorem 4.3)
$QP = (QP^-Q)P$
$= QP^-(QP) = (QP^-)(PQ)$
$= (QP^-Q)Q$
$= QQ = Q^2$
Similarly ,$PQ = Q^2$
Hence $QP = PQ = Q^2 \Rightarrow P \overset{\#}{>} Q$ (By Theorem 4.3) □

Theorem 4.5. *For $Q \in {(IF)^-}_{mn}$, and $P \in (IF)_{mn}$ we have the following.*
$R(P) \subseteq R(Q) \Leftrightarrow C(P^{'}) \subseteq C(Q^{'})$

Proof. $R(P) \subseteq R(Q) \Leftrightarrow P = PPQ^-Q$ (Taking Transpose on both sides)
$\Leftrightarrow P^{'} = Q^{'}(Q^-)^{'}P^{'}$
$\Leftrightarrow P^{'} = Q^{'}(Q^{'})^-P^{'}$
$\Leftrightarrow C(P^{'}) \subseteq C(Q)^{'}$ □

Theorem 4.6. *Let $P, Q \in {(IF)^{\#}}_n$, If $P \overset{\#}{>} Q$ then $P \geq Q$ and $PQ^{\#}P = Q$. Conversely $P \geq Q$, $C(PQ^{\#}P) \subseteq C(Q)$ and $R(PQ^{\#}P) \subseteq R(Q)$ imply $Q \overset{\#}{>} P$.*

Proof. Clearly. $P \overset{\#}{>} Q \Rightarrow P > Q$ with respect to $Q^{\#}$ and $PQ^{\#}Q = Q$.
Now assume $Q > P$ and $C(PQ^{\#}P) \subseteq C(Q)$ hold. By Theorem. Since $Q > P$ and $P^{\#} \in P\{1\}$, we have, $QP^{\#}Q = Q, QP^{\#}P = PP^{\#}Q = Q$
$P(PQ^{\#}P) \subseteq C(Q) \Rightarrow QQ^{\#}(PQ^{\#}P) = PQ^{\#}P$
$\Rightarrow QP^{\#}(QQ^{\#}PQ^{\#}P) = QP^{\#}(PQ^{\#}P)$ (Premultiply by $QP^{\#}$)
$\Rightarrow QQ^{\#}PQ^{\#}P = QQ^{\#}P = PQ^{\#}P$
$\Rightarrow QQ^{\#}P(P^{\#}Q) = PQ^{\#}P(P^{\#}Q)$ (Premultiply by $P^{\#}Q$)
$\Rightarrow Q = QQ^{\#}Q = PQ^{\#}Q$
$\Rightarrow QQ^{\#} = PQ^{\#}$ (Premultiply by $Q^{\#}$)
Thus $C(PQ^{\#}P) \subseteq C(Q) \Rightarrow QQ^{\#} = PQ^{\#}$..........(1)
Similarly, $R\left[(PQ^{\#}P)^{'}\right] \subseteq R(Q)$ and $Q \geq P$
$\Rightarrow C\left[(PQ^{\#}P)^{'}\right] \subseteq C(Q)^{'}$ and $Q^{'} \geq P^{'}$

$\Rightarrow Q^{'}(Q^{'})^{\#} = P^{'}(Q^{'})^{\#}$ (By 1)
$\Rightarrow Q^{'}(Q^{\#})^{'} = p^{'}(Q^{\#})^{'}$
$\Rightarrow Q^{\#}Q = Q^{\#}P$
Hence $Q \geq P$
In general $(QP)^{\#} \neq P^{\#}Q^{\#}$. This is illustrated in the given example. □

Example 4.1. $P = \begin{bmatrix} <1,0> & <1,0> \\ <0,1> & <0,1> \end{bmatrix}$,

$$Q = \begin{bmatrix} <1,0> & <0,1> \\ <1,0> & <0,1> \end{bmatrix}.$$

Here $QP = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$

since Q, P *and* QP *are idempotent,* $Q^{\#} = Q, P^{\#} = P$ *and* $(QP)^{\#} = QP$*. But*

$$P^{\#}Q^{\#} = \begin{bmatrix} <1,0> & <1,0> \\ <0,1> & <0,1> \end{bmatrix} \begin{bmatrix} <1,0> & <0,1> \\ <1,0> & <0,1> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <0,1> \\ <0,1> & <0,1> \end{bmatrix} \neq \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix} = (QP)^{\#}$$

Hence $(QP)^{\#} \neq P^{\#}Q^{\#}$ *Theorem 2.6 can be restated involving group inverse in the following.*

Theorem 4.7. *Let* $P, Q \in (IF)^{\#}{}_{n}$*, If* $P \overset{\#}{>} Q$ *then* $P \geq Q$ *and* $P^{\#}QP^{\#} = Q^{\#}$*. Conversely* $P \geq Q$*,* $C(P^{\#}QP^{\#}) \subseteq C(Q^{\#})$ *and* $R(P^{\#}QP^{\#}) \subseteq R(Q^{\#})$ *then* $Q \overset{\#}{>} P$*.*

Proof. Clearly. $P \overset{\#}{>} Q \Rightarrow P > Q$ with respect to $Q^{\#}$ and $P^{\#} \overset{\#}{>} Q^{\#}$ follows by the definition 3.2 Theorem 4.2
Now $P^{\#} \overset{\#}{>} Q^{\#} \Rightarrow Q^{\#} = Q^{\#}QQ^{\#} = P^{\#}QP^{\#}$.
Conversely,
$P \geq Q \Rightarrow QP^{\#}Q = Q, QP^{\#}P = P^{\#}PQ = Q$ (By theorem 4.1)
$C(P^{\#}QP^{\#}) \subseteq C(P^{\#}) \Rightarrow Q^{\#}QP^{\#}QP^{\#} = P^{\#}QP^{\#}$
$\Rightarrow Q^{\#}(QP^{\#}Q)P^{\#}Q = P^{\#}(QP^{\#}Q)$ (Post multiply by Q)
$\Rightarrow Q^{\#}(QP^{\#}Q) = P^{\#}Q$
$\Rightarrow Q^{\#}Q = P^{\#}Q$
Similarly, $R(P^{\#}QP^{\#}) \subseteq R(P^{\#}) \Rightarrow C\left[\left(P^{\#}QP^{\#}\right)^{'}\right] \subseteq C\left[(P^{\#})^{'}\right]$
$\Rightarrow (Q^{'})^{\#}Q^{'} = (P^{'})^{\#}Q^{'}$
$\Rightarrow QQ^{\#} = QP^{\#}$
Therefore $P^{\#} \overset{\#}{>} Q^{\#}$ which implies $P \overset{\#}{>} Q$ (By Theorem 4.2)
Hence, $P \overset{\#}{>} Q$ □

Theorem 4.8. *Let* $Q \in (IF)^{\#}{}_{n}$*, and* $P \in (IF)_{n}$*, if both* Q *and* P *are symmetric IFM then* $P \geq Q = P^{2} \Rightarrow Q^{2} = Q \overset{\#}{>} P$

Proof. $P \geq Q = Q^{2} \Rightarrow QP^{-}P = PP^{-}Q = QP^{-}Q$
$P \geq Q$ with P idempotent which implies Q is idempotent
Now $PQ = P(PP^{-}Q)$

$PQ = P^2P^-A = PP^-Q = Q = Q' = (PQ)' = Q'P' = QP$
Hence, $QP = PQ = Q = Q^2$
$\Rightarrow P \overset{\#}{>} Q$ (By Theorem 4.3) □

Theorem 4.9. *For $P, Q \in (IF)^{\#}{}_n$, if Q is symmetric with Q^+ exists then $P \overset{\#}{>} Q \Leftrightarrow Q \overset{T}{>} P$.*

Proof. Q is symmetric which implies Q is range symmetric. We know that Q is range symmetric and Q^+ exists imply $Q^\#$ exists and $Q^\# = Q^t$

Thus, $P \overset{\#}{>} Q \Leftrightarrow Q \overset{T}{>} P$ holds □

5. Reverse Left–T and Right–T Partial Ordering

Definition 5.1. [7] *Let $P, Q \in (IF)_{mn}$ We say that P and Q with respect to the left-T ordering if $Q'Q = Q'P$ and $C(Q) \subseteq C(P)$ and is denoted as $Qt > P$. We say that Q is below P with respect to the right -T ordering if $QQ' = PQ'$ and $R(Q) \subseteq R(P)$ and is denoted as $Q > tP$.*

Example 5.1. *Let us consider* $P = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}$,

$$Q = \begin{bmatrix} <1,0> & <1,0> \\ <0,1> & <0,0> \end{bmatrix}.$$

$$Q^TQ = \begin{bmatrix} <1,0> & <0,1> \\ <1,0> & <0,0> \end{bmatrix} \begin{bmatrix} <1,0> & <1,0> \\ <0,1> & <0,0> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$

$$Q^TP = \begin{bmatrix} <1,0> & <0,1> \\ <1,0> & <0,0> \end{bmatrix} \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$

$$Q = Py = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix} \begin{bmatrix} <0,1> & <0,0> \\ <1,0> & <1,0> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}.$$

$Q^TQ = Q^TP$ *and* $C(Q) \subseteq C(P)$

$$PQ^T = \begin{bmatrix} <1,0> & <0,1> \\ <1,0> & <0,0> \end{bmatrix} \begin{bmatrix} <1,0> & <0,1> \\ <1,0> & <0,0> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <0,0> \\ <1,0> & <0,0> \end{bmatrix}.$$

$Q^TQ \neq Q^TP$ *and* $R(Q) \not\subseteq R(P)$.

Example 5.2. *Let us consider* $P = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}$,

$$Q = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$

$$Q'Q = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$$

$$Q^{'}P = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$$
$$y = \begin{bmatrix} <1,0> & <0,0> \\ <1,0> & <1,0> \end{bmatrix}$$
$$Q = Py = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix} \begin{bmatrix} <1,0> & <0,0> \\ <1,0> & <1,0> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$
$$Q = yP = \begin{bmatrix} <1,0> & <0,0> \\ <1,0> & <1,0> \end{bmatrix} \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$

In particular $P, Q \in (IF)_{mn}$ since by $P^{+} = P^{T}$ Definition is equal to the following
$Q^{+}Q = Q^{+}P$ *and* $C(Q) \subseteq C(P)$
$QQ^{+} = PQ^{+}$ *and* $R(Q) \subseteq R(P)$

Example 5.3. *Let us consider* $P = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}$,

$$Q = \begin{bmatrix} <0,1> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$
$$Q^{'}Q = \begin{bmatrix} <0,1> & <1,0> \\ <1,0> & <1,0> \end{bmatrix} \begin{bmatrix} <0,1> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$
$$Q^{'}P = \begin{bmatrix} <0,1> & <1,0> \\ <1,0> & <1,0> \end{bmatrix} \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$

$Q^{'}Q = Q^{'}P$ *but* $C(Q) \nsubseteq C(P)$

$$PQ^{'} = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix} \begin{bmatrix} <0,1> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$$
$$= \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$

$Q^{'}Q = PQ^{'}$ *but* $R(Q) \nsubseteq R(P)$

Hence T - ordering need not imply both left - T and right - T orderings. In the following we discuss relationship between the left - T and right - T orderings with T - ordering.

Theorem 5.1. *let $Q \in (IF)^{+}{}_{mn}, P \in (IF)_{mn}, Qt > P$ and $Q > tP \Leftrightarrow Q \overset{T}{>} P$*

Proof. $Qt > P$ and $Q > tP$ implies $Q^{'}Q = Q^{'}P$ and $QQ^{'} = PQ^{'} \Rightarrow Q \overset{T}{>} P$

Conversely, $Q \overset{T}{>} P \Rightarrow QQ^{'} = Q^{'}P$ and $QQ^{'} = PQ^{'}$
$Q^{+}Q = Q^{+}P$ and $QQ^{+} = PQ^{+}$
$Q \overset{T}{>} P \Rightarrow Q^{+}Q = Q^{+}P \Rightarrow QQ^{+}Q = QQ^{+}P$
$\Rightarrow Q = XP$ where $X = QQ^{+}$
$R(Q) \subseteq R(P)$
$QQ^{+} = PQ^{+} \Rightarrow Q = PQ^{+}Q \Rightarrow Q = PY$
$C(Q) \subseteq C(P)$

Thus, $Q \overset{T}{>} P \Rightarrow Qt > P$ and $Pt > Q$

Hence, $Q \in (IF)^{+}{}_{mn}, P \in (IF)_{mn}, Qt > P$ and $Q > tP \Leftrightarrow Q \overset{T}{>} P$ ☐

Theorem 5.2. *Let $Q \in (IF)^{+}{}_{mn}, P \in (IF)^{-}{}_{mn}$, if either $Qt > P$ or $Q > tP$ then $Q \geq P$*

Proof. $Qt > P \Rightarrow Q^{'}Q = Q^{'}P$
$\Rightarrow QQ^{+}Q = QQ^{+}P$ (Pre multiply by Q and replace Q^{t} by Q^{+})
$\Rightarrow Q = (QQ^{+}P)$
$\Rightarrow R(Q) \subseteq R(P)$
Now, $Q^{'}Q = Q^{'}P \Rightarrow Q^{+}QP^{-}Q = Q^{+}PP^{-}Q$ (Post multiply by PQ and replace Q^{t} by Q^{+})
$\Rightarrow (QQ^{+}Q)P^{-}Q = QQ^{+}(PP^{-}Q)$ (Pre multiply by Q)
$\Rightarrow QP^{-}Q = Q$
Hence, $Qt > P \Rightarrow C(Q) \subseteq C(P), R(Q) \subseteq R(P)$ and $QP^{-}Q = Q$
$\Rightarrow Q \geq P$
Proof of $Q > tP \Rightarrow Q \geq P$ can be proved in the same manner. ☐

Theorem 5.3. *For $P, Q \in (IF)^{+}{}_{mn}$ we have*
(i) $Qt > P \Leftrightarrow Q^{+}t > P^{+}$
(ii) $Q > tP \Leftrightarrow Q^{+} > tP^{+}$

Proof. $Dt > C \Rightarrow Q^{'}Q = Q^{'}P$ and $C(Q) \subseteq C(P)$
Now, $C(Q) \subseteq C(P) \Rightarrow Q = PP^{+}Q$
$\Rightarrow Q = PP^{'}Q$
$\Rightarrow Q^{'} = Q^{'}PP^{'}$
$\Rightarrow Q^{'} = (P^{'}Q)P^{'}$ ($Q^{t}P$ is symmetric)
$\Rightarrow Q^{'} = P^{'}(QP^{'})$
$\Rightarrow C(Q^{'}) \subseteq C(P^{'})$
$QQ^{'} = Q^{'}P \Rightarrow C(Q^{'}Q)P^{'} = Q(Q^{'}P)P^{'}$
$\Rightarrow QP^{'} = Q(PP^{'}Q)^{'}$
$\Rightarrow QP^{'} = QQ^{'}$ (By $Q = PP^{'}Q$)
$\Rightarrow QP^{+} = QQ^{+}$
Thus $(Q^{+})^{'}P^{+} = (Q^{+})^{'}Q^{+}$ and $C(Q^{+}) \subseteq C(P^{+}) \Rightarrow Q^{+}t > P^{+}$
Converse follows from above part by using $(Q^{+})^{+} = Q$
Proof of (ii) is similar. ☐

Theorem 5.4. *For $P, Q \in (IF)^{+}{}_{mn}$ we have*
(i) $Qt > P \Leftrightarrow Q^{+}Q = QP^{+}$ and $R(Q) \subseteq R(P) \Leftrightarrow Q > tP$
(ii) $Q > tP \Leftrightarrow Q^{+}Q = P^{+}Q$ and $C(Q) \subseteq C(P) \Leftrightarrow Qt > P$

Proof. $(i)Qt > P \Leftrightarrow\Leftrightarrow Q^{+}t > P^{+}$
$\Leftrightarrow (Q^{+})^{'}Q^{+} = (Q^{+})^{'}P^{+}$ and $C(Q^{+}) \subseteq C(P^{+})$
$\Leftrightarrow QQ^{'} = QP^{'}$ and $R(Q) \subseteq R(P)$
$\Leftrightarrow QQ^{'} = PQ^{'}$ and $R(Q) \subseteq R(P)$ (QP^{t} is symmetric)
$\Leftrightarrow Q > tP$
Proof of (ii) is same and hence omitted. ☐

Theorem 5.5. *For $P, Q \in (IF)^{+}{}_{mn}$ the following are equivalent*
(i) $Qt > P$
(ii) $Q > tP$
(iii) $Q \geq P$ and $Q^{+}P$ is symmetric
(iv) $Q \geq P$ and QP^{+} is symmetric

Proof. (i)$\Leftrightarrow$ (ii) Follow from theorem 5.4

$(i) \Leftrightarrow (iii) Qt > P \Rightarrow Q \geq P$ (By Theorem 5.2)

$Qt > P \Rightarrow Q^{'}Q = Q^{'}P$

$\Rightarrow Q^{'}P$ symmetric

$\Rightarrow Q^{+}P$ symmetric (By using $Q^{+} = Q^{'}$)

$(iii) \Leftrightarrow (i)$ If $Q^{+}P$ is symmetric then $Q^{+}P = P^{+}Q$ follows by replacing P^{t} by P^{+}

Since $Q \geq P$ from theorem (5.2) we have $PP^{+}Q = Q = QP^{+}Q$

$Q^{T}Q = Q^{+}Q = Q^{+}(PP^{+}Q) = (Q^{+}P)(P^{+}DQ) = P^{t}(QP^{+}Q) = P^{+}Q = Q^{t}P$

Hence $Q > P$

$(iii) \Leftrightarrow (iv)$ $Q \geq P$ and $Q^{+}P$ is symmetric

$QP^{+} = (PP^{+}Q)P^{+} = (PP^{t}Q)P^{t} = PP^{t}QP^{t} = P(P^{t}Q)P^{t} = P(Q^{t}PP^{t}) = P(PP^{t}Q)^{t} = PQ^{t} = (QP^{t})^{t} = (QP^{+})^{t}$

$(iv) \Leftrightarrow (iii) : Q^{t}P = (QP^{+}P)P = P^{'}(PQ^{'})^{'}P = P^{'}(QP^{T}P) = P^{'}Q$

Hence $Q^{t}P$ is symmetric

In the above theorem (5.5), the condition $Q^{+}P$ and QP^{+} are symmetric is essential. □

Example 5.4. *Let us consider* $P = \begin{bmatrix} <1,0> & <1,0> \\ <0,1> & <0,1> \end{bmatrix}$,

$Q = \begin{bmatrix} <1,0> & <1,0> \\ <0,1> & <0,1> \end{bmatrix}$.

Clearly $Q \geq P$ *with respect to* $A = A^{-}$.

Here $Q^{'}P = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$ *and*

$QP^{'} = \begin{bmatrix} <1,0> & <1,0> \\ <0,1> & <0,1> \end{bmatrix}$ *are not symmetric.*

Here $Qt \not\geq P$.

Theorem 5.6. *For* $Q \in (IF)_{mn}$, *if* $Q^{(1,4)}$ *and* $Q^{(1,3)}$ *then exists and* $Q^{+} = Q^{(1,4)}QQ^{(1,3)}$

Proof. Let $Y = Q^{(1,4)}QQ^{(1,3)}$ one can easily verify that $Y \in Q\{1,2\}$

$QY = Q\Big(Q^{(1,4)}QQ^{(1,3)}\Big) = \Big(QQ^{(1,4)}Q\Big)Q^{(1,3)} = QQ^{(1,3)}$

$(QY)^{'} = (QQ^{(1,3)})^{'} = QQ^{(1,3)} = QY$

$YQ = (QQ^{(1,3)}Q) = Q^{(1,4)}(QQ^{(1,3)}Q) = Q^{(1,4)}Q$

$(YQ)^{'} = (Q^{(1,4)}Q)^{'} = Q^{(1,4)}Q = YQ$

Thus, $Y \in Q\{1,2,3,4\}$

Hence $Y = Q^{(1,4)}QQ^{(1,3)} = Q^{+}$ □

Definition 5.2. *For* $Q \in (IF)^{+}_{mn}$, *and* $P \in (IF)_{mn}$, *we say that* Q *is below* P *with respect to the minus - k ordering, where* $k = 3$ *or* 4 *denoted as* $P \underset{k}{\geq} Q$ *is defined as* $UQ = UB$ *and* $QV = PV$ *for some* $U, V \in Q\{1,k\}$.

Theorem 5.7. *For* $Q \in (IF)^{+}_{mn}$ *the following are equivalent (K=3 or 4)*

(i) $P \underset{k}{\geq} Q$

(ii) $LQ = LP$ *and* $QM = PM$ *for some* $L, M \in Q\{1,2,k\}$

(iii) $NQ = NP$ *and* $QN = PN$ *for some* $N \in Q\{1,2,k\}$

(iv) $RQ = RP$ *and* $RM = RM$ *for some* $R \in Q\{1,k\}$

Proof. (i) implies (ii) $P \underset{k}{\geq} Q \Rightarrow UQ = UP$ and $PV = QV$ for some $U, V \in Q\{1,k\}$

$\Rightarrow (UQ)UQ = (UQ)UP$ and $QV(QV) = PV(QV)$
(Pre multiply the former condition UP and post multiply the later condition by QV)
$\Rightarrow (UQU)Q = (UQU)P$ and $Q(VQV) = P(VQV)$
Chose $L = UPU$ and $M = VPV$, clearly $L, M \in Q\{1, 2, k\}$
Implies $LQ = LP$ and $QM = PM$ for $L, M \in Q\{1, 2, k\}$
Thus (ii) holds
(ii) implies (iii) Pre-multiply the former condition in (ii) by MP, postmultiply the later condition in (ii) by PL and choose $N = MQL$
Clearly, $N \in P\{1, 2, k\}$.
Hence (ii) implies (iii) holds
The proof of (iii) implies (iv) implies (i) is obvious, hence omitted. □

Theorem 5.8. *For $P, Q \in (IF)^+_{mn}$ the following are equivalent*
(i) $P \underset{3}{\geq} Q$
(ii) $Pt > Q$

Proof. (i) $P \underset{3}{\geq} Q \Rightarrow UQ = UP$ and $QV = PV$ for some $U, V \in Q\{1, 3\}$
$UQ = UP \Rightarrow Q^+Q(UQ) = Q^+Q(UP)$
$\Rightarrow (QUQ) = (Q^+QU)P$
$\Rightarrow Q^+Q = Q^+P$
$\Rightarrow Q^{'}Q = Q^{'}P$
$QV = PV = QVQ = PVQ$
$\Rightarrow Q = P(VQ)$
$C(Q) \subseteq C(P)$
Thus, $Q^{'}Q = Q^{'}P$ and $C(Q) \subseteq C(P) \Rightarrow Pt > Q$
Conversely: $Pt > Q \Rightarrow Q^+Q = Q^+P \Rightarrow UQ = UB$ where $U = A^+$
$pt > Q \Rightarrow P \geq Q$ (by theorem 5.2)
$\Rightarrow Q = QP^+Q$ (by theorem 5.2)
Now, $C(Q) \subseteq C(P) \Rightarrow Q = PP^+Q$
$\Rightarrow QP^+Q = PP^+Q$
$\Rightarrow QP^+QQ^+ = PP^+QQ^+$ (Postmultiply by Q^+)
$\Rightarrow Q(P^+QQ^+) = P(P^+QQ^+)$
$pt > Q \Rightarrow P^+ \in Q\{1, 3, 4\}$
$\Rightarrow P^+ \in Q\{1, 3\}$
$Q^+ \in Q\{1, 3\}$ and $P^+ \in Q\{1, 3\} \Rightarrow P^+QQ^+ \in Q\{1, 3\}$
Choose, $V = P^+QQ^+ \Rightarrow QV = PQ$
Hence, $UQ = UP$ and $QV = PV$ for $U, V \in Q\{1, 3\}$
Hence, $P \underset{3}{\geq} Q$ □

6. Conclusions

We established that for commuting pairs of matrices, sharp ordering and minus ordering are identical. We prove that under certain conditions sharp ordering reduces to the T-ordering on IFM. We establish a set of necessary condition for IFM with specified row and column spaces to be under sharp order. The concept of left-T and right-T orderings for IFM as an analogue of left-star and right-star partial orderings for complex matrices. We show that these ordering preserve its Moore-penrose inverse property. By using various generalized inverses the new type of minus orderings are discussed. Finally, we show that these ordering are identical for certain class of IFM.

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References

[1] Atanassov, K., (1983), Intuitionistic Fuzzy Sets, Fuzzy Sets and System., 20, pp. 87- 96.

[2] Atanassov, K., (2005), Intuitionistic fuzzy implications and modus ponens, In Notes on Intuitionistic Fuzzy Sets, 11(1), pp. 1-5.

[3] Atanassov, K., (2005), On some types of fuzzy negations, In Notes on Intuitionistic Fuzzy Sets, 11(4), pp. 170-172.

[4] Jian Miao Chen, (1982), Fuzzy matrix partial orderings and generalized inverses, Fuzzy Sets Sys., 105, pp. 453 - 458.

[5] Meenakshi, A. R., Inbam, C., (2004), The minus partial order in fuzzy matrices, The Journal of Fuzzy Matrices, 12(3), pp. 695 - 700.

[6] Meenakshi, A. R., (2008), Fuzzy Matrix – Theory and its Applications, MJP Publishers.

[7] Muthu Guru Packiam, K., Krishna Mohan, K. S., (2019), Partial orderings on k-idempotent fuzzy matrices, 15(5), pp. 733-741.

[8] Padder, R. A., Murugadas, P., (2022), Algorithm for controllable and nilpotent intuitionistic fuzzy matrices, Afr. Mat., 33, Article no: 84.

[9] Padder, R. A., Murugadas, P., (2019), Determinant theory for intuitionistic fuzzy matrices, Afr. Mat., 30, pp. 943-955.

[10] Punithavalli, G., (2020), The partial orderings of m-symmetric fuzzy matrices, International journal of Mathematics Trends and Technology., 66(4), pp.705-803.

[11] Punithavalli, G., Anandhkumar, M., (2022) , Partial Orderings On K-Idempotent Intuitionistic Fuzzy Matrices., 54(2), pp. 2096-3246.

[12] Pradhan, R., Pal, M., (2014), The generalized inverse of Atanassov's intuitionistic fuzzy matrices, International Journal of Computational Intelligence Systems., 7(6), pp. 1083-1095.

[13] Shyamal, A. K., Pal, M., (2002), Distance between intuitionistic fuzzy matrices, V. U. J. Physical Sciences., 8, pp. 81-91.

[14] Sriram, S., Murugadas, P., (2010), The Moore-Penrose inverse of intuitionistic fuzzy matrices, Int. Journal of Math. Analysis., 4(36), pp. 1779 -1786.

[15] Thomson, M. G., (1977), Convergence of power of a fuzzy matrix, Journal of Mathematical Analysis and Applications, 57, pp. 476-480.

[16] Zadeh, L. A., (1965), Fuzzy Sets, Information and Control, 8, pp. 338-353.

[21] Ivaz K., Asadpour Fazlallahi M., Khastan A., Nieto J.J., (2023), Fuzzy one-phase Stefan problem. Appl. Comput.Math., V.22, N.1, pp.66-79.

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