TWMS J. App. and Eng. Math. V.15, N.5, 2025, pp. 1063-1073

ZWEIER IDEAL CONVERGENCE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION ON NEUTROSOPHIC NORMED SPACES

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Abstract. The present article is introduced to study the concept of Zweier ideal convergent sequence space which is characterised by Orlicz function on neutrosophic normed spaces. Some important topological and algebraic properties of ideal convergent sequence space have been analyzed using the Zweier transformation and Orlicz function. Also, we have proved a few relations that are related to new sequence spaces.

Keywords: Ideal convergence, neutrosophic normed spaces, Orlicz functions

AMS Subject Classification: 40A35, 40C05, 46A45, 46S20

1. Introduction

Kostryko *et al.* [17] established the concept of ideal convergence (I-convergence) by generalizing the idea on statistical convergence [4] of the sequences. Statistical convergence can be seen as a special case of ideal convergence where the ideal is the set of all subsets of $\mathbb{N}$ that have natural density zero. Ideal convergence inspired many researchers to do work in different directions in recent years as it provides a broader framework to study the convergence behaviour of sequences by leveraging the concept of ideals, allowing for more flexible and generalized notions of convergence [1, 2, 3, 16, 18, 20, 21, 22, 28].

The concept of Zweier sequence spaces was defined by Şengönül [25] using Z^p-transforms $(1 \leq p < \infty)$ of a sequence under the matrix domain with limitation method. Zweier sequence spaces offer a rich structure for analyzing sequences, generalizing many properties of classical sequence spaces while introducing a Hilbert space framework through their specific norm. This makes them valuable in various areas of analysis like in approximation theory for studying errors and convergence rates of sequences of approximating functions. Later, Hazarika *et al.* [8] associated the work on Zweier sequence spaces with statistical convergence. Further, Khan *et al.* [11] described Zweier ideal convergence, and later extended to different setups by many more [5, 6, 7, 9, 10, 12, 29, 30].

The Orlicz sequence spaces were given by Lindenstrauss and Tzafriri [19] as a remarkable extension of Orlicz spaces, which has an important role in Banach space theory. An Orlicz

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§ Manuscript received: February 16, 2024; accepted: May 28, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.5;

sequence space must have a subspace that is isomorphic to space $l_p\,(1 \le p < \infty)$. Several types of effective sequence spaces have been explored, modified and introduced by using Orlicz functions by various researchers [6, 10, 31].

Kirişci and Şimşek [14] presented the idea of neutrosophic normed spaces as a notable consideration of neutrosophic metric spaces [13]. Neutrosophic sets [26, 27] provide the ideas to solve real-life situations, where indeterminacy occurs, powerful advancement of the intuitionistic fuzzy sets, that established the concept for the classical sets, fuzzy sets, vague sets etc., for non-standard analysis. As every element of the neutrosophic set has a truth value, a false value and an indeterminacy value, which lies in the non-standard unit interval. Consequently, because of this nature neutrosophic concept is a more adjustable and efficient tool because of its ability to handle, not only the free components of information but also partially independent and dependent information. Neutrosophic normed spaces provide more robust and flexible approaches for analyzing and solving problems in various mathematical and applied disciplines where uncertainty is a key factor. This leads us to investigate the concept of Zweier ideal convergence on neutrosophic normed spaces by the Orlicz function.

Before defining this concept in the main section firstly we will give some introductory notions related to ideal convergence, Orlicz functions, Zweier sequence spaces and neutrosophic normed spaces.

2. Preliminaries

The essential terminology, definitions and few well-known notions are reviewed in this section for developing our concept.

Definition 2.1. [17] *A family $I \subseteq P(\mathbb{S})$, where $\mathbb{S} \neq \emptyset$ and $P(\mathbb{S})$ is a power set for the set $\mathbb{S}$, has been called an ideal on $\mathbb{S}$ provided, (i) $\emptyset \in I$, (ii) $R, S \in I$ then $R \cup S \in I$, (iii) To every $R \in I, S \subset R$ then $S \in I$.*

If $I \neq P(\mathbb{S})$, then I termed as non-trivial ideal and further any non-trivial ideal I has been considered as an admissible ideal on $\mathbb{S}$ provided all the possibles singleton sets are contained in I *i.e* $I \supset \{\{x\} : x \in \mathbb{S}\}$. Consider I as non-trivial admissible ideal throughout this article.

Definition 2.2. [17] *A family $\mathbb{F} \subset P(\mathbb{S})$, where $\mathbb{S} \neq \emptyset$ and $\mathbb{F} \neq \emptyset$, has been called filter on $\mathbb{S}$ provided, (i) $\emptyset \notin \mathbb{F}$, (ii) $R, S \in \mathbb{F}$ then $R \cap S \in \mathbb{F}$, (iii) For each $R \in \mathbb{F}, R \subset S$ then $S \in \mathbb{F}$. Every ideal I is associated with a filter $\mathbb{F}(I)$, where $\mathbb{F}(I) = \{A \subseteq \mathbb{S} : A^c \in I\}$.*

Definition 2.3. [17] *Consider I as an admissible ideal. Any sequence $x = \{x_k\}$ from $\mathbb{S}$ has been ideal convergent (I-convergent) to ξ from $\mathbb{S}$ provided for every $\epsilon > 0$ have $\{k \in \mathbb{N} : |x_k - \xi| \ge \epsilon\} \in I$. Here, ξ is termed as I-limit for a sequence $x = \{x_k\}$.*

Next, we mention about neutrosophic normed space using continuous t -norm and continuous t -conorm along with convergence of sequence in this space.

Definition 2.4. [24] *A continuous t-norm is the mapping $\oplus : [0,1] \times [0,1] \to [0,1]$ such that*
(i) $\oplus$ is continuous, associative, commutative and with identity 1,
(ii) $a_1 \oplus b_1 \le a_2 \oplus b_2$ whenever $a_1 \le a_2$ and $b_1 \le b_2$, for $a_1, a_2, b_1, b_2 \in [0,1]$.

Definition 2.5. [24] *A continuous t-conorm is the mapping $\odot : [0,1] \times [0,1] \to [0,1]$ such that*
(i) $\odot$ is continuous, associative, commutative and with identity 0,
(ii) $a_1 \odot b_1 \le a_2 \odot b_2$ whenever $a_1 \le a_2$ and $b_1 \le b_2$, for $a_1, a_2, b_1, b_2 \in [0,1]$.

Definition 2.6. [14] *A neutrosophic normed space(NNS) is 4-tuple* $(\mathbb{S}, \aleph, \oplus, \odot)$ *with vector space* $\mathbb{S}$*, normed space* $\aleph = \{< \tau(a), \upsilon(a), \eta(a) > : a \in \mathbb{S}\}$ *such that* $\aleph : \mathbb{S} \times \mathbb{R}^+ \to [0,1]$*, continuous t-norm* $\oplus$ *and continuous t-conorm* $\odot$*, if for each* $x, y \in \mathbb{S}$ *and* $s, t > 0$*, satisfying*
(i) $0 \leq \tau(x,t), \upsilon(x,t), \eta(x,t) \leq 1$,
(ii) $\tau(x,t) + \upsilon(x,t) + \eta(x,t) \leq 3$,
(iii) $\tau(x,t) = 1$, $\upsilon(x,t) = 0$ *and* $\eta(x,t) = 0$ *for* $t > 0$ *iff* $x = 0$,
(iv) $\tau(x,t) = 0$, $\upsilon(x,t) = 1$ *and* $\eta(x,t) = 1$ *for* $t \leq 0$,
(v) $\tau(\alpha x, t) = \tau\left(x, \frac{t}{|\alpha|}\right)$, $\upsilon(\alpha x, t) = \upsilon\left(x, \frac{t}{|\alpha|}\right)$ *and* $\eta(\alpha x, t) = \eta\left(x, \frac{t}{|\alpha|}\right)$ *for* $\alpha \neq 0$,
(vi) $\tau(x, .)$ *is continuous non-decreasing function,*
(vii) $\tau(x,s) \oplus \tau(y,t) \leq \tau(x+y, s+t)$,
(viii) $\upsilon(x, .)$ *is continuous non-increasing function,*
(ix) $\upsilon(x,s) \odot \upsilon(y,t) \geq \upsilon(x+y, s+t)$,
(x) $\eta(x, .)$ *is continuous non-increasing function,*
(xi) $\eta(x,s) \odot \eta(y,t) \geq \eta(x+y, s+t)$,
(xii) $\lim_{t\to\infty} \tau(x,t) = 1$, $\lim_{t\to\infty} \upsilon(x,t) = 0$ *and* $\lim_{t\to\infty} \eta(x,t) = 0$.
Here, (τ, υ, η) *is considered as neutrosophic norm.*

Example 2.1. [14] *Consider* $(\mathbb{S}, \| . \|)$ *as a normed space. If for each* $t > 0$ *and all* $x \in \mathbb{S}$*, we take*
(i) $\tau(x,t) = \frac{t}{t+\|x\|}$, $\upsilon(x,t) = \frac{\|x\|}{t+\|x\|}$ *and* $\eta(x,t) = \frac{\|x\|}{t}$ *when* $t > \|x\|$,
(ii) $\tau(x,t) = 0$, $\upsilon(x,t) = 1$ *and* $\eta(x,t) = 1$ *when* $t \leq \|x\|$.
Also, $a_1 \oplus a_2 = a_1 a_2$ *and* $a_1 \odot a_2 = a_1 + a_2(1 - a_1)$ *for* $a_1, a_2 \in [0,1]$.
Then, a 4-tuple $(\mathbb{S}, \aleph, \oplus, \odot)$ *is a NNS which satisfies above mentioned conditions.*

Further, Kirişci and Şimşek [14] have established the overview of convergence for sequences on NNS as given below.

Definition 2.7. [14] *Let* $(\mathbb{S}, \aleph, \oplus, \odot)$ *be a NNS with neutrosophic norm* (τ, υ, η)*. Sequence* $x = \{x_k\}$ *from* $\mathbb{S}$ *has been convergent to* $\xi \in \mathbb{S}$ *corresponding to neutrosophic norm* (τ, υ, η) *provided for* $\epsilon > 0$ *and* $t > 0$ *we can find* $k_0 \in \mathbb{N}$ *satisfying* $\tau(x_k - \xi, t) > 1 - \epsilon$, $\upsilon(x_k - \xi, t) < \epsilon$ *and* $\eta(x_k - \xi, t) < \epsilon$ *for* $k \geq k_0$*. It is represented symbolically by* (τ, υ, η)-$\lim_{k\to\infty} x_k = \xi$ *or* $x_k \xrightarrow{(\tau,\upsilon,\eta)} \xi$.

Further overview of ideal convergence was given by Kişi[15]

Definition 2.8. [15] *Let* $(\mathbb{S}, \aleph, \oplus, \odot)$ *be a NNS with neutrosophic norm* (τ, υ, η) *and* I *is an admissible ideal. A sequence* $x = \{x_k\}$ *from* $\mathbb{S}$ *has been ideal convergent to* $\xi \in \mathbb{S}$ *corresponding to neutrosophic norm* (τ, υ, η) *provided for every* $\epsilon > 0$ *and* $t > 0$ *satisfying*

$$\{k \in \mathbb{N} : \tau(x_k - \xi, t) \leq 1 - \epsilon, \upsilon(x_k - \xi, t) \geq \epsilon \text{ or } \eta(x_k - \xi, t) \geq \epsilon\} \in I.$$

It is represented symbolically by $I_{(\tau,\upsilon,\eta)}$-$\lim_{k\to\infty} x_k = \xi$ *or* $x_k \xrightarrow{I_{(\tau,\upsilon,\eta)}} \xi$.

Definition 2.9. [23] *A mapping* $M : [0,\infty) \to [0,\infty)$ *is called an Orlicz function provided* M *is non-decreasing, continuous, and convex function satisfying (i)* $M(0) = 0$*, (ii)* $M(x) > 0$ *for* $x > 0$*, and (iii)* $M(x) \to \infty$ *as* $x \to \infty$.

If we use inequality $M(x + y) \leq M(x) + M(y)$ in place of convexity for function M then, M is said to be a modulus function.

An Orlicz function M holds $\triangle^2$-condition, if we can get some constant $Q > 0$, satisfying $M(Ls) \leq QLM(s)$, for some $s > 0$ and $L > 1$.

Definition 2.10. [19] *The sequence space l_M using Orlicz function M is given by*

$$l_M = \left\{ \{x_k\} \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\mu}\right) < \infty, \text{ where } \mu > 0 \right\}$$

is reduced to Banach space using following norm, named as Orlicz sequence space.

$$\|\{x_k\}\| = \inf\left\{ \mu > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\mu}\right) \le 1 \right\}.$$

The properties of l_M space have a close association with l_p space for $M(x) = x^p$, where $1 \le p < \infty$.

Definition 2.11. [25] *Zweier sequence spaces for single sequences were defined considering the sequence $y = \{y_k\}$, which is obtained by Z^p-transformation on any sequence $x = \{x_k\}$ i.e $y_k = px_k + (1-p)x_{k-1}$ and $y_{-1} = 0$, $1 < p < \infty$. The transformation Z^p represents a matrix $Z^p = \{z_{jk}\}$, where*

$$z_{jk} = \begin{cases} p & j = k \\ 1-p & j-1 = k \\ 0 & otherwise \end{cases}$$

With Z^p-transformation the Z and Z_0 Zweier sequence spaces are given by

$$Z = \{\{x_k\} \in w : Z^p x \in c, \ x = \{x_k\}\},$$

$$Z_0 = \{\{x_k\} \in w : Z^p x \in c_0, \ x = \{x_k\}\}.$$

3. Main Results

This section endeavor to define Zweier ideal convergence on a NNS $(\mathbb{S}, \aleph, \oplus, \odot)$ and establishing some of its important properties. First, we present Zweier ideal convergent sequence spaces $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ and $Z^{I-NNS}_{0(\tau,\upsilon,\eta)}(M)$, which are defined using Orlicz function M as given below:

$$Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M) = \left\{ \{x_k\} \in w : \left\{ \begin{array}{c} k \in \mathbb{N} : M\left(\frac{\tau(x_k - L, t)}{\mu}\right) \le 1 - \epsilon, \\ \text{or } M\left(\frac{\upsilon(x_k - L, t)}{\mu}\right) \ge \epsilon, \\ M\left(\frac{\eta(x_k - L, t)}{\mu}\right) \ge \epsilon, \text{for some } \mu > 0 \text{ and } L \end{array} \right\} \in I \right\},$$

$$Z^{I-NNS}_{0(\tau,\upsilon,\eta)}(M) = \left\{ \{x_k\} \in w : \left\{ \begin{array}{c} k \in \mathbb{N} : M\left(\frac{\tau(x_k, t)}{\mu}\right) \le 1 - \epsilon, \text{or } M\left(\frac{\upsilon(x_k, t)}{\mu}\right) \ge \epsilon, \\ M\left(\frac{\eta(x_k, t)}{\mu}\right) \ge \epsilon, \text{ for some } \mu > 0 \text{ and } L \end{array} \right\} \in I \right\}.$$

The following results are established on $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ and $Z^{I-NNS}_{0(\tau,\upsilon,\eta)}(M)$ sequence spaces.

Theorem 3.1. *Let $(\mathbb{S}, \aleph, \oplus, \odot)$ be a NNS with neutrosophic norm (τ, υ, η) and I be an admissible ideal. Then spaces $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ and $Z^{I-NNS}_{0(\tau,\upsilon,\eta)}(M)$ are linear spaces.*

Proof. First we establish result for space $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ and then for $Z^{I-NNS}_{0(\tau,\upsilon,\eta)}(M)$ result can be easily available. Let α and β be scalars. Take $x = \{x_k\}$ and $y = \{y_k\}$ sequences from $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$. For given $\epsilon > 0$ and $t > 0$ we have $A_1, A_2 \in I$, where

$$A_1 = \left\{ \begin{array}{l} k \in \mathbb{N} : M\left(\dfrac{\tau(x_k - \xi_1, \frac{t}{2|\alpha|})}{\mu_1}\right) \leq 1 - \epsilon, \text{ or } M\left(\dfrac{\upsilon(x_k - \xi_1, \frac{t}{2|\alpha|})}{\mu_1}\right) \geq \epsilon, \\ \qquad M\left(\dfrac{\eta(x_k - \xi_1, \frac{t}{2|\alpha|})}{\mu_1}\right) \geq \epsilon, \text{ for some } \mu_1 > 0 \end{array} \right\},$$

$$A_2 = \left\{ \begin{array}{l} k \in \mathbb{N} : M\left(\dfrac{\tau(x_k - \xi_2, \frac{t}{2|\beta|})}{\mu_2}\right) \leq 1 - \epsilon, \text{ or } M\left(\dfrac{\upsilon(x_k - \xi_2, \frac{t}{2|\beta|})}{\mu_2}\right) \geq \epsilon, \\ \qquad M\left(\dfrac{\eta(x_k - \xi_2, \frac{t}{2|\beta|})}{\mu_2}\right) \geq \epsilon, \text{ for some } \mu_2 > 0 \end{array} \right\}.$$

Therefore, $A_1^c, A_2^c \in \mathbb{F}(I)$. Take $A_3 = A_1 \cup A_2$ so that $A_3 \in I$. Consequently, $A_3 \neq \emptyset$ in $\mathbb{F}(I)$. For given $\epsilon > 0$ choose $\lambda > 0$ with $(1-\epsilon) \oplus (1-\epsilon) > 1 - \lambda$ and $\epsilon \odot \epsilon < \lambda$. Now for $\mu_3 = \max\{\, 2\,|\alpha|\,\mu_1,\, 2\,|\beta|\,\mu_2 \,\}$ we will show that

$$\begin{aligned} A_3^c \subset \{k \in \mathbb{N} : & M\left(\frac{\tau(\alpha x_k + \beta y_k - (\alpha\xi_1 + \beta\xi_2), t)}{\mu_3}\right) > 1 - \lambda, \text{or} \\ & M\left(\frac{\upsilon(\alpha x_k + \beta y_k - (\alpha\xi_1 + \beta\xi_2), t)}{\mu_3}\right) < \lambda, \\ & M\left(\frac{\eta(\alpha x_k + \beta y_k - (\alpha\xi_1 + \beta\xi_2), t)}{\mu_3}\right) < \lambda\} \in I. \end{aligned}$$

Let $m \in A_3^c$

$$M\left(\frac{\tau(x_m - \xi_1, \frac{t}{2|\alpha|})}{\mu_3}\right) > 1 - \epsilon, \text{or } M\left(\frac{\upsilon(x_m - \xi_1, \frac{t}{2|\alpha|})}{\mu_3}\right) < \epsilon, M\left(\frac{\eta(x_m - \xi_1, \frac{t}{2|\alpha|})}{\mu_3}\right) < \epsilon,$$

and

$$M\left(\frac{\tau(y_m - \xi_2, \frac{t}{2|\beta|})}{\mu_3}\right) > 1 - \epsilon, \text{or } M\left(\frac{\upsilon(y_m - \xi_2, \frac{t}{2|\beta|})}{\mu_3}\right) < \epsilon, M\left(\frac{\eta(y_m - \xi_2, \frac{t}{2|\beta|})}{\mu_3}\right) < \epsilon.$$

Now, we have

$$\begin{aligned} M\left(\frac{\tau\left(\alpha x_k + \beta y_k - (\alpha\xi_1 + \beta\xi_2), t\right)}{\mu_3}\right) &\geq M\left(\frac{\tau\left(\alpha x_m - \alpha\xi_1, \frac{t}{2}\right)}{\mu_3}\right) \oplus M\left(\frac{\tau\left(\beta y_m - \beta\xi_2, \frac{t}{2}\right)}{\mu_3}\right) \\ &= M\left(\frac{\tau\left(x_m - \xi_1, \frac{t}{2|\alpha|}\right)}{\mu_3}\right) \oplus M\left(\frac{\tau\left(y_m - \xi_2, \frac{t}{2|\beta|}\right)}{\mu_3}\right) \\ &> (1-\epsilon) \oplus (1-\epsilon) \\ &> 1 - \lambda, \end{aligned}$$

$$
\begin{aligned}
M\left(\frac{\upsilon(\alpha x_k+\beta y_k-(\alpha\xi_1+\beta\xi_2),t)}{\mu_3}\right) &\leq M\left(\frac{\upsilon\left(\alpha x_m-\alpha\xi_1,\frac{t}{2}\right)}{\mu_3}\right)\odot M\left(\frac{\upsilon\left(\beta y_m-\beta\xi_2,\frac{t}{2}\right)}{\mu_3}\right)\\
&= M\left(\frac{\upsilon\left(x_m-\xi_1,\frac{t}{2|\alpha|}\right)}{\mu_3}\right)\odot M\left(\frac{\upsilon\left(y_m-\xi_2,\frac{t}{2|\beta|}\right)}{\mu_3}\right)\\
&< \epsilon\odot\epsilon\\
&< \lambda,
\end{aligned}
$$

and

$$
\begin{aligned}
M\left(\frac{\eta(\alpha x_k+\beta y_k-(\alpha\xi_1+\beta\xi_2),t)}{\mu_3}\right) &\leq M\left(\frac{\eta\left(\alpha x_m-\alpha\xi_1,\frac{t}{2}\right)}{\mu_3}\right)\odot M\left(\frac{\eta\left(\beta y_m-\beta\xi_2,\frac{t}{2}\right)}{\mu_3}\right)\\
&= M\left(\frac{\eta\left(x_m-\xi_1,\frac{t}{2|\alpha|}\right)}{\mu_3}\right)\odot M\left(\frac{\eta\left(y_m-\xi_2,\frac{t}{2|\beta|}\right)}{\mu_3}\right)\\
&< \epsilon\odot\epsilon\\
&< \lambda.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
A_3^c\subset\{k\in\mathbb{N}: &M\left(\frac{\tau(\alpha x_k+\beta y_k-(\alpha\xi_1+\beta\xi_2),t)}{\mu_3}\right)>1-\lambda,\ \text{or}\\
&M\left(\frac{\upsilon(\alpha x_k+\beta y_k-(\alpha\xi_1+\beta\xi_2),t)}{\mu_3}\right)<\lambda,\\
&M\left(\frac{\eta(\alpha x_k+\beta y_k-(\alpha\xi_1+\beta\xi_2),t)}{\mu_3}\right)<\lambda\}\in I.
\end{aligned}
$$

Hence, $Z_{(\tau,\upsilon,\eta)}^{I-NNS}(M)$ proved to be a linear space. □

Theorem 3.2. *Let $(\mathbb{S},\aleph,\oplus,\odot)$ be a NNS with neutrosophic norm (τ,υ,η) and I be an admissible ideal. Then, every open ball $B_x(r,t)(M)$ at centre x with radius r corresponding to t is an open set in a $Z_{(\tau,\upsilon,\eta)}^{I-NNS}(M)$.*

Proof. Consider $y'\in B_x^c(r,t)(M)$,

$$
B_x(r,t)(M)=\left\{y\in\mathbb{S}:\left\{\begin{matrix} k\in\mathbb{N}: M\left(\frac{\tau(y-x,t)}{\mu}\right)\leq 1-r, \text{or}\, M\left(\frac{\upsilon(y-x,t)}{\mu}\right)\geq r,\\ M\left(\frac{\eta(y-x,t)}{\mu}\right)\geq r\end{matrix}\right\}\in I\right\}.
$$

Then

$$
M\left(\frac{\tau(y'-x,t)}{\mu}\right)>1-r, M\left(\frac{\upsilon(y'-x,t)}{\mu}\right)<r \text{ and } M\left(\frac{\eta(y'-x,t)}{\mu}\right)<r.
$$

Also, there exists $t_0\in(0,1)$

$$
M\left(\frac{\tau(y'-x,t_0)}{\mu}\right)>1-r, M\left(\frac{\upsilon(y'-x,t_0)}{\mu}\right)<r \text{ and } M\left(\frac{\eta(y'-x,t_0)}{\mu}\right)<r.
$$

Choose $r_0=M\left(\frac{\tau(y'-x,t_0)}{\mu}\right)$ so that $r_0>1-r$.
Further, we can get $r'\in(0,1)$ with $r_0>1-r'>1-r$.
In-case $r_0>1-r'$ we can get $r_1,r_2\in(0,1)$ with $r_0\oplus r_1>1-r'$ and $(1-r_0)\odot(1-r_1)\leq r'$.

Take $r_3 = \max\{r_1, r_2\}$ and consider a ball $B^c_{y'}(1 - r_0, t - t_0)(M)$.
For result we try to obtain $B^c_{y'}(1 - r_3, t - t_0)(M) \subset B^c_x(r, t)(M)$.
Consider $z \in B^c_{y'}(1 - r_3, t - t_0)(M)$. Then

$$M\left(\frac{\tau(z - y', t - t_0)}{\mu}\right) > r_3, M\left(\frac{\upsilon(z - y', t - t_0)}{\mu}\right) < 1 - r_3 \text{ and } M\left(\frac{\eta(z - y', t - t_0)}{\mu}\right) < 1 - r_3.$$

$$\begin{aligned} M\left(\frac{\tau(z - x, t)}{\mu}\right) &\geq M\left(\frac{\tau(y' - x, t_0)}{\mu}\right) \oplus M\left(\frac{\tau(z - y', t - t_0)}{\mu}\right) \\ &\geq r_0 \oplus r_3 \\ &\geq r_0 \oplus r_1 \\ &> 1 - r' \\ &> 1 - r, \\ M\left(\frac{\upsilon(z - x, t)}{\mu}\right) &\leq M\left(\frac{\upsilon(y' - x, t_0)}{\mu}\right) \odot M\left(\frac{\upsilon(z - y', t - t_0)}{\mu}\right) \\ &\leq (1 - r_0) \odot (1 - r_3) \\ &\leq 1 - r_0 \odot 1 - r_2 \\ &< r' \\ &< r, \end{aligned}$$

and

$$\begin{aligned} M\left(\frac{\eta(z - x, t)}{\mu}\right) &\leq M\left(\frac{\eta(y' - x, t_0)}{\mu}\right) \odot M\left(\frac{\eta(z - y', t - t_0)}{\mu}\right) \\ &\leq (1 - r_0) \odot (1 - r_3) \\ &\leq 1 - r_0 \odot 1 - r_2 \\ &< r' \\ &< r. \end{aligned}$$

Hence, $z \in B^c_x(r, t)(M)$. Thus, $B^c_{y'}(1 - r_3, t - t_0)(M) \subset B^c_x(r, t)(M)$. □

Definition 3.1. *Let $(\mathbb{S}, \aleph, \oplus, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). Define*

$$\tau_{(\tau,\upsilon,\eta)} = \{A \subset Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M) : \forall x \in A\ \exists\ t > 0 \text{ and } r \in (0, 1)\ s.t.\ B^c_x(r, t)(M) \subset A\}$$

which is topology on $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$.

Theorem 3.3. *Let $(\mathbb{S}, \aleph, \oplus, \odot)$ be a NNS with neutrosophic norm (τ, υ, η) and I be an admissible ideal. Then topology $\tau_{(\tau,\upsilon,\eta)}$ on $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ is first countable.*

Proof. Consider set $\{B_x\left(\frac{1}{n}, \frac{1}{n}\right)(M) : n = 1, 2, 3 \ldots\}$, which is a local base at x. Hence, topology $\tau_{(\tau,\upsilon,\eta)}$ on $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ becomes first countable. □

Theorem 3.4. *$Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ and $Z^{I-NNS}_{0(\tau,\upsilon,\eta)}(M)$ are Hausdorff spaces.*

Proof. First we establish our result for $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ and then extend to $Z^{I-NNS}_{0(\tau,\upsilon,\eta)}(M)$ as a particular case.
Take $x, y \in Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$; $x \neq y$. Then $0 < M\left(\frac{\tau(x-y,t)}{\mu}\right) < 1$, $0 < M\left(\frac{\upsilon(x-y,t)}{\mu}\right) < 1$ and $0 < M\left(\frac{\eta(x-y,t)}{\mu}\right) < 1$.

Put $r_1 = M\left(\frac{\tau(x-y,t)}{\mu}\right)$, $r_2 = M\left(\frac{\upsilon(x-y,t)}{\mu}\right)$ and $r_3 = M\left(\frac{\eta(x-y,t)}{\mu}\right)$.
Further, we choose $r = \max\{r_1, 1-r_2, 1-r_3\}$. For $r_0 \in (0,1)$ there exists r_4 and r_5 along $r_4 \oplus r_5 \geq r_0$ and $(1-r_4) \odot (1-r_5) \leq 1-r_0$.
Take $r_6 = \max\{r_4, r_5\}$ and consider $B_x(1-r_6, t/2)(M)$ and $B_y(1-r_6, t/2)(M)$.
Clearly, $B_x(1-r_6, t/2)(M) \cap B_y(1-r_6, t/2)(M) \neq \emptyset$.
For $z \in B_x(1-r_6, t/2)(M) \cap B_y(1-r_6, t/2)(M)$ we have

$$\begin{aligned} r_1 = M\left(\frac{\tau(x-y,t)}{\mu}\right) &\geq M\left(\frac{\tau(z-x,t/2)}{\mu}\right) \oplus M\left(\frac{\tau(z-y,t/2)}{\mu}\right) \\ &\geq r_6 \oplus r_6 \\ &\geq r_4 \oplus r_5 \\ &\geq r_0 \\ &> r, \end{aligned}$$

$$\begin{aligned} r_2 = M\left(\frac{\upsilon(x-y,t)}{\mu}\right) &\leq M\left(\frac{\upsilon(z-x,t/2)}{\mu}\right) \odot M\left(\frac{\upsilon(z-y,t/2)}{\mu}\right) \\ &\leq (1-r_6) \odot (1-r_6) \\ &\leq (1-r_4) \odot (1-r_5) \\ &\leq 1-r_0 \\ &< r, \end{aligned}$$

and

$$\begin{aligned} r_3 = M\left(\frac{\eta(x-y,t)}{\mu}\right) &\leq M\left(\frac{\eta(z-x,t/2)}{\mu}\right) \odot M\left(\frac{\eta(z-y,t/2)}{\mu}\right) \\ &\leq (1-r_6) \odot (1-r_6) \\ &\leq (1-r_4) \odot (1-r_5) \\ &\leq 1-r_0 \\ &< r. \end{aligned}$$

Which leads to a contradiction. This provides, $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ is Hausdorff space. ☐

Theorem 3.5. *Let $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ be a NNS and $\tau_{(\tau,\upsilon,\eta)}$ be topology on $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$. Then sequence $x = \{x_k\}$ from $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ converges to ξ iff $M\left(\frac{\tau(x_k-\xi,t)}{\mu}\right) \to 1$, $M\left(\frac{\upsilon(x_k-\xi,t)}{\mu}\right) \to 0$ and $M\left(\frac{\eta(x_k-\xi,t)}{\mu}\right) \to 0$ as $k \to \infty$.*

Proof. Suppose sequence $x = \{x_k\}$ converges to ξ. Then we get $n_0 \in \mathbb{N}$ for some $r \in (0,1)$, which gives $\{x_k\} \in B_\xi(r,t)(M)$ for $k \geq n_0$.

$$A(r,t)(M) = \left\{ \begin{array}{c} k \in \mathbb{N} : M\left(\frac{\tau(x_k-\xi,t)}{\mu}\right) \leq 1-r, \text{ or } M\left(\frac{\upsilon(x_k-\xi,t)}{\mu}\right) \geq r, \\ M\left(\frac{\eta(x_k-\xi,t)}{\mu}\right) \geq r, \text{ for some } \mu > 0 \end{array} \right\}.$$

such that $A(r,t)(M) \in I$ *i.e* $A^c(r,t)(M) \in \mathbb{F}(I)$.
Then we have $M\left(\frac{\tau(x_k-\xi,t)}{\mu}\right) > 1-r$, $M\left(\frac{\upsilon(x_k-\xi,t)}{\mu}\right) < r$ and $M\left(\frac{\eta(x_k-\xi,t)}{\mu}\right) < r$,
i.e $M\left(\frac{\tau(x_k-\xi,t)}{\mu}\right) \to 1$, $M\left(\frac{\upsilon(x_k-\xi,t)}{\mu}\right) \to 0$ and $M\left(\frac{\eta(x_k-\xi,t)}{\mu}\right) \to 0$ as $k \to \infty$.

Conversely, if for every $t > 0$ we obtain $M\left(\frac{\tau(x_k-\xi,t)}{\mu}\right) \to 1$, $M\left(\frac{\upsilon(x_k-\xi,t)}{\mu}\right) \to 0$ and $M\left(\frac{\eta(x_k-\xi,t)}{\mu}\right) \to 0$ as $k \to \infty$. Consequently, for $r \in (0,1)$ we get $n_0 \in \mathbb{N}$ satisfying $M\left(\frac{\tau(x_k-\xi,t)}{\mu}\right) > 1-r$, $M\left(\frac{\upsilon(x_k-\xi,t)}{\mu}\right) < r$ and $M\left(\frac{\tau(x_k-\xi,t)}{\mu}\right) < r$ for $k \geq n_0$.
Therefore, $x_k \in B^c_\xi(r,t)(M)$ for $k \geq n_0$, and consequently, sequence $x = \{x_k\}$ converges to ξ. □

Theorem 3.6. *Sequence $x = \{x_k\}$ from $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ is I-convergent to ξ iff for each $\epsilon > 0$ and $t > 0$ there exists $k_0(x,\epsilon,t)$ with*

$$\left\{\begin{array}{c} k_0 \in \mathbb{N} : M\left(\dfrac{\tau(x_{k_0}-\xi,t)}{\mu}\right) > 1-\epsilon, \textit{ and } M\left(\dfrac{\upsilon(x_{k_0}-\xi,t)}{\mu}\right) < \epsilon, \\ M\left(\dfrac{\eta(x_{k_0}-\xi,t)}{\mu}\right) < \epsilon, \textit{ for some } \mu > 0 \end{array}\right\} \in \mathbb{F}(I).$$

Proof. Let $I_{(\tau,\upsilon,\eta)}\text{-}\lim_{k\to\infty} x_k = \xi$. For given $\epsilon > 0$ choose $\lambda > 0$ with $(1-\epsilon) \oplus (1-\epsilon) > 1-\lambda$ and $\epsilon \odot \epsilon < \lambda$. Then for $x = \{x_k\}$ from $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$, define

$$A(\epsilon,t)(M) = \left\{\begin{array}{c} k \in \mathbb{N} : M\left(\dfrac{\tau(x_k-\xi,\frac{t}{2})}{\mu}\right) \leq 1-\epsilon, \text{ or } M\left(\dfrac{\upsilon(x_k-\xi,\frac{t}{2})}{\mu}\right) \geq \epsilon, \\ M\left(\dfrac{\eta(x_k-\xi,\frac{t}{2})}{\mu}\right) \geq \epsilon, \text{ for some } \mu > 0 \end{array}\right\},$$

such that $A(\epsilon,t)(M) \in I$ which implies $A^c(\epsilon,t)(M) \in \mathbb{F}(I)$, where

$$A^c(\epsilon,t)(M) = \left\{\begin{array}{c} k \in \mathbb{N} : M\left(\dfrac{\tau(x_k-\xi,\frac{t}{2})}{\mu}\right) > 1-\epsilon, \text{and } M\left(\dfrac{\upsilon(x_k-\xi,\frac{t}{2})}{\mu}\right) < \epsilon, \\ M\left(\dfrac{\eta(x_k-\xi,\frac{t}{2})}{\mu}\right) < \epsilon, \text{ for some } \mu > 0 \end{array}\right\}.$$

Conversely, choose $k_0 \in A^c(\epsilon,t)(M)$. Then $M\left(\frac{\tau(x_{k_0}-\xi,\frac{t}{2})}{\mu}\right) > 1-\epsilon$, $M\left(\frac{\upsilon(x_{k_0}-\xi,\frac{t}{2})}{\mu}\right) < \epsilon$ and $M\left(\frac{\eta(x_{k_0}-\xi,\frac{t}{2})}{\mu}\right) < \epsilon$.
Now we want to show that a number k_0 exists provided

$$\left\{\begin{array}{c} k \in \mathbb{N} : M\left(\dfrac{\tau(x_k-x_{k_0},t)}{\mu}\right) \leq 1-\lambda, \text{or } M\left(\dfrac{\upsilon(x_k-x_{k_0},t)}{\mu}\right) \geq \lambda, \\ M\left(\dfrac{\eta(x_k-x_{k_0},t)}{\mu}\right) \geq \lambda, \text{ for some } \mu > 0 \end{array}\right\} \in I.$$

Hence, we define for each $x = \{x_k\} \in Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$

$$B(\lambda,t)(M) = \left\{\begin{array}{c} k \in \mathbb{N} : M\left(\dfrac{\tau(x_k-x_{k_0},t)}{\mu}\right) \leq 1-\lambda, \text{or } M\left(\dfrac{\upsilon(x_k-x_{k_0},t)}{\mu}\right) \geq \lambda, \\ M\left(\dfrac{\eta(x_k-x_{k_0},t)}{\mu}\right) \geq \lambda, \text{ for some } \mu > 0 \end{array}\right\}.$$

Now we want to get $B(\lambda,t)(M) \subset A(\epsilon,t)(M)$. Consider $B(\lambda,t)(M) \subset A^c(\epsilon,t)(M)$. This provides $m \in B(\lambda,t)(M)$ but $x \notin A(\epsilon,t)(M)$ which gives $M\left(\frac{\tau(x_m-x_{k_0},t)}{\mu}\right) < 1-\lambda$, and

$M\left(\frac{\tau(x_m-\xi,\frac{t}{2})}{\mu}\right) > 1-\epsilon$.
Therefore,

$$1-\lambda \geq M\left(\frac{\tau(x_m - x_{k_0}, t)}{\mu}\right) \geq M\left(\frac{\tau(x_m - \xi, \frac{t}{2})}{\mu}\right) \oplus M\left(\frac{\tau(x_{k_0} - \xi, \frac{t}{2})}{\mu}\right)$$
$$\geq (1-\epsilon) \oplus (1-\epsilon) > 1-\lambda$$

which fails to exist.

Also $M\left(\frac{\upsilon(x_m-x_{k_0},t)}{\mu}\right) \geq \lambda$, and $M\left(\frac{\upsilon(x_m-\xi,\frac{t}{2})}{\mu}\right) < \epsilon$.

$$\lambda \leq M\left(\frac{\upsilon(x_m - x_{k_0}, t)}{\mu}\right) \leq M\left(\frac{\upsilon(x_m - \xi, \frac{t}{2})}{\mu}\right) \odot M\left(\frac{\upsilon(x_{k_0} - \xi, \frac{t}{2})}{\mu}\right)$$
$$\leq \epsilon \odot \epsilon < \lambda$$

which fails to exist.

Further $M\left(\frac{\eta(x_m-x_{k_0},t)}{\mu}\right) < 1-\lambda$, and $M\left(\frac{\tau(x_m-\xi,\frac{t}{2})}{\mu}\right) > 1-\epsilon$.

$$\lambda \leq M\left(\frac{\eta(x_m - x_{k_0}, t)}{\mu}\right) \leq M\left(\frac{\eta(x_m - \xi, \frac{t}{2})}{\mu}\right) \odot M\left(\frac{\eta(x_{k_0} - \xi, \frac{t}{2})}{\mu}\right)$$
$$\leq \epsilon \odot \epsilon < \lambda$$

which fails to exist.
Hence, $B(\lambda,t)(M) \subset A(\epsilon,t)(M)$. Therefore, $B(\lambda,t)(M) \in I$ as $A(\epsilon,t)(M) \in I$. $\square$

4. Conclusions

The above article presents the sequence space on Zweier ideal convergence defined by Orlicz function on neutrosophic normed spaces. The various topological properties related Zweier ideal convergent sequence spaces $Z^{I-NNS}_{(\tau,\upsilon,\eta)}(M)$ and $Z^{I-NNS}_{0(\tau,\upsilon,\eta)}(M)$ has been discussed.

References

[1] Ahmad, M., Idrisi, M. I., Sirohi, A. K., (2023), Generalized neutrosophic ideal convergent sequence spaces, J. Math. Computer Sci., 30, pp. 332–339.
[2] Balcerzak, M., Dems, K., Komisarski, A., (2007), Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl., 32(1), pp. 715–729.
[3] Banerjee, A. K., Banerjee, A., (2018), A study on I-Cauchy sequences and I-divergence in S-metric spaces, Malaya J. Mat., 6(2), pp. 326–330.
[4] Fast H., (1951), Sur la convergence statistique, Colloq. Math., 2(3-4), pp. 241–244.
[5] Hazarika, B., Tamang, K., Singh, B. K., (2014), Zweier ideal convergent sequence spaces defined by Orlicz function, J. Math. Comp. Sci., 8(3), pp. 307–318.
[6] Hazarika, B., Tamanag, K., (2017), On Zweier generalized difference ideal convergent sequences in a locally convex space defined by Musielak-Orlicz function, Bol. Soc. Parana. Mat., 35(2), pp. 19–37.
[7] Hazarika, B., Tamanag, K., (2016), On Zweier Sequence Spaces and de la Vallée-Poussin Mean of Order α, Natl. Acad. Sci. Lett., 39, pp. 371–374.
[8] Hazarika, B., Tamanag, K., (2016), On Zweier statistically convergent sequences of fuzzy numbers and de La Vallée-Poussin mean of order α, Ann. Fuzzy Math. Inform., 11(4), pp. 541–555.
[9] Khan, V. A. and Ahmad, M., (2020), On (λ;μ)-Zweier ideal convergence in intuitionistic fuzzy normed space, Yugosl. J. Oper. Res., 30(4), pp. 413–427.
[10] Khan, V. A., Ebadullah, K., (2013), Zweier I-convergent sequence spaces defined by Orlicz function, Analysis, 3, pp. 251–261.
[11] Khan, V. A., Ebadullah, K., Yasmeen, (2014), On Zweier I-convergent sequence spaces, Proyecciones (Antofagasta), 33(3), pp. 259–276.

[12] Khan, V. A., Yasmeen, Fatima, H., Ahamd, A., (2016), Intuitionistic fuzzy Zweier I-convergent double sequence spaces defined by modulus function, Cogent Mathematics, 3(1), pp. 1–9.
[13] Kirişci, M., Şimşek, N., (2020), Neutrosophic metric spaces, Math. Sci., 14, pp. 241–248.
[14] Kirişci, M., Şimşek, N., (2020), Neutrosophic normed spaces and statistical convergence, J. Anal., 28(4), pp. 1059–1073.
[15] Kişi, Ö., (2021), Ideal convergence of sequences in neutrosophic normed spaces, J. Intell. Fuzzy Syst., 41(2), pp. 2581–2590.
[16] Kostyrko, P., Máčaj, M., Šalát, T., Sleziak, M., (2005), I-convergence and extremal I-limit points, Math. Slovaca, 55(4), pp. 443–464.
[17] Kostyrko P., Šalát T. Wilczyński W., (2000), I-convergence, Real analysis exchange, 26(2), pp. 669–685.
[18] Lahiri, B. K., Das, P., (2003), Further results on I-limit superior and limit inferior, Math. Commun., 2003, 8(2), pp. 151–156.
[19] Lindenstrauss, J., Tzafriri, L., (1971), Orlicz sequence spaces, Classical Banach Spaces I.
[20] Mohiuddine, S. A., Hazarika, B., (2017), Some classes of ideal convergent sequences and generalized difference matrix operator, Filomat, 31(6), pp. 1827–1834.
[21] Mursaleen, M., Alotaibi, A., (2011), On I-convergence in random 2-normed spaces, Math. Slovaca, 2011, 61(6), pp. 933–940.
[22] Mursaleen, M., Mohiuddine, S. A., (2012), On ideal convergence in probabilistic normed spaces, Math. Slovaca, 62(1), pp. 49–62.
[23] Orlicz, W., (1936), Über raume (LM), Bull. Int. Acad. Polon. Sci. A, 93, pp. 107.
[24] Schweizer, B., Sklar, A., (1960), Statistical metric spaces, Pac. J. Math., 10, pp. 313-334.
[25] Şengönül, M., (2007), On the Zweier sequence space, Demonstratio Math., 40(1), pp. 181–196.
[26] Smarandache, F., (2006), Neutrosophic set-a generalization of the intuitionistic fuzzy set, 2006 IEEE international conference on granular computing, pp. 38–42.
[27] Smarandache, F., (2013), Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability.
[28] Subramanian, N., Altin, Y., Bektas, C., (2009), Generalized difference sequence spaces defined by Orlicz functions, Tamsui Oxf. J. Math. Sci., 23(3), pp. 325–342.
[29] Tamang, K. and Hazarika, B., (2016), Some geometric properties of lacunary Zweier Sequence Spaces of order α, Proyecciones (Antofagasta), 35(4), pp. 483–492.
[30] Tamang, K. and Hazarika, B., (2017), On generalized difference Zweier ideal convergent sequences space defined by Musielak-Orlicz functions, Bol. da Soc. Parana. de Mat., 35(2), pp. 19–37.
[31] Tripathy, B. C., Altin, Y., Et, M., (2008), Generalized difference sequence spaces on seminormed space defined by Orlicz functions, Math. Slovaca, 58(3), pp. 315–324.

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