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A VARIABLE SAMPLING PLAN BASED ON QUADRATIC RANDOM DECISION FUNCTION USING CLASSICAL AND BAYESIAN ESTIMATES UNDER TYPE II HYBRID CENSORED SAMPLES

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Abstract. This article endeavors to develop a single variable sampling plan when the lifetime of product follows the exponential distribution. The sampling plan has been developed by assuming that the life test is under type II hybrid censoring. Based on a two-sided decision function with a quadratic random doubt zone and a suitable loss function, an explicit expression for the Bayes risk has been determined using classical and Bayesian estimate with the Linex loss function. In order to obtain the optimal sampling plan, a simple algorithm based on the grid search method was provided. Finally, a numerical simulation with extensive tables and a comparison of performance have been provided to illustrate the proposed model.

Keywords: Two-sided decision function, quadratic random doubt zone, Linex loss function, optimal sampling plan, type II hybrid censoring.

AMS Subject Classification: 62D05, 62F15, 62N05.

1. Introduction

Acceptance sampling plan plays a vital role in quality control engineering as it can determine the good quality of both products and processes. In other terms, the objective of acceptance sampling is twofold: drawing the items' optimal number and determining the batch's quality. There are different criteria that develop sampling plans. Decision theory approach is a very effective method to quality control. Such that, the sampling plan is determined by making an optimal decision on the basis of maximizing the return or minimizing the loss.

In recent years, there has been an increasing interest in Bayesian sampling plans based on the lifetime censored data, such as, the sampling plan based on type II censored sample is discussed in Lam (1990), the sampling plan based on type I censored sample is described

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in Lam (1994). Interval censored sample in Chen et al. (2015). The type I hybrid censored sample was initially introduced by Epstein (1954). Child et al. (2003) computed the exact distribution of the maximum likelihood estimator (MLE) of the expected lifetime for the case when the lifetime of components follows exponential distribution under type I and type II hybrid censoring are provided. Lin et al. (2008) have studied the type I and type II hybrid censoring for quadratic loss function. Also, Lin et al. (2011) developed Bayesian sampling plan for the exponential distribution with progressive hybrid censoring based on the classical one-sided decision function and the quadratic loss function. The type I hybrid censored sample was initially introduced by Liang et al. (2013). Further, with a curtailed decision function, Chen et al. (2017) discussed Bayesian sampling based on type II censored samples. Modified type II hybrid censoring where the inspection interrupted at the censoring time $\tau = min\left\{max\left(X_{(m)}, t\right), X_{(n)}\right\}$ has been provided by Yang et al. (2017). For exponential distribution under type I censoring and type I hybrid censoring a new shrinkage estimator for the expected lifetime studied by Prajapati et al. (2018), which always exists even if no failure occurs at the censoring time $\tau = min\left(X_{(m)}, t\right)$. Furthermore Prajapati et al. (2020) developed generalized type II hybrid censoring for the exponential distribution case, they employed the posterior expectation to construct a Bayesian one-sided decision function. Aslam (2019a, 2019b) have proposed acceptance sampling plan for variable and attribute using the neutrosophic statistics. Recently, Prajapati et al. (2021) discussed the Bayesian sampling plan under the balanced joint censoring scheme, where the lifetime of items exponentially distributed. This type of sampling plan can be applied where several types of products must be studied simultaneously. Aslam et al (2022) studied the acceptance sampling plan of a two-step process loss using neutrosophic statistics based on the operating characteristic curve function and the producer's and the consumer's risks. Chen et al. (2022, 2023) investigate Bayesian sampling plans for simple and random step-stress of accelerated life test based on censored data, which can provide more accurate and reliable results compared to traditional methods. The Bayesian sampling plan under competing risks data has been developed by Prajapati et al. (2023a). In addition, Prajapati et al. (2023b) investigated the Bayesian sampling plan based on random stress-change time model.

However, several single variable sampling plans have been improved in recent years, most improvements have been achieved by considering the one-sided decision function. Nonetheless, it is key to take into account that a doubt zone existed in the decision interval i.e. the minimum acceptable T_0 and the maximum rejectable T_1 surviving time are not equal. As given in the Figure 1:

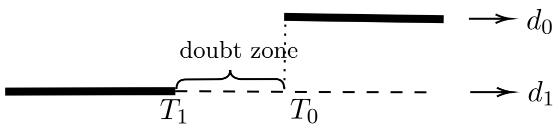

FIGURE 1. Schematic representation of a decision function with doubt zone

where d_0 and d_1 represent respectively the decisions of accepting and rejecting the batch. Belbachir and Benahmed (2022a, 2022b) investigated Bayesian sampling plans for exponential and Weibull distributions based on a linear random decision function.

In this paper, we develop a variable sampling plan using maximum likelihood and Bayesian estimates for the exponential distribution under type II hybrid censoring and based on a two-sided decision function with a quadratic doubt zone, such that the transition from d_1 to d_0 is done by a quadratic random function, as shown in Figure 2: The

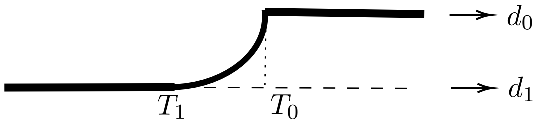

FIGURE 2. Schematic representation of a decision function with quadratic doubt zone

doubt zone is a new contribution essentially with the quadratic form, which moves towards acceptance in a smooth manner compared to the linear situation as given in Belbachir and Benahmed (2022a, 2022b). Furthermore, in this model, we will use two forms of estimators, one of which is Bayesian with the Linex loss function.

The rest of this paper is organized in the following way. In Section 2, the proposed model and all necessary computation of the different estimators for type II hybrid censoring are described. In Section 3, an explicit expressions for the Bayes risk based on the proposed decision function is obtained. In Section 4, an approximation method is suggested along with a finite algorithm for finding an optimal sampling plan. In Section 5, an illustrative example for the quadratic loss function is provided. In addition, some numerical results are introduced followed by a comparison of performance between the proposed sampling plan and the one of Yang et al. (2017). Finally, concluding remarks are given in Section 6.

2. Formulation of the model

Suppose that we have a batch of items prepared for inspection. The lifetime of each item is a random variable X which follows an exponential distribution $Exp(\lambda)$ with the following pdf:

$$f(x|\lambda) = \begin{cases} \lambda \exp(-\lambda x), & \text{for } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where the scale parameter λ is unknown. We suppose that λ has a conjugate prior gamma distribution, α and β are known, with the pdf:

$$g(\lambda; \alpha, \beta) = \begin{cases} \lambda^{\alpha-1} \exp(-\beta\lambda)\beta^{\alpha}/\Gamma(\alpha), & \text{for } \lambda > 0, \\ 0, & \text{otherwise,} \end{cases}$$

Let $(X_1, X_2, ..., X_n)$ be a sample of size n selected from a batch for life testing. Let $(X_{(1)}, X_{(2)}, ..., X_{(n)})$ be the corresponding order statistics. Here, we use the type II hybrid censored in this instance, and the observed data we obtain can be classified as either of the following two cases:

Data:

$$\begin{cases} \text{Case I: } \left\{X_{(1)} < X_{(2)} < \cdots < X_{(m)}\right\} & \text{if } X_{(m)} \geq t, \\ \text{Case II: } \left\{X_{(1)} < X_{(2)} < \cdots < X_{(D)}\right\} & \text{if } X_{(m)} < t, \text{where } m < D < n, X_{(D)} \leq t < X_{(D+1)} \end{cases}$$

Thus, the life test is terminated at the random time $\tau = min\left\{max\left(X_{(m)}, t\right), X_{(n)}\right\}$ where $X_{(m)}$ be the time of $m(\leq n)$-th failure, and D is the observed failures before time t.

Using the type II hybrid censored life testing, the likelihood function can be represented as:

$$l\left(Data|\lambda\right) = \begin{cases} \dfrac{n!}{(n-m)!}\lambda^m e^{-\lambda\left[\sum_{i=1}^m X_{(i)} + (n-m)X_{(m)}\right]} & \text{for } D = 0, 1, ..., m, \\ \dfrac{n!}{(n-D)!}\lambda^D e^{-\lambda\left[\sum_{i=1}^D X_{(i)} + (n-D)t\right]} & \text{for } D = m+1, m+2, ..., n \end{cases} \tag{1}$$

Hence, the MLE of the expected lifetime $\theta = 1/\lambda$ is obtained as:

$$\hat{\theta}^{ML} = \begin{cases} \frac{1}{m}\left(\sum_{i=1}^{m} X_{(i)} + (n-m)X_{(m)}\right) & \text{for } D = 0,1,...,m, \\ \frac{1}{D}\left(\sum_{i=1}^{D} X_{(i)} + (n-D)t\right) & \text{for } D = m+1, m+2, ..., n. \end{cases} \quad (2)$$

The distribution of $\hat{\theta}^{ML}$:

$$f_{\hat{\theta}^{ML}}(y) = \sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^j \binom{n}{d}\binom{d}{j} e^{-\lambda t(n-d+j)} g\left(y - a_{j,M}; M, \lambda M\right), \quad (3)$$

where $a_{j,M} = (n-d+j)t/M$ and $M = max\{d,m\}$. For more details see Childs et al. (2003).

In order to obtain a Bayesian estimation for the expected lifetime θ, we use the Linex loss function, that is, an extension of the squared loss function. The Linex loss function is applied as follows:

$$L(\delta) = exp(a\delta) - a\delta - 1;\ \delta = \hat{\theta} - \theta,\ \text{with } a \neq 0. \quad (4)$$

Here, we use the loss function $L(\delta)$; with $\delta = \frac{\hat{\theta}}{\theta} - 1$; this loss function has been used by several authors, e.g. see Boudjerda et al. (2017).

To compute the Bayes estimator, we shall minimize the posterior expected loss.
The posterior density function under the conjugate prior $g(\lambda;\alpha,\beta)$ is:

$$\pi(\lambda|Data) = \frac{l(Data|\lambda)\, g(\lambda;\alpha,\beta)}{\int_0^\infty l(Data|\lambda)\, g(\lambda;\alpha,\beta)d\lambda}$$

Thus, for Case I

$$\pi(\lambda|Data) = g\left(\lambda; \alpha+m, \beta + \sum_{i=1}^{m} X_{(i)} + (n-m)X_{(m)}\right),$$

and for Case II

$$\pi(\lambda|Data) = g\left(\lambda; \alpha+D, \beta + \sum_{i=1}^{D} X_{(i)} + (n-D)t\right),$$

Hence, the posterior expected loss is given by:

$$\rho = \int_0^\infty \left[exp(a\delta) - a\delta - 1\right]\pi(\lambda|Data)d\lambda$$

Substituting $\delta = \hat{\theta}/\theta - 1$, $\theta = 1/\lambda$ and $\hat{\theta} = 1/\hat{\lambda}$ we obtain:
For Case I

$$\begin{aligned}\rho &= \int_0^\infty \left[exp\left(a\frac{\lambda}{\hat{\lambda}} - a\right) - a\frac{\lambda}{\hat{\lambda}} + a - 1\right]\lambda^{\alpha+m-1}e^{-\lambda(\beta+Y_m)}(\beta+Y_m)^{\alpha+m}/\Gamma(\alpha+m)d\lambda \\ &= e^{-a}\left(\tfrac{\beta+Y_m}{\beta+Y_m-a/\hat{\lambda}}\right)^{\alpha+m} - \tfrac{a}{\hat{\lambda}}\tfrac{\alpha+m}{\beta+Y_m} + a - 1,\end{aligned}$$

where $Y_m = \beta + \sum_{i=1}^{m} X_{(i)} + (n-m)X_{(m)}$.
Derivating ρ with respect to $\hat{\lambda}$, we get:

$$\frac{\partial\rho}{\partial\hat{\lambda}} = -a(\beta+Y_m)e^{-a}\tfrac{(\beta+Y_m)/\hat{\lambda}^2}{(\beta+Y_m-a/\hat{\lambda})^2}\left(\tfrac{\beta+Y_m}{\beta+Y_m-a/\hat{\lambda}}\right)^{\alpha+m-1} - \tfrac{a}{\hat{\lambda}^2}\tfrac{\alpha+m}{\beta+Y_m}.$$

By solving $\frac{\partial \rho}{\partial \hat{\lambda}} = 0$, we obtain the Bayes estimator for θ as:

$$\hat{\theta}^{BL} = \frac{1}{\hat{\lambda}} = \frac{\beta + \sum_{i=1}^{m} X_{(i)} + (n-m)X_{(m)}}{a}\left[1 - e^{\frac{-a}{\alpha+m+1}}\right].$$

Similarly, for the Case II:

$$\hat{\theta}^{BL} = \frac{1}{\hat{\lambda}} = \frac{\beta + \sum_{i=1}^{D} X_{(i)} + (n-D)t}{a}\left[1 - e^{\frac{-a}{\alpha+D+1}}\right].$$

Hence

$$\hat{\theta}^{BL} = \begin{cases} \frac{\beta+\sum_{i=1}^{m} X_{(i)}+(n-m)X_{(m)}}{a}\left[1 - e^{\frac{-a}{\alpha+m+1}}\right] & \text{for } D = 0, 1, ..., m, \\ \frac{\beta+\sum_{i=1}^{D} X_{(i)}+(n-D)t}{a}\left[1 - e^{\frac{-a}{\alpha+D+1}}\right] & \text{for } D = m+1, m+2, ..., n. \end{cases} \tag{5}$$

Note that, $\hat{\theta}^{BL}$ can be written as:

$$\hat{\theta}^{BL} = \frac{\beta + M\hat{\theta}^{ML}}{a}\left[1 - e^{\frac{-a}{\alpha+M+1}}\right] \tag{6}$$

Using the linear transformation of random variables, the distribution of $\hat{\theta}^{BL}$ can be easily constructed as:

$$f_{\hat{\theta}^{BL}}(y) = \sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^j \binom{n}{d}\binom{d}{j} e^{-\lambda t(n-d+j)} g\left(y - a_{j,A,\beta}; M, \lambda A\right), \tag{7}$$

with $a_{j,A,\beta} = ((n-d+j)t + \beta)/A$ and $A = a\left[1 - e^{\frac{-a}{\alpha+M+1}}\right]^{-1}$.

2.1. **Random decision function and loss function.** Based on the observed data $\underline{x} = (x_{(1)}, x_{(2)}, ..., x_{(n)})$, a decision function $\delta(\underline{x})$ is made. We consider the following quadratic two-sided decision function:

$$\delta(\underline{x}) = \begin{cases} d_0, & \text{for } \hat{\theta} \geq T_0, \\ \begin{cases} d_1, & \text{with probability } p_{\hat{\theta}} \\ d_0, & \text{with probability } 1 - p_{\hat{\theta}} \end{cases} & \text{for } T_1 \leq \hat{\theta} < T_0, \\ d_1, & \text{for } \hat{\theta} < T_1, \end{cases} \tag{8}$$

where $p_{\hat{\theta}} = \left(\frac{T_0 - \hat{\theta}}{T_0 - T_1}\right)^2$, and $T_1 < T_0$.

In order to construct the sampling plan (n, m, t, T_0, T_1), the following loss function is considered:

$$L(\lambda, \delta(\underline{x})) = \begin{cases} nC_s - (n - D_{n,m,t})v_s + C_t\tau_{n,m,t} + \sum_{i=0}^{k} a_i\lambda^i, & \text{for } \delta(\underline{x}) = d_0, \\ nC_s - (n - D_{n,m,t})v_s + C_t\tau_{n,m,t} + C_r, & \text{for } \delta(\underline{x}) = d_1, \end{cases} \tag{9}$$

where, the random variable $\tau_{n,m,t} = min\left\{max\left(X_{(m)}, t\right), X_{(n)}\right\}$ is the censoring time and the random variable $D_{n,m,t}$ is the number of observed failures at the time $\tau_{n,m,t}$. The parameters C_s, C_t and C_r are positive constants and represent respectively the unit inspection cost, the cost per unit of time used for the test and the loss due to rejection of the batch, the quantity $a_0 + a_1\lambda + \cdots + a_k\lambda^k$ denotes the loss of accepting the batch and be positive and increasing in λ. When the life test was finished, the unfailure items can be reused and therefore have the salvage value v_s, where $v_s < C_s$.

3. Computation of the Bayes risk

With the loss function in (4), the Bayes risk of a sampling plan $R(n,m,t,T_0,T_1)$ can be computed as follows:

$$R(n,m,t,T_0,T_1) = E\{E\left[L(\lambda,\delta(\underline{x}))\right]\}$$
$$= E\left\{E\left[nC_s - (n-D_{n,m,t})v_s + C_t\tau_{n,m,t} + d_1C_r + (1-d_1)\sum_{i=0}^{k} a_i\lambda^i|\lambda\right]\right\}$$
$$= n\left(C_s - v_s\right) + v_sE\left[E\left(D_{n,m,t}|\lambda\right)\right] + C_tE\left[E\left(\tau_{n,m,t}|\lambda\right)\right] + \sum_{i=0}^{k} a_i\gamma^i$$
$$+ \sum_{i=0}^{k} E\left\{C_i\lambda^iE\left[1_{\{\hat{\theta}<T_1\}} + p_{\hat{\theta}}1_{\{T_1\le\hat{\theta}<T_0\}}|\lambda\right]\right\}$$
$$= n\left(C_s - v_s\right) + v_sE\left[E\left(D_{n,m,t}|\lambda\right)\right] + C_tE\left[E\left(\tau_{n,m,t}|\lambda\right)\right] + \sum_{i=0}^{k} a_i\gamma_i + r(n,m|d_1),$$

where $C_0 = C_r - a_0, C_i = -a_i$, for $i = 1,...,k$, and γ_i represents the i-th moment of λ.

3.1. **Bayes risk under ML estimator.** The expression of $R^{ML}(n,m,t,T_0,T_1)$ under ML estimation can be described in the following way:

$$R(n,m,t,T_0,T_1) = E\{E\left[L(\lambda,\delta(\underline{x}))\right]\}$$
$$= n\left(C_s - v_s\right) + v_sE\left[E\left(D_{n,m,t}|\lambda\right)\right] + C_tE\left[E\left(\tau_{n,m,t}|\lambda\right)\right] + \sum_{i=0}^{k} a_i\gamma_i + r^{ML}(n,m|d_1),$$

such that

$$r^{ML}(n,m|d_1) = \sum_{i=0}^{k} E\left\{C_i\lambda^iE\left[1_{\{\hat{\theta}<T_1\}} + p_{\hat{\theta}}1_{\{T_1\le\hat{\theta}<T_0\}}|\lambda\right]\right\}$$
$$= \sum_{i=0}^{k} C_i\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-\beta\lambda}\lambda^{\alpha+i-1}\left[\int_0^{T_1} f_{\hat{\theta}}(y)\,dy + \int_{T_1}^{T_0}\left(\frac{T_0-y}{T_0-T_1}\right)^2 f_{\hat{\theta}}(y)\,dy\right]d\lambda$$
$$= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}(-1)^jC_i\binom{n}{d}\binom{d}{j}\frac{\beta^\alpha M^M}{\Gamma(\alpha)\Gamma(M)}\int_0^\infty\left[\int_{a_{j,M}}^{T_1}\left(y-a_{j,M}\right)^{M-1}e^{-(\beta+My)\lambda}\lambda^{\alpha+M+i-1}dy\right.$$
$$\left.+\int_{T_1}^{T_0}\left(\frac{T_0-y}{T_0-T_1}\right)^2\left(y-a_{j,M}\right)^{M-1}e^{-(\beta+My)\lambda}\lambda^{\alpha+M+i-1}dy\right]d\lambda$$
$$= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}(-1)^jC_i\binom{n}{d}\binom{d}{j}\frac{\beta^\alpha M^M}{\Gamma(\alpha)\Gamma(M)}\int_0^\infty\left[\int_0^{T_1-a_{j,M}} y^{M-1}e^{-\left(\beta+My+Ma_{j,M}\right)\lambda}\lambda^{\alpha+M+i-1}dy\right.$$
$$\left.+\int_{T_1-a_{j,M}}^{T_0-a_{j,M}}\left(\frac{T_0-a_{j,M}-y}{T_0-T_1}\right)^2 y^{M-1}e^{-\left(\beta+My+Ma_{j,M}\right)\lambda}\lambda^{\alpha+M+i-1}dy\right]d\lambda$$
$$= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}(-1)^jC_i\binom{n}{d}\binom{d}{j}\frac{\beta^\alpha M^M\Gamma(M+\alpha+i)}{\Gamma(\alpha)\Gamma(M)}\left[\int_0^{T_1-a_{j,M}}\frac{y^{M-1}}{\left(\beta+Ma_{j,M}+My\right)^{\alpha+M+i}}dy\right.$$
$$\left.+\int_{T_1-a_{j,M}}^{T_0-a_{j,M}}\left(\frac{T_0-a_{j,M}-y}{T_0-T_1}\right)^2\frac{y^{M-1}}{\left(\beta+Ma_{j,M}+My\right)^{\alpha+M+i}}dy\right],$$

using the transformation $z=\frac{My}{My+\beta+Ma_{j,M}}$, with $My+\beta+Ma_{j,M}=\frac{\beta+Ma_{j,M}}{1-z}$ and $y=\frac{(\beta+Ma_{j,M})z}{M(1-z)}$. We get:

$$
\begin{aligned}
r^{ML}(n,m|d_1) &= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k} C_i\binom{n}{d}\binom{d}{j}\frac{(-1)^j\beta^\alpha\Gamma(M+\alpha+i)}{\Gamma(\alpha)\Gamma(M)\left(\beta+Ma_{j,M}\right)^{\alpha+i}}\left[\int_0^{q_1} z^{M-1}\left(1-z\right)^{\alpha+i-1}dz\right.\\
&+\left(\tfrac{T_0-a_{j,M}}{T_0-T_1}\right)^2\int_{q_1}^{q_0} z^{M-1}\left(1-z\right)^{\alpha+i-1}dz-2\frac{\left(\beta+Ma_{j,M}\right)\left(T_0-a_{j,M}\right)}{M(T_0-T_1)^2}\\
&\left.\times\int_{q_1}^{q_0} z^{M}\left(1-z\right)^{\alpha+i-2}dz+\left(\tfrac{\beta+Ma_{j,M}}{M(T_0-T_1)}\right)^2\int_{q_1}^{q_0} z^{M+1}\left(1-z\right)^{\alpha+i-3}dz\right]\\
&=\sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k} C_i\binom{n}{d}\binom{d}{j}\frac{(-1)^j\beta^\alpha\Gamma(M+\alpha+i)}{\Gamma(\alpha)\Gamma(M)\left(\beta+Ma_{j,M}\right)^{\alpha+i}}\left[B_1\left(M,\alpha+i\right)I_{q_1}\left(M,\alpha+i\right)+\left(\tfrac{T_0-a_{j,M}}{T_0-T_1}\right)^2\right.\\
&\times B_1\left(M,\alpha+i\right)\left\{I_{q_0}\left(M,\alpha+i\right)-I_{q_1}\left(M,\alpha+i\right)\right\}-2\frac{\left(\beta+Ma_{j,M}\right)\left(T_0-a_{j,M}\right)B_1(M+1,\alpha+i-1)}{M(T_0-T_1)^2}\times\\
&\left\{I_{q_0}\left(M+1,\alpha+i-1\right)-I_{q_1}\left(M+1,\alpha+i-1\right)\right\}+\left(\tfrac{\beta+Ma_{j,M}}{M(T_0-T_1)}\right)^2 B_1\left(M+2,\alpha+i-2\right)\\
&\left.\times\left\{I_{q_0}\left(M+2,\alpha+i-2\right)-I_{q_1}\left(M+2,\alpha+i-2\right)\right\}\right],
\end{aligned}
$$

where $B_x\left(a,b\right)=\int_0^x t^{a-1}(1-t)^{b-1}dt, I_x\left(a,b\right)=B_x\left(a,b\right)/B_1\left(a,b\right)$, denote the incomplete Beta function and Beta ratio respectively. Hence:

$$
\begin{aligned}
r^{ML}(n,m|d_1) &= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}(-1)^j C_i\binom{n}{d}\binom{d}{j}\frac{\beta^\alpha\Gamma(\alpha+i)}{\Gamma(\alpha)\left(\beta+Ma_{j,M}\right)^{\alpha+i}}\left\{I_{q_1}\left(M,\alpha+i\right)+\left(\tfrac{T_0-a_{j,M}}{T_0-T_1}\right)^2\right.\\
&\times\left[I_{q_0}(M,\alpha+i)-I_{q_1}(M,\alpha+i)\right]-2\frac{\left(\beta+Ma_{j,M}\right)\left(T_0-a_{j,M}\right)}{(\alpha+i-1)(T_0-T_1)^2}\\
&\left[I_{q_0}(M+1,\alpha+i-1)-I_{q_1}(M+1,\alpha+i-1)\right]+\frac{(M+1)\left(\beta+Ma_{j,M}\right)^2}{M(\alpha+i-1)(\alpha+i-2)(T_0-T_1)^2}\\
&\left.\times\left[I_{q_0}(M+2,\alpha+i-2)-I_{q_1}(M+2,\alpha+i-2)\right]\right\}.
\end{aligned}
\tag{10}
$$

While

$$
E\left[E\left(D_{n,m,t}|\lambda\right)\right]=\sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^{d-j}M\binom{n}{d}\binom{d}{j}\left(\tfrac{\beta}{\beta+(n-j)t}\right)^\alpha. \tag{11}
$$

Therefore, the Bayes risk $R^{ML}\left(n,m,t,T_0,T_1\right)$ is given by:

$$
\begin{aligned}
&R^{ML}(n,m,t,T_0,T_1)=\\
&n\left(C_s-v_s\right)+v_s\sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^{d-j}M\binom{n}{d}\binom{d}{j}\left(\tfrac{\beta}{\beta+(n-j)t^\mu}\right)^\alpha+\sum_{i=0}^{k}a_i\gamma_i+\tau^*C_t\\
&+\sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}(-1)^j C_i\binom{n}{d}\binom{d}{j}\frac{\beta^\alpha\Gamma(\alpha+i)}{\Gamma(\alpha)\left(\beta+Ma_{j,M}\right)^{\alpha+i}}\left\{I_{q_1}\left(M,\alpha+i\right)+\left(\tfrac{T_0-a_{j,M}}{T_0-T_1}\right)^2\right.\\
&\times\left[I_{q_0}(M,\alpha+i)-I_{q_1}(M,\alpha+i)\right]-2\frac{\left(\beta+Ma_{j,M}\right)\left(T_0-a_{j,M}\right)}{(\alpha+i-1)(T_0-T_1)^2}\\
&\left[I_{q_0}(M+1,\alpha+i-1)-I_{q_1}(M+1,\alpha+i-1)\right]+\frac{(M+1)\left(\beta+Ma_{j,M}\right)^2}{M(\alpha+i-1)(\alpha+i-2)(T_0-T_1)^2}\\
&\left.\times\left[I_{q_0}(M+2,\alpha+i-2)-I_{q_1}(M+2,\alpha+i-2)\right]\right\},
\end{aligned}
$$

where, for $m < n$

$$\tau^* = E\left[E\left(\tau_{n,m,t}|\lambda\right)\right] = m\binom{n}{m}\sum_{j=0}^{m-1}(-1)^{m-j-1}\binom{m-1}{j}\frac{\alpha\beta}{(n-j)^2}B_{1-q*}\left(2,\alpha-1\right)$$
$$+\frac{tn!}{(m-1)!(n-m-1)!}\sum_{i=0}^{m-1}\sum_{j=0}^{n-m-1}\Big[(-1)^{n-i-j}\binom{m-1}{i}\binom{n-m-1}{j}$$
$$\times\frac{\beta^{\alpha}}{(m+j-i)(n-m-j)}\left(\frac{1}{((n-m-j)t+\beta)^{\alpha}}-\frac{1}{((n-i)t+\beta)^{\alpha}}\right)\Big]$$
$$+n\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j}\frac{\alpha\beta}{(n-j)^2}B_{q*}\left(2,\alpha-1\right),$$

and, for $n = m$

$$\tau^* = E\left[E\left(\tau_{n,m,t}|\lambda\right)\right] = \frac{n\beta}{\alpha-1}\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j}\frac{1}{(n-j)^2},$$

$q* = \frac{(n-j)t}{\beta+(n-j)t}$, $q_i = \frac{M\left(T_i-a_{j,M}\right)}{\beta+M\left(T-a_{j,M}\right)+Ma_{j,M}}$.
For the computation of $E\left\{E\left(\tau_{n,m,t}|\lambda\right)\right\}$ and $E\left\{E\left(D_{n,m,t}|\lambda\right)\right\}$ See Belbachir and Benahmed (2022b).

3.2. **Bayes risk under Bayes estimator.** In this section, a Bayesian sampling plan is designed using the Bayes estimator of θ under the Linex loss. The expression of $R^{BL}(n,m,t,T_0,T_1)$ given by:

$$R^{BL}(n,m,t,T_0,T_1) = E\{E\left[L(\lambda,\delta(\underline{x}))\right]\}$$
$$= n\left(C_s-v_s\right)+v_sE\left[E\left(D_{n,m,t}|\lambda\right)\right]+C_tE\left[E\left(\tau_{n,m,t}|\lambda\right)\right]+\sum_{i=0}^{k}a_i\gamma_i+r^{BL}(n,m|d_1),$$

Similarly

$$R^{BL}(n,m,t,T_0,T_1) =$$
$$n\left(C_s-v_r\right)+v_s\sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^{d-j}M\binom{n}{d}\binom{d}{j}\left(\frac{\beta}{\beta+(n-j)t^{\mu}}\right)^{\alpha}+\sum_{i=0}^{k}a_i\gamma_i+\tau^*C_t$$
$$+\sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}\binom{n}{d}\binom{d}{j}\frac{(-1)^jC_i\beta^{\alpha}\Gamma(\alpha+i)}{\Gamma(\alpha)(\beta+(n-d+j)t)^{\alpha+i}}\Bigg\{I_{q_1'}\left(M,\alpha+i\right)+\left(\frac{T_0-a_{j,A,\beta}}{T_0-T_1}\right)^2$$
$$\times\left[I_{q_0'}(M,\alpha+i)-I_{q_1'}(M,\alpha+i)\right]-2\frac{M(\beta+(n-d+j)t)\left(T_0-a_{j,A,\beta}\right)}{A(\alpha+i-1)(T_0-T_1)^2}$$
$$\left[I_{q_0'}(M+1,\alpha+i-1)-I_{q_1'}(M+1,\alpha+i-1)\right]+\frac{M(M+1)(\beta+(n-d+j)t)^2}{A^2(\alpha+i-1)(\alpha+i-2)(T_0-T_1)^2}$$
$$\times\left[I_{q_0'}(M+2,\alpha+i-2)-I_{q_1'}(M+2,\alpha+i-2)\right]\Bigg\},$$

with, $q_i' = \frac{A\left(T_i-a_{j,A,\beta}\right)}{\beta+A\left(T-a_{j,A,\beta}\right)+(n-d+j)t}$.

4. Computation of the optimal sammling plan

For finding an optimal sampling plan, we suggest finite algorithm which is presented below. Furthermore, the optimal size of the sample is bounded by:

$$N = \min\left\{\left[\frac{C_r}{C_s-v_s}\right],\left[\frac{\sum_{i=0}^{k}a_i\gamma_i}{C_s-v_s}\right]\right\}. \tag{12}$$

where $[x]$ is the integer part of x. For the proof see Belbachir and Benahmed (2022a).

The expression $R(n, m, t, T_0, T_1)$ is quite complicated. So, as given by prajapati et al. (2020), we assume T_0 has an upper bound since $0 < T_0 < T_0^*$, and for t, we obtain a confidence interval $[t_L, t_U]$ where $P(X > t_U) = \eta/2$, $P(X < t_L) = \eta/2$, with $\eta = 0.05$. For searching the optimal sampling plan (n, m, t, T_0, T_1), we consider the following algorithm:

(1) Start with $(n, m, t) = (0, 0, 0)$, compute N from Equation (9) and compute $R\left(0, 0, 0, T'_{0,(n,m,t)}, T'_{1,(n,m,t)}\right)$ $= min\left\{R(0, 0, 0, \infty, \infty) = C_r, R(0, 0, 0, 0, 0) = \sum_{i=0}^{k} a_i\gamma_i\right\}$.

(2) For fixed (n, m, t), compute the optimal $T'_{0,(n,m,t)}$ and $T'_{1,(n,m,t)}$ using grid search method, such that $R\left(n, m, t, T'_{0,(n,m,t)}, T'_{1,(n,m,t)}\right)$ $= \underset{0<T_1<T_0\leq T^*}{min} R(n, m, t, T_1, T_0)$, with the grid size 0.0125.

(3) For fixed (n, m), compute the optimal $t'_{(n,m)}$ using grid search method, such that $R\left(n, m, t'_{(n,m)}, T'_{0,(n,m,t)}, T'_{1,(n,m,t)}\right)$ $= \underset{t_L\leq t\leq t_U}{min} R\left(n, m, t, T'_{0,(n,m,t)}, T'_{1,(n,m,t)}\right)$, with the grid size $\frac{t_U - t_L}{100}$.

(4) For $0 \leq m \leq n \leq N$, choose (n', m', t', T'_0, T'_1) which corresponds to the smallest value of the Bayes risks $R\left(n, m, t'_{(n,m)}, T'_{0,(n,m,t)}, T'_{1,(n,m,t)}\right)$.

5. Simulation study

To illustrate the proposed model, we assume that the loss function is a quadratic function with $(k = 2)$, such that the calculations can be done in a similar way for higher degree. The evaluation of the Bayes risk is done numerically based on the upper bound of sample size and the grid search method, i.e. we can get an optimal sampling plan in a finite number of search steps. Various numerical examples are tabulated in Tables 1-4, in each table we denote the optimal plan under both estimators by S_0^{ML} and S_0^{BL}, and their Bayes risk by R_0^{ML} and R_0^{BL}. Further, we indicate the expected number of failures by $E[D]$, and the expected censoring time by $E[\tau]$.

The selection of parameters and coefficients of the loss function is the main factor that controls the value of the minimum Bayes risk. To maintain the sensitivity analysis of the risk function, we vary one(two) coefficient(s) or parameter(s) and we fix the others. As the true values of parameters and coefficients of the model for which we made the calculations, we choose $\alpha = 2.5$, $\beta = 0.8$, $a_0 = a_1 = a_2 = 2$, $C_s = 0.5$, $v_s = 0.3$, $C_t = 1$, $C_r = 30$ and $a = -10$. For the grid search method we take $T_0^* = 2$.

For the standard values mentioned above, the optimal sampling plan under ML estimator is $S_0^{ML} = (4, 1, 0.1427, 0.8500, 0.3000)$, which means, we put 4 items for the test and the life test terminates after the maximum between the fourth failure time and $t = 0.1427$. the batch is accepted if the estimator of the average lifetime $\hat{\theta}$ is greater than or equal 0.8500. When $\hat{\theta}$ is between 0.3000 and 0.8500, the batch is accepted with probability $1 - p_{\hat{\theta}}$, with the Bayes risk $R_0^{ML} = 25.1898$.

Table 1 illustrates that, the Bayes risks R_0^{ML} (resp. R_0^{BL}) is decreasing when α fixed and β increases, while the optimal sample size n' decreases for both plans, while $R_0^{BL} < R_0^{ML}$. Furthermore, for the variations of (α, β), $E[D] \geq m'$ and this is indicated that S_0^{ML} (resp. S_0^{BL}) can give a sufficient information about the lifetime of the items put in the life testing. In addition, we observe that, $R_0^{BL} < R_0^{ML}$ for most variation of a_2 and C_r, with

$E[D] \geq m'$. From Table 1, it can be seen that the optimal sampling plan S_0^{ML} (resp. S_0^{BL}) occurs at no sampling case for some selected values. For instance, when $(\alpha, \beta) = (3.5, 0.6)$, $S_0^{ML} = S_0^{BL} = (0, 0, 0, \infty, \infty)$, which means that the decision made without takes a sample and the batch is rejected without any sample cost. When $C_r = 100$, the optimal sampling plans takes the form $S_0^{ML} = S_0^{BL} = (0, 0, 0, 0.0000, 0.0000)$, indicating that, we should accept the batch with the Bayes risk $R_0^{ML} = R_0^{BL} = 35.5938$.

5.1. **Comparison with the sampling plan of Yang et al. (2017).** In this Section, we consider a comparison between our proposed sampling plan and the one of Yang et al. (2017), we denote their sampling plan by $S_L \equiv (n_L, m_L, t_L)$ and it's related Bayes risk by $R_L \equiv R(n_L, m_L, t_L)$. We indicate the expected number of failures by $E[D_L]$ and the expected time censoring by $E[\tau_L]$. Various numerical results are tabulated in Table 2 under setting $a_0 = a_1 = a_2 = 2$, $C_s = 0.5$, $v_s = 0.3$ and $C_r = 30$ while α, β and C_t vary.

In Table 2, We observe that the Bayes risks R_0^{ML} and R_0^{BL} are less than R_L for the most selected values of (α, β, C_t), especially, when $C_t = 1, 5, 8$ while (α, β) vary, where S_0^{BL} get more samples than S_0^{ML} with $R_0^{BL} \leq R_0^{ML}$, which indicates that, the optimal sampling plan under the Bayesian estimator of LINEX loss is more efficient than the one under the maximum likelihood estimator. Furthermore, the deference between the Bayes risks R_0^{ML} and R_0^{BL} of the proposed model and R_L is increasing when C_t increases. On the other hand, in most cases the expected number of failures $E[D]$ and $E[D_L]$ close to each other, where the sample $n^{ML} \geq n_L$(resp. $n^{BL} \geq n_L$) except the cases $\alpha = 3$, $\beta = 0.6$ and $C_t = 0.5$ with $n^{ML} = n^{BL} = 2$ and $n_L = 3$.

6. Conclusion

A number of variable sampling plans have been developed in the past few years. The majority of the elaborations were produced by taking the one-sided decision function into account. There are some industrial processes or the quality characteristic data may originate from an unpredictable environment or from a complex production process (i.e. the lowest acceptable and the maximum rejectable survival time are not equal). As such, it is crucial to investigate the uncertainty zone and how it affects the best possible sampling strategy. This paper's goal is to illustrate the doubt zone's characteristic using a different strategy. We created Bayesian sampling strategies based on a two-sided decision function with a quadratic random doubt zone. Also, we construct Bayesian sampling plans using maximum likelihood and Bayesian estimates for mean lifetime of items that is follows an exponential distribution. In addition, we found an explicit form for the Bayes risk using a suitable polynomial loss function. It is observed that the resulting Bayesian risk expression is very intricate. Therefore, in order to provide the performance of the Bayes risk, we have presented a discretization method. The discretization method's results demonstrate that numerical techniques can effectively approximate the Bayes risk.

We realized a simulation study to examine the performance of the proposed model. Based on the results, the plan based on the Bayesian estimate under the Linex loss can give a minimum Bayes risk compared with the plan under maximum likelihood estimate. Further, it can be concluded that, our proposed sampling plan as good as the one of Yang et al. (2017) in terms of information obtained about the expected lifetime of the batch items, with a preference to our model in terms of minimum Bayes risk.

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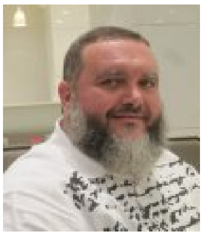

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TABLE 1. Optimal sampling plans and it's Bayes risk for α, β, a_2 and C_r vary.

α	β	a_2	C_r	Plan	n'	m'	t'	T_0'	T_1'	$E[D]$	$E[\tau]$	$R_0^{ML}(R_0^{BL})$
2.5	0.6	2	30	S^{ML}	3	1	0.2483	1.1875	0.3125	1.8706	0.2623	28.2112
				S^{BL}	4	1	0.1877	1.4125	0.8125	2.1061	0.2102	28.2026
	0.8			S^{ML}	4	1	0.1427	0.8500	0.3000	1.6064	0.2008	25.1898
				S^{BL}	7	2	0.1427	1.3500	0.5125	2.8315	0.2158	25.0944
	1.0			S^{ML}	6	1	0.0774	0.6500	0.3000	1.4059	0.1401	21.6799
				S^{BL}	6	1	0.0774	1.3250	1.0875	1.4059	0.1401	21.5688
	1.2			S^{ML}	7	1	0.0526	0.5000	0.2500	1.2239	0.1291	18.1699
				S^{BL}	7	1	0.0526	1.3875	1.2000	1.2239	0.1291	18.0978
3.0	0.6	2	30	S^{ML}	2	2	0.0485	0.5625	0.5125	2.0000	0.4500	29.9277
				S^{BL}	1	1	0.0196	0.9000	0.8500	1.0000	0.3000	29.8896
	0.8			S^{ML}	4	1	0.1804	1.0625	0.3875	1.9721	0.2035	27.9927
				S^{BL}	4	1	0.1804	1.1875	0.7625	1.9721	0.2035	27.9961
	1.0			S^{ML}	5	1	0.1290	0.8625	0.4375	1.7505	0.1657	25.2194
				S^{BL}	6	2	0.1532	1.1750	0.4750	2.6110	0.2321	25.0692
	1.2			S^{ML}	5	1	0.0970	0.6625	0.3125	1.4011	0.1578	21.9093
				S^{BL}	5	1	0.0970	1.1875	0.9750	1.4011	0.1578	21.9717
3.5	0.6	2	30	S^{ML}	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
				S^{BL}	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
	0.8			S^{ML}	3	1	0.2739	1.3125	0.3375	2.0139	0.2571	29.7173
				S^{BL}	3	1	0.2441	1.0750	0.5125	1.9216	0.2392	29.6992
	1.0			S^{ML}	4	1	0.1748	1.0125	0.3875	1.8804	0.1981	27.8107
				S^{BL}	6	2	0.1748	1.0500	0.4500	2.9270	0.2182	27.7605
	1.2			S^{ML}	4	1	0.1428	0.8125	0.3375	1.5571	0.1872	25.2051
				S^{BL}	6	2	0.1428	1.0500	0.4625	2.5157	0.2179	25.0902
2.5	0.8	0.5	30	S^{ML}	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	15.0859
				S^{BL}	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	15.0859
		1.0		S^{ML}	5	1	0.0620	0.4375	0.1875	1.2918	0.1272	20.4955
				S^{ML}	5	1	0.0620	1.0125	0.8250	1.2918	0.1272	20.5576
		1.5		S^{ML}	6	1	0.0889	0.7000	0.3750	1.6680	0.1301	23.3785
				S^{BL}	5	1	0.0889	1.1625	0.8875	1.4890	0.1438	23.3542
		2.0		S^{ML}	4	1	0.1427	0.8500	0.3000	1.6064	0.2008	25.1898
				S^{BL}	7	2	0.1427	1.3500	0.5125	2.8315	0.2158	25.0944
		3.0		S^{ML}	4	1	0.2234	1.2875	0.5125	1.9922	0.2602	27.2440
				S^{BL}	4	1	0.2234	1.7125	1.0750	1.9922	0.2602	27.2270
		5.0		S^{ML}	3	1	0.4117	1.9625	0.5125	2.0344	0.3962	29.1198
				S^{BL}	3	1	0.3310	2.1250	0.8375	1.8706	0.34978	29.1211
		8.0		S^{ML}	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
				S^{BL}	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
2.5	0.8	2.0	20	S^{ML}	3	1	0.3041	1.4625	0.3625	1.8085	0.3335	19.2725
				S^{BL}	3	1	0.2772	1.8250	0.8625	1.7424	0.3169	19.2675
			25	S^{ML}	4	1	0.1965	1.1375	0.4375	1.8707	0.2400	22.4705
				S^{BL}	4	1	0.1965	1.5875	1.0250	1.8707	0.2400	22.4690
			30	S^{ML}	4	1	0.1427	0.8500	0.3000	1.6064	0.2008	25.1898
				S^{BL}	7	2	0.1427	1.3500	0.5125	2.8315	0.2158	25.0944
			35	S^{ML}	6	1	0.0889	0.7125	0.3625	1.6680	0.1301	27.3293
				S^{BL}	7	2	0.1158	1.1875	0.4750	2.6046	0.1993	27.1533
			40	S^{ML}	7	1	0.0620	0.5625	0.3125	1.5295	0.1017	29.1029
				S^{BL}	8	2	0.0889	1.1125	0.4500	2.5271	0.1662	28.9976
			100	S^{ML}	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	35.5938
				S^{BL}	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	35.5938

TABLE 2. Comparison of performance with the plan of Yang et al. (2017) for $a_0 = a_1 = a_2 = 2$, $C_s = 0.5, v_s = 0.3$ and $C_r = 30$.

C_t	α	β	S_0^{ML}	$E[D]$	$E[\tau]$	R_0^{ML}	S_0^{BL}	$E[D]$	$E[\tau]$	R_0^{BL}	S_L	$E[D_L]$	$E[\tau_L]$	R_L
0.5	2.5	0.6	(3,1,0.2483,1.1875,0.3125)	1.8706	0.2623	28.0801	(3,1,0.2483,1.4875,0.7375)	1.8706	0.2623	28.0804	(3,2,0.9463)	2.7912	0.5559	27.8387
		0.8	(4,1,0.1427,0.8500,0.3000)	1.6064	0.2008	25.0894	(4,1,0.1696,1.4500,0.9875)	1.7418	0.2202	25.0754	(4,2,0.7698)	3.3541	0.6299	24.9619
		1.0	(4,1,0.1111,0.6500,0.2500)	1.3249	0.2068	21.6001	(6,1,0.0774,1.3250,1.0875)	1.4059	0.1401	21.4988	(4,1,0.5794)	2.7739	0.5349	21.7061
	3.0	0.6	(2,2,0.0919,0.5500,0.5000)	2.0000	0.4500	29.7042	(2,2,0.0051,0.7500,0.7000)	2.0000	0.4500	29.7038	(3,2,1.1011)	2.8942	0.4722	29.6200
		0.8	(4,1,0.1804,1.0750,0.3750)	1.9721	0.2035	27.8928	(5,2,0.2383,1.2500,0.4625)	2.9935	0.2833	27.8891	(3,2,0.8864)	2.7576	0.5579	27.7033
		1.0	(5,1,0.1290,0.8750,0.4250)	1.7505	0.1657	25.1304	(6,2,0.1532,1.1750,0.4750)	2.6110	0.2321	24.9531	(4,1,0.7117)	3.2200	0.5635	25.0772
1.0	2.5	0.6	(3,1,0.2483,1.1875,0.3125)	1.8706	0.2623	28.2112	(4,1,0.1877,1.4125,0.8125)	2.1061	0.2102	28.2026	(3,2,0.9204)	2.7829	0.5519	28.1161
		0.8	(4,1,0.1427,0.8500,0.3000)	1.6064	0.2008	25.1898	(7,2,0.1427,1.3500,0.5125)	2.8315	0.2158	25.0944	(4,1,0.6659)	3.1456	0.5397	25.2605
		1.0	(6,1,0.0774,0.6500,0.3000)	1.4059	0.1401	21.6799	(6,1,0.0774,1.3250,1.0875)	1.4059	0.1401	21.5688	(5,1,0.3596)	2.7566	0.3758	21.9437
	3.0	0.6	(2,2,0.0485,0.5625,0.5125)	2.0000	0.4500	29.9277	(1,1,0.0196,0.9000,0.8500)	1.0000	0.3000	29.8896	(3,2,0.0001)	2.0000	0.2500	29.7805
		0.8	(4,1,0.1804,1.0625,0.3875)	1.9721	0.2035	27.9927	(4,1,0.1804,1.1875,0.7625)	1.9721	0.2035	27.9961	(3,2,0.8864)	2.7576	0.5579	27.9822
		1.0	(5,1,0.1290,0.8625,0.4375)	1.7505	0.1657	25.2194	(6,2,0.1532,1.1750,0.4750)	2.6110	0.2321	25.0692	(4,1,0.6192)	3.0816	0.5173	25.3476
5.0	2.5	0.6	(5,1,0.1272,1.0125,0.2625)	2.0722	0.1532	28.9536	(5,1,0.1474,1.3500,0.8500)	2.2475	0.1696	28.9608	(4,2,0.0001)	2.0000	0.2333	29.3372
		0.8	(7,1,0.0889,0.8125,0.4375)	1.8579	0.1210	25.7473	(7,1,0.0889,1.3375,1.0125)	1.8579	0.1210	25.7701	(6,1,0.1889)	2.5783	0.2113	26.5029
		1.0	(7,1,0.0774,0.7125,0.3875)	1.5295	0.1272	22.2053	(6,1,0.0774,1.3250,1.0875)	1.4059	0.1401	22.1293	(6,1,0.1485)	1.9589	0.1910	23.0443
	3.0	0.6	(0,0,0.0000,∞,∞)	0.0000	0.0000	30.0000	(0,0,0.0000,∞,∞)	0.0000	0.0000	30.0000	(0,0,0.0000)	0.0000	0.0000	30.0000
		0.8	(5,1,0.1418,1.0125,0.4125)	2.0847	0.1633	28.7118	(6,1,0.1225,1.1375,0.8250)	2.2288	0.1403	28.7115	(5,2,0.0001)	2.0000	0.1800	29.2166
		1.0	(6,1,0.1049,0.8375,0.4375)	1.7833	0.1363	25.8076	(6,1,0.1049,1.1750,0.9125)	1.7833	0.1363	25.7803	(7,1,0.1481)	2.4929	0.1652	26.5398
8.0	2.5	0.6	(7,1,0.0868,0.9625,0.2625)	2.1810	0.1068	29.3903	(6,1,0.0666,1.1875,0.4625)	1.6680	0.0976	29.3604	(6,2,0.0001)	2.0000	0.1467	29.7795
		0.8	(7,1,0.0889,0.8125,0.4375)	1.8579	0.1210	26.1103	(7,1,0.0889,1.3375,1.0125)	1.8579	0.1210	26.1331	(7,1,0.1514)	2.5828	0.1726	27.0606
		1.0	(7,1,0.0774,0.7125,0.3875)	1.5295	0.1272	22.5869	(6,1,0.0774,1.3250,1.0875)	1.4059	0.1401	22.5497	(8,1,0.1032)	1.9638	0.1370	23.5890
	3.0	0.6	(0,0,0.0000,∞,∞)	0.0000	0.0000	30.0000	(0,0,0.0000,∞,∞)	0.0000	0.0000	30.0000	(0,0,0.0000)	0.0000	0.0000	30.0000
		0.8	(7,1,0.1032,0.9625,0.4875)	2.2814	0.1189	29.1230	(7,1,0.1032,1.1250,0.8250)	2.2814	0.1189	29.1265	(6,2,0.0001)	2.0000	0.1467	29.6384
		1.0	(7,1,0.0808,0.7625,0.3875)	1.7163	0.1100	26.1424	(7,1,0.0808,1.1250,0.8875)	1.7163	0.1100	26.1423	(7,1,0.1431)	2.4382	0.1608	27.0284