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DESIGNS ARISING FROM PRODUCTS OF HYPERGRAPHS OF CYCLES

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ABSTRACT. In this paper, we have considered many standard graph products viz. cartesian product, direct product, strong product and lexicographic product, and extended these graph products to hypergraphs which are natural generalizations of ususal graphs where edges may consist of more than two vertices. We have constructed a hypergraph of a graph by considering hyperedges as closed neighbourhood of each vertex in the graph. As the product of any two hypergraphs is again a hypergraph, we have obtained designs arising from products of hypergraphs where blocks are hyperedges of a hypergraph obtained by taking standard products of hypergraphs of cycles.

Keywords: PBIB-designs, Hypergraphs, Cartesian product, Direct product, Strong product, Lexicographic product.

AMS Subject Classification: 05C12, 05C50, 92E10

1. INTRODUCTION

Combinatorial design theory is a part of combinatorics that deals with existence, construction and properties of systems of finite sets whose arrangements satisfy certain conditions. Balanced incomplete block (BIB)-designs and partially balanced incomplete block (PBIB)-designs are two major subfields finding a wide range of applications in various fields of studies and experimentations. Balanced incomplete block designs are connected and efficiency balanced, in the sense that all treatment differences are estimated with the same accuracy. But they exist only for certain parameters and a major disadvantange of using BIB-designs in experimentation is that it requires a large number of replication of treatments. PBIB-designs help in reducing the number of replications by compromising on the property of balanced efficiency. Therefore, PBIB-designs are not balanced but partially balanced although they are connected. Thus, they find more applications in real world problems than BIB-designs. Various designs have been constructed from graphs taking blocks to be certain subsets of the vertex set in [14], [15], [16], [29] etc.

Hypergraph theory was introduced in 1960s as a generalization of graph theory. The expository text, 'Graphs and Hypergraphs' by Berge [2] in 1973 introduces the concept lucidly. The generalization of graph problems to hypergraphs brings a number of new perspectives to the field

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of graph theory. Research into the theories of set systems and hypergraphs provide a valuable basis to various fields of mathematics such as matroids, designs, combinatorial probability and Ramsey theory for infinite sets. Hypergraph theory studies a mathematical structure on a set of elements with a relation, as a recognised discipline in a relatively new era. In recent years, theory of hypergraphs have proved to be of major interest in applications to real world problems. Recent developments in this comparatively younger theory have played a major part in revealing hypergraphs as a prominent mathematical tool in a variety of applications in the fields of engineering, particularly in computer science, software engineering, image processing, molecular biology, and related businesses and industries, chemistry and so on [4], [19], [20], [21], etc. Hypergraphs can represent group relationships and thus have many uses in solving technical problems. Real world examples of hypergraphs are social networks like Facebook or Linkedin wherein each user is a vertex that could be a part of a group which is the hyperedge.

Every branch of mathematics employs one notion of a product that enables the combination or decomposition of its elemental structures. In graph theory, we can find four main products each with their own sets of applications and theoretical interpretations. Graph products are natural structures in discrete mathematics that arise in a variety of different contexts from computer science, computational engineering to theoretical biology [1], [10]. They are viewed as a convenient language which is used to describe various structures. Computer science is one of the many fields in which graph products is prevalent and one of its application is load balancing for massively parallel computer architectures. Besides, it finds applications in chemical graph theory and dynamic location problem. The applications of median graphs in human genetics and powers of direct products to model large networks are in use owing to its significance. Detailed literature and applications of products of graphs has been given in monograph 'Handbook of Product Graphs' by Hammack et al. [11]. Many researchers have extended the concepts and algorithms developed in graphs analogously to hypergraps. Products of hypergraphs is one such captivating topic having wide applications in network theory. A survey on hypergraph products by Hellmuth et al. [13] has described briefly different graph products that can be extended to hypergraphs and the properties associated with them. Bretto et al. [5] deduced new properties and algorithms concerning aspects of cartesian product of hypergraphs. They also extended a classical prime factorization algorithm initially designed for graphs to connect conformal hypergraphs using 2-sections of hypergraphs. This concept was further generalized for directed hypergraphs by Ostermeier [22] by showing that every simple (weakly) connected, possibly directed and infinite hypergraph has a unique prime factor decomposition with respect to the (weak) cartesian product even if it has infinitely many factors. Bruce et al. [7] extended the properties of lexicographic products to lexicographic products of r-uniform hypergraphs and gave a generalization for new multicolor inequality for hypergraph Ramsey numbers. Bounds on chromatic number of direct product of hypergraphs is due to Sterboul [27]. Bretto et al. [6] developed an algorithm which factorizes any hypergraph into its prime factors in $O(nm)$ time where n and m are order and size of hypergraph, respectively. Later on, Hellmuth et al. [12] have showed that every connected hypergraph has a unique prime factorization with respect to the normal and strong hypergraph products using cartesian skeleton of hypergraph. They have also developed algorithms in order to prove their results. Also, a lot of research has been carried out in perfect matchings in hypergraphs. Keevash [18] gave a link between perfect matching and designs arising from hypergraphs. Here, the author developed necessary and sufficient conditions for the existence of a perfect matching in block designs. The designs considered are Steiner systems and (n, q, r, λ) design which can be thought of as q-uniform hypergraphs.

The use of 2-associate class of PBIB-design is common in experimental work. However, PBIB-designs with more than two or three associate classes are not widely used because of the complicated nature of analysis and construction involved. In literature, several papers are available on construction of various block designs with two and three class association schemes from other existing designs and from graphs, but very few on construction of designs with more than 4-class association schemes [24], [25], [28]. Therefore, in this paper, we have constructed a few class of PBIB-designs with more than four associate classes arising from hypergraphs. The entire paper is split into four sections. First section deals with introduction to hypergraph products, combinatorial design theory and some of its applications. Preliminary definitions and some known results are listed in second section. In the third section, we construct PBIB-designs arising from hypergraph products of neighbourhood hypergraphs of cycles. We have also defined different association schemes for each of the products, followed by conclusion in the fourth section.

2. PRELIMINARIES

Undefined graph theoretical terms are used in the sense of Buckley and Harary [8] and undefined design theoretical terms are in the sense of Colbourn et al. [9].

Definition 2.1. [23] *Given a set $\{1, 2, 3, \ldots, v\}$ of v elements, a relation satisfying the following conditions is said to be an association scheme with m classes.*
(i) Any two elements α and β are i^{th} associates for some i with $1 \leq i \leq m$ and this relation of being i^{th} associates is symmetric.
(ii) The number of i^{th} associates of each element is n_i.
(iii) If α and β are two elements which are i^{th} associates, then the number of elements which are j^{th} associates of α and k^{th} associates of β is p^i_{jk} and is independent of the pair of i^{th} associates α and β.

Definition 2.2. [29] [3] *Consider a set $V = \{1, 2, \ldots, v\}$ and an association scheme with m classes on V. A partially balanced incomplete block (PBIB)-design represented as $(v, b, r, k, \lambda_1, \ldots, \lambda_m)$ is a collection of b subsets of V called blocks, each of them containing k elements $(k < v)$ such that every element occurs in r blocks and any two elements α and β which are i^{th} associates occur together in λ_i blocks, the number λ_i being independent of the choice of the pair α and β.*

The numbers $v,\ b,\ r,\ k,\ \lambda_i\ (i = 1, 2, \ldots, m)$ are called the parameters of first kind and $n_i^{\prime}s$ and p^i_{jk} are called the parameters of second kind.

Hypergraphs are natural generalization of undirected graphs in which edges may consist of more than two vertices.

Definition 2.3. [13] *A (finite) hypergraph $H = (V, E)$ consists of a (finite) set V and a collection E of non-empty subsets of V.*
The elements of V are called vertices and elements of E are called hyperedges.
A hypergraph $H = (V, E)$ is simple if no hyperedge is contained in any other hyperedge and $|e| \geq 2$ for all $e \in E$.
Two vertices u and v are adjacent in $H = (V, E)$ if there is a hyperedge $e \in E$ such that $u,\ v \in e$.
Two hyperedges $e,\ f \in E$ are adjacent if $e \cap f \neq \phi$.
A vertex v and a hyperedge e of H are incident if $v \in e$.
The degree $deg(v)$ of a vertex $v \in V$ is the number of hyperedges incident to v.
The rank of a hypergraph $H = (V, E)$ is $r(H) = max(|e|)$ and the antirank is $s(H) = min(|e|)$ where $e \in E$.
A hypergraph is said to be uniform if $r(H) = s(H)$.
A simple uniform hypergraph of rank r is called r-uniform hypergraph. A 2-uniform hypergraph is the ordinary graph.

Remark 2.1. *Product of two r-uniform hypergraphs is again a r-uniform hypergraph.*

Definition 2.4. [26] *Let $G = (V(G), E(G))$ be a graph with n vertices numbered arbitrarily by numbers $1, 2, 3, \ldots, n$, then the hypergraph $\mathcal{H}_k = (V(\mathcal{H}_k), E(\mathcal{H}_k)),\ k \geq 1$ is such that $V(\mathcal{H}_k) =$*

$V(G)$ and $E(\mathcal{H}_k) = \{e_1, e_2, e_3, \ldots, e_p\}$, e_i ={set of vertices j: $d(i,j) \leq k$} where $d(i,j)$ is the distance between vertices i and j in G.

Remark 2.2. *$\mathcal{H}_k$ is basically a neighbourhood hypergraph of order k. When $k = 1$, it is called simply as a neighbouhood hypergraph.*

Now we define four graph products known as the standard graph products, viz., cartesian product $\Box$, direct product $\times$, strong product $\boxtimes$ and lexicographic product $\circ$, which are extendible to hypergraphs as well.

Definition 2.5. [13] *Let $\circledast_{i=1}^{n} H_i = (V, E) = (\times_{i=1}^{n} V(H_i), E(\circledast_{i=1}^{n} H_i))$ be an arbitrary hypergraph product. The projection $p_j : V \to V(H_j)$ is defined by $(v = v_1, v_2, \ldots v_n) \mapsto v_j$. v_j is called the j^{th} coordinate of the vertex $v \in V$. Products of simple hypergraphs are simple.*

Definition 2.6. [13] *Cartesian product $H = H_1 \Box H_2$ of two hypergraphs H_1 and H_2 has vertex set $V(H) = V(H_1) \times V(H_2)$ and the edge set*
$E(H) = \{\{x\} \times f : x \in V(H_1), f \in E(H_2)\} \cup \{e \times \{y\} : e \in E(H_1), y \in V(H_2)\}$.
Cartesian product of hypergraphs can be described in terms of projections as follows:
For $H = H_1 \Box H_2$, with $H_i = (V_i, E_i)$ for $i = 1, 2$ and $e \subset V(H)$, we have $e \in E(H)$ if and only if there is an $i \in \{1, 2\}$, such that
(i) $p_i(e) \in E_i$ (ii) $|p_j(e)| = 1$ for $i \neq j$.
Furthermore, $|p_i(e)| = |e|$.

Definition 2.7. [13] *Direct product $H = H_1 \times H_2$ of two hypergraphs H_1 and H_2 has vertex set $V(H) = V(H_1) \times V(H_2)$ and the edge set*
$E(H) = \{e \in V(H_1) \times V(H_2) \ : \ p_1(e) \in E(H_1),\ p_2(e) \in E(H_2)$ *and* $|\,e\,| = max\,(|p_1(e)|, |p_2(e)|)\}$.
For r-uniform hypergraphs H_1 and H_2, $\{(x_1, y_1),\ (x_2, y_2), \ldots, (x_r, y_r)\} \in E(H)$ if and only if
(i) $\{x_1, x_2, \ldots, x_r\} \in E(H_1)$ and there exist a hyperedge $e_1 \in E(H_2)$ such that $\{y_1, y_2, \ldots, y_r\}$ is a family of all elements of e_1 or
(ii) $\{y_1, y_2, \ldots, y_r\} \in E(H_2)$ and there exist a hyperedge $e_2 \in E(H_1)$ such that $\{x_1, x_2, \ldots, x_r\}$ is a family of all elements of e_2.

Definition 2.8. [13] *Strong product of hypergraphs can be interpreted as a superposition of edges of cartesian product and direct product.*
Strong product $H = H_1 \boxtimes H_2$ of two hypergraphs H_1 and H_2 has vertex set $V(H) = V(H_1) \times V(H_2)$ and the edge set $E(H) = E(H_1 \Box H_2) \cup E(H_1 \times H_2)$.

Definition 2.9. [13] *Lexicographic product $H = H_1 \circ H_2$ of two hypergraphs H_1 and H_2 has vertex set $V(H) = V(H_1) \times V(H_2)$ and edge set*
$E(H) = \{e \subseteq V(H) \ : \ p_1(e) \in E(H_1),\ |p_1(e)| = |e|\} \cup \{\{x\} \times e_2 \ : \ x \in V(H_1),\ e_2 \in E(H_2)\}$.
Since $|p_1(e)| = |e|$, there are $|e|$ vertices of e that have pairwise different first coordinates.
Lexicographic product of two r-uniform hypergraphs H_1 and H_2 is again a r-uniform hypergraph H with vertex set $V(H) = V(H_1) \times V(H_2)$ and edge set $\{(x_1, y_1),\ (x_2, y_2), \ldots, (x_r, y_r) :$ $\{x_1, x_2, \ldots, x_r\} \in E(H_1)$ or $x_1 = x_2 = \cdots = x_r$ and $\{y_1, y_2, \ldots, y_r\} \in E(H_2)\}$.

Remark 2.3. *A simple graph is a 2-uniform hypergraph. Consequently, the products defined above are appropriate for graphs as well.*

Definition 2.10. [8] *A cycle is a closed path. Therefore, cycle is 2-regular. A cycle on n vertices is denoted as C_n.*

There are several definitions for hypergraph cycle. Below, we define hypergraph cycle $\mathcal{C}$ for a 3-uniform hypergraph.

Definition 2.11. [17] *A hypergraph cycle of length n, denoted as $\mathcal{C}_n$, is called a loose cycle if it has vertices $\{v_1, v_2, \ldots, v_n\}$ and hyperedges $\Big\{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \ldots, \{v_{n-1}, v_n, v_1\}\Big\}$ when n is even.*

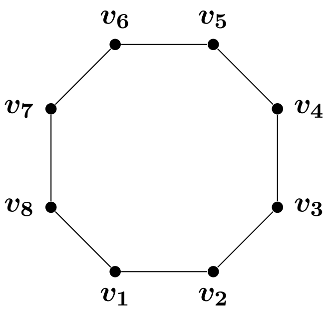

FIGURE 1. Cycle graph C_8

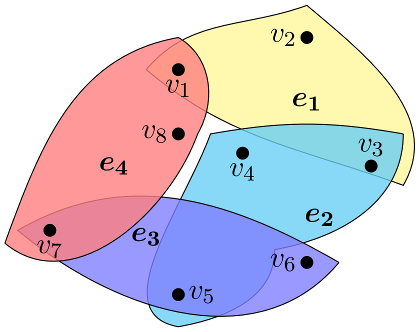

FIGURE 2. Loose cycle hypergraph $\mathcal{C}_8$

Next section deals with main results of the paper.

3. RESULTS

Consider any graph G. Let $\mathcal{H}_1(G)$ be the hypergraph obtained from G by taking each hyperedge to be closed neighbourhood of each vertex of G, then the number of distinct hyperedges in hypergraph $\mathcal{H}_1$ of G is atmost equal to order of G. Note that for cycle on three vertices, C_3 with vertex set $\{a_1, a_2, a_3\}$, its hypergraph $\mathcal{H}_1$ has a single hyperedge $\{a_1, a_2, a_3\}$.

In all four subsections below, we have considered hypergraphs of cycles. Cycles being 2-regular, closed neighbourhood of each vertex will have 3 elements. Thus, their hypergraphs $\mathcal{H}_1$ are 3-uniform as all hyperedges are of size 3 and hence the product of 3-regular hypergraphs is again a 3-regular hypergraph.

3.1. **Cartesian product of hypergraphs.**

In this subsection we consider cartesian product. Below we give an illustration for cartesian product of hypergraph $\mathcal{H}_1$ of cycles C_3 and C_4.

Illustration: Consider two cycles C_3 and C_4.

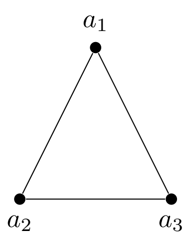

FIGURE 3. Cycle C_3

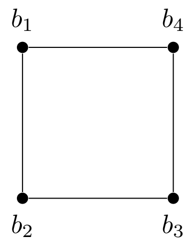

FIGURE 4. Cycle C_4

Let $\mathcal{H}_1(C_3)$ be hypergraph of cycle C_3 and $\mathcal{H}_1(C_4)$ be hypergraph of cycle C_4. Vertex sets of $\mathcal{H}_1(C_3)$ and $\mathcal{H}_1(C_4)$ are $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3, b_4\}$, respectively, and hyperedge sets are $\{a_1, a_2, a_3\}$ and $\Big\{\{b_1, b_2, b_3\}, \{b_2, b_3, b_4\}, \{b_1, b_3, b_4\}, \{b_1, b_2, b_4\}\Big\}$, respectively.

Consider hypergraph H, where $H = \mathcal{H}_1(C_3) \square \mathcal{H}_1(C_4)$. In H, we represent a vertex (a_i, b_j) such that $a_i \in V(C_3)$ and $b_j \in V(C_4)$ by $a_i b_j$. Then from Definition 2.6 [13],
$V(H) = \{a_1b_1, a_1b_2, a_1b_3, a_1b_4, a_2b_1, a_2b_2, a_2b_3, a_2b_4, a_3b_1, a_3b_2, a_3b_3, a_3b_4\}$ and
$E(H) = \Big\{\{a_1b_1, a_1b_2, a_1b_3\}, \{a_1b_2, a_1b_3, a_1b_4\}, \{a_1b_1, a_1b_3, a_1b_4\}, \{a_1b_1, a_1b_2, a_1b_4\},$
$\{a_2b_1, a_2b_2, a_2b_3\}, \{a_2b_2, a_2b_3, a_2b_4\}, \{a_2b_1, a_2b_3, a_2b_4\}, \{a_2b_1, a_2b_2, a_2b_4\}, \{a_3b_1, a_3b_2, a_3b_3\},$
$\{a_3b_2, a_3b_3, a_3b_4\}, \{a_3b_1, a_3b_3, a_3b_4\}, \{a_3b_1, a_3b_2, a_3b_4\}, \{a_1b_1, a_2b_1, a_3b_1\}, \{a_1b_2, a_2b_2, a_3b_2\},$
$\{a_1b_3, a_2b_3, a_3b_3\}, \{a_1b_4, a_2b_4, a_3b_4\}\Big\}.$
The hypergraphs $\mathcal{H}_1(C_3)$, $\mathcal{H}_1(C_4)$ and $H = \mathcal{H}_1(C_3)\square\mathcal{H}_1(C_4)$ are given in Figures 5, 6 and 7 respectively.

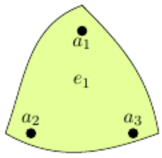

FIGURE 5. $\mathcal{H}_1(C_3)$

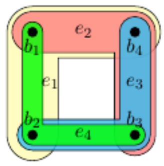

FIGURE 6. $\mathcal{H}_1(C_4)$

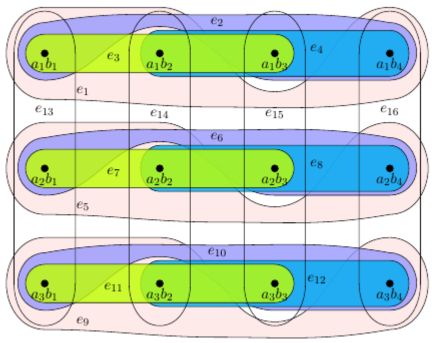

FIGURE 7. $\mathcal{H}_1(C_3)\square\mathcal{H}_1(C_4)$

Below we give the association scheme for design arising from cartesian product of hypergraphs $\mathcal{H}_1$ of two cycles C_m and C_n.
The number of blocks containing a pair of vertices whose
(i) first coordinates are same and second coordinates are at distance 1 in C_n is λ_1.
(ii) first coordinates are same and second coordinates are at distance 2 in C_n is λ_2.
(iii) first coordinates are at distance 1 in C_m and second coordiantes are same is λ_3.
(iv) first coordinates are at distance 2 in C_m and second coordinates are same is λ_4.
(v) one of the coordinates are same and the other is at distance greater than or equal to three in their respective graphs or both the coordinates are different is λ_5^*.

Using the above association scheme, we construct a design arising from cartesian product of hypergraphs of two cycles where hyperedges are considered as blocks.

Theorem 3.1. *The collection of all hyperedges of cartesian product, $H = \mathcal{H}_1(C_m) \square \mathcal{H}_1(C_n)$, of hypergraphs $\mathcal{H}_1$ of two cycles C_m and C_n forms a partially balanced incomplete block (PBIB)-design with 5-class association scheme having parameters (v, b, r, k, λ_i) for $1 \le i \le 5$ as follows:*

(i) $(9, 6, 2, 3, 1, 0, 1, 0, 0)$ *when $m = 3$ and $n = 3$.*

(ii) $(12, 16, 4, 3, 2, 2, 1, 0, 0)$ *when $m = 3$ and $n = 4$.*

(iii) $(3n, 4n, 4, 3, 2, 1, 1, 0, 0)$ *when* $m = 3$ *and* $n \geq 5$.
(iv) $(12, 16, 4, 3, 1, 0, 2, 2, 0)$ *when* $m = 4$ *and* $n = 3$.
(v) $(3m, 4m, 4, 3, 1, 0, 2, 1, 0)$ *when* $m \geq 5$ *and* $n = 3$.
(vi) $(16, 32, 6, 3, 2, 2, 2, 2, 0)$ *when* $m = 4$ *and* $n = 4$.
(vii) $(4n, 8n, 6, 3, 2, 1, 2, 2, 0)$ *when* $m = 4$ *and* $n \geq 5$.
(viii) $(4m, 8m, 6, 3, 2, 2, 2, 1, 0)$ *when* $m \geq 5$ *and* $n = 4$.
(ix) $(mn, 2mn, 6, 3, 2, 1, 2, 1, 0)$ *when* $m \geq 5$ *and* $n \geq 5$.

Proof. Consider two cycles C_m and C_n where m and n are greater than 3. Clearly, $\mathcal{H}_1(C_m)$ and $\mathcal{H}_1(C_n)$ has m and n distinct hyperedges respectively. For a cycle on 3 vertices, C_3, closed neighbourhood of all its vertices are same. Hence $\mathcal{H}_1(C_3)$ has a unique hyperedge.

Let $H = \mathcal{H}_1(C_m) \,\square\, \mathcal{H}_1(C_n)$. It is well known from hypergraph theory that vertices of H are cartesian product of vertex set of $\mathcal{H}_1(C_m)$ and $\mathcal{H}_1(C_n)$. Hence, the order of hypergraph H is mn.

Now, let us count the number of hyperedges $|E(H)|$ in H. For this, we partition the hyperedges of H into two subsets E_1 and E_2 where
$E_1 = \{\{x\} \times e \;:\; x \in V(\mathcal{H}_1(C_m)),\; e \in E(\mathcal{H}_1(C_n))\}$ and
$E_2 = \{e \times \{u\} \;:\; e \in E(\mathcal{H}_1(C_m)),\; u \in V(\mathcal{H}_1(C_n))\}$.
Clearly, $|E_1| = |V(\mathcal{H}_1(C_m))| \times |E(\mathcal{H}_1(C_n))|$ and $|E_2| = |E(\mathcal{H}_1(C_m))| \times |V(\mathcal{H}_1(C_n))|$. Therefore, $|E(H)| = |E_1| + |E_2|$ as E_1 and E_2 are disjoint, that is,
$|E(H)| = (|V(\mathcal{H}_1(C_m))| \times |E(\mathcal{H}_1(C_n))|) \;+\; (|E(\mathcal{H}_1(C_m))| \times |V(\mathcal{H}_1(C_n))|)$.
For $m \geq 4$ and $n \geq 4$, $|E(H)| = 2mn$.
When $m = 3$ and $n \geq 4$, $|E(H)| = 4n$ and for $m \geq 4$ and $n = 3$, $|E(H)| = 4m$.
When $m = 3$ and $n = 3$, $|E(H)| = 6$.

Taking each hyperedge of H to be a block of a design, we now find parameters of the design obtained. From Remark 2.1, we see that $H = \mathcal{H}_1(C_m) \,\square\, \mathcal{H}_1(C_n)$ is a 3-uniform hypergraph. Hence, block size k is 3. Each vertex in H is a 2-tuple where first element is a vertex of $\mathcal{H}_1(C_m)$ and second element is a vertex of $\mathcal{H}_1(C_n)$. To find repetition number of the design, let us consider a vertex, say, ab in H. From the compositions of hyperedges in subsets E_1 and E_2, it is clear that vertex ab appears atmost thrice in E_1 as vertex b is present in atmost 3 hyperedges in $\mathcal{H}_1(C_n)$. Similarly, vertex ab appears atmost 3 times in E_2 as vertex a appears thrice in $E(\mathcal{H}_1(C_m))$. Hence, ab appears atmost in 6 hyperedges of H. If $m = 3$ and $n \geq 4$ (or $m \geq 4$ and $n = 3$), repetition number of the design becomes 4 and when both m and n are equal to 3, repetition number reduces to 2.

To obtain the values of λ_i, for $1 \leq i \leq 5$, we consider different cases. Let $a_i \in V(\mathcal{H}_1(C_m))$ and $b_i \in V(\mathcal{H}_1(C_n))$. λ_1 gives the number of blocks in H containing a pair of vertices ab_1 and ab_2 where vertices b_1 and b_2 are at distance 1 in C_n. If b_1 and b_3 are vertices at distance 2 in C_n, then the number of blocks containing pair of vertices ab_1 and ab_3 gives the value of λ_2. Clearly, hyperedges containing the pairs of vertices (ab_1, ab_2) and (ab_1, ab_3) belong to set E_1. Similarly, λ_3 is the number of blocks containing a pair of vertices a_1b and a_2b in H where vertices a_1 and a_2 are at distance 1 in C_m and λ_4 gives the number of blocks containing vertices a_1b and a_3b where vertices a_1 and a_3 are at distance 2 in C_m. Clearly, hyperedges containing pairs of vertices (a_1b, a_2b) and (a_1b, a_3b) belong to set E_2. It is obvious that λ_5^* is always 0 as there is no hyperedge in hypergraph $\mathcal{H}_1$ of cycles containing a pair of vertices at distance greater than 2 contained in cycle or vertices in H wherein both the coordinates are differnt.

Case i) : $m = 3$ *and* $n = 3$.
As $\mathcal{H}_1$ of cycle C_3 has a single hyperedge and all vertices are at distance 1 in C_3, we get the design parameters as (9, 6, 2, 3, 1, 0, 1, 0, 0).

Case ii) : $m = 3$ *and* $n = 4$.
$\mathcal{H}_1(C_4)$ has four hyperedges such that a pair of vertices at distance 1 as well as a pair of vertices

at distance 2 in C_4 appear twice in $E(\mathcal{H}_1(C_4))$. Thus we get the values of λ_1 and λ_2 as 2. Thus, the design parameters are (12, 16, 4, 3, 2, 2, 1, 0, 0).

Case iii) : $m = 3$ and $n \geq 5$.

There are two hyperedges in $\mathcal{H}_1(C_n)$ containing a pair of vertices which are at distance 1 in C_n and one hyperedge in $\mathcal{H}_1(C_n)$ wherein a pair of vertices at distance 2 in C_n occurs together. Hence $\lambda_1 = 2$ and $\lambda_2 = 1$. Thus, design parameters are $(3n, 4n, 4, 3, 2, 1, 1, 0, 0)$.

Case iv) : $m = 4$ and $n = 3$.

Clearly, values of $v,\ b,\ r,\ k$ remains same as in $Case\ ii)$, only values of λ_i interchanges with respect to m and n values. Thus, parameters of design are (12, 16, 4, 3, 1, 0, 2, 2, 0).

Case v) : $m \geq 5$ and $n = 3$.

The values of $\lambda_1,\ \lambda_2$ interchanges with the values of $\lambda_3,\ \lambda_4$, respectively, as m and n is interchanged from that in $Case\ iii)$. Thus, design parameters are $(3m, 4m, 4, 3, 1, 0, 2, 1, 0)$.

Case vi) : $m = 4$ and $n = 4$.

There are two hyperedges each in H containing a pair of vertices which are distance 1 as well as a pair of vertices at distance 2 in C_4. Hence, we get the values of λ_i for $1 \leq i \leq 4$ as 2. Thus, design parameters are (16, 32, 6, 3, 2, 2, 2, 2, 0).

Case vii) : $m = 4$ and $n \geq 5$.

Any pair of vertices at distance 1 in C_n occur together twice in $E(\mathcal{H}_1(C_n))$. Similarly, a pair of vetices at distance 2 in C_n occur together exactly once in $E(\mathcal{H}_1(C_n))$. Thus $\lambda_1 = 2$ and $\lambda_2 = 1$. Since $m = 4$, values of λ_3 and λ_4 are 2. Thus, design parameters are $(4n, 8n, 6, 3, 2, 1, 2, 2, 0)$.

Case viii) : $m \geq 5$ and $n = 4$.

Parameters of the design obtained in this case is similar to that in $Case\ vii)$ where values of λ_1, λ_2 interchanges with the values of $\lambda_3,\ \lambda_4$, respectively, as values of m and n are interchanged. Thus, we get the parameters of design as $(4m, 8m, 6, 3, 2, 2, 2, 1, 0)$.

Case ix) : $m \geq 5$ and $n \geq 5$.

This is the general case for cartesian product of hypergraph $\mathcal{H}_1$ of cycles of order greater 4. Clearly, design parameters obtained are $(mn, 2mn, 6, 3, 2, 1, 2, 1, 0)$. □

In proof of Theorem 3.1, we have given parameters of first kind of the PBIB-designs constructed. Let us now see parameters of second kind.

Remark 3.1. *In the association scheme discussed above, we can split λ_5^* into various cases which include all other remaining pairs of vertices not applicable in first four cases, such as pairs of vertices whose first coordinates are same and second coordinates are at distances 3, 4 and so on upto diameter of C_n which is $\lfloor n/2 \rfloor$, first coordinates at distances 1, 2, and so on upto $\lfloor m/2 \rfloor$ and second coordinate same, both the coordinates different and at varying distances in their respective graphs with atleast one of the distances greater than 2. Thus we get a total of $(\lfloor m/2 \rfloor \times \lfloor n/2 \rfloor) + (\lfloor m/2 \rfloor \times \lfloor n/2 \rfloor)$ associate classes. Hence parameters of second kind is different for each design depending on the values of m and n.*

Illustration : Below we give an illustration taking the example of cartesian product of hypergraph $\mathcal{H}_1$ of cycles C_4 and C_6. The vertex sets of $\mathcal{H}_1(C_4)$ and $\mathcal{H}_1(C_6)$ are $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$, respectively, and hyperedge sets are $\Big\{\{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_2, a_3, a_4\}, \{a_1, a_3, a_4\}\Big\}$ and $\Big\{\{b_1, b_2, b_3\}, \{b_2, b_3, b_4\}, \{b_3, b_4, b_5\}, \{b_4, b_5, b_6\}, \{b_1, b_5, b_6\}, \{b_1, b_2, b_6\}\Big\}$, respectively.

Consider hypergraph H, where $H = \mathcal{H}_1(C_4) \square \mathcal{H}_1(C_6)$. Then from Definition 2.6,
$V(H) = \{a_1b_1, a_1b_2, a_1b_3, a_1b_4, a_1b_5, a_1b_6, a_2b_1, a_2b_2, a_2b_3, a_2b_4, a_2b_5, a_2b_6, a_3b_1, a_3b_2, a_3b_3, a_3b_4, a_3b_5, a_3b_6, a_4b_1, a_4b_2, a_4b_3, a_4b_4, a_4b_5, a_4b_6\}$ and
$E(H) = \Big\{\{a_1b_1, a_1b_2, a_1b_3\}, \{a_1b_2, a_1b_3, a_1b_4\}, \{a_1b_3, a_1b_4, a_1b_5\}, \{a_1b_4, a_1b_5, a_1b_6\},$

$\{a_1b_1, a_1b_5, a_1b_6\}$, $\{a_1b_1, a_1b_2, a_1b_6\}$, $\{a_2b_1, a_2b_2, a_2b_3\}$, $\{a_2b_2, a_2b_3, a_2b_4\}$, $\{a_2b_3, a_2b_4, a_2b_5\}$, $\{a_2b_4, a_2b_5, a_2b_6\}$, $\{a_2b_1, a_2b_5, a_2b_6\}$, $\{a_2b_1, a_2b_2, a_2b_6\}$, $\{a_3b_1, a_3b_2, a_3b_3\}$, $\{a_3b_2, a_3b_3, a_3b_4\}$, $\{a_3b_3, a_3b_4, a_3b_5\}$, $\{a_3b_4, a_3b_5, a_3b_6\}$, $\{a_3b_1, a_3b_5, a_3b_6\}$, $\{a_3b_1, a_3b_2, a_3b_6\}$, $\{a_4b_1, a_4b_2, a_4b_3\}$, $\{a_4b_2, a_4b_3, a_4b_4\}$, $\{a_4b_3, a_4b_4, a_4b_5\}$, $\{a_4b_4, a_4b_5, a_4b_6\}$, $\{a_4b_1, a_4b_5, a_4b_6\}$, $\{a_4b_1, a_4b_2, a_4b_6\}$, $\{a_1b_1, a_2b_1, a_3b_1\}$, $\{a_1b_1, a_2b_1, a_4b_1\}$, $\{a_2b_1, a_3b_1, a_4b_1\}$, $\{a_1b_1, a_3b_1, a_4b_1\}$, $\{a_1b_2, a_2b_2, a_3b_2\}$, $\{a_1b_2, a_2b_2, a_4b_2\}$, $\{a_2b_2, a_3b_2, a_4b_2\}$, $\{a_1b_2, a_3b_2, a_4b_2\}$, $\{a_1b_3, a_2b_3, a_3b_3\}$, $\{a_1b_3, a_2b_3, a_4b_3\}$, $\{a_2b_3, a_3b_3, a_4b_3\}$, $\{a_1b_3, a_3b_3, a_4b_3\}$, $\{a_1b_4, a_2b_4, a_3b_4\}$, $\{a_1b_4, a_2b_4, a_4b_4\}$, $\{a_2b_4, a_3b_4, a_4b_4\}$, $\{a_1b_4, a_3b_4, a_4b_4\}$, $\{a_1b_5, a_2b_5, a_3b_5\}$, $\{a_1b_5, a_2b_5, a_4b_5\}$, $\{a_2b_5, a_3b_5, a_4b_5\}$, $\{a_1b_5, a_3b_5, a_4b_5\}$, $\{a_1b_6, a_2b_6, a_3b_6\}$, $\{a_1b_6, a_2b_6, a_4b_6\}$, $\{a_2b_6, a_3b_6, a_4b_6\}$, $\{a_1b_6, a_3b_6, a_4b_6\}\Big\}$.

The association scheme is given explicitly as follows.
The number of blocks containing a pair of vertices whose
(i) first coordinates are same and second coordinates are at distance 1 in C_6 is λ_1.
(ii) first coordinates are same and second coordinates are at distance 2 in C_6 is λ_2.
(iii) first coordinates are at distance 1 in C_4 and second coordinates are same is λ_3.
(iv) first coordinates are at distance 2 in C_4 and second coordinates are same is λ_4.
(v) first coordinates are same and second coordinates are at distance 3 in C_6 is λ_5.
(vi) first coordinates are at distance 1 in C_4 and second coordinates are at distance 1 in C_6 is λ_6.
(vii) first coordinates are at distance 1 in C_4 and second coordinates at distance 2 in C_6 is λ_7.
$(viii)$ first coordinates are at distance 1 in C_4 and second coordinates at distance 3 in C_6 is λ_8.
(ix) first coordinates are at distance 2 in C_4 and second coordinates are at distance 1 in C_6 is λ_9.
(x) first coordinates are at distance 2 in C_4 and second coordinates are at distance 2 in C_6 is λ_{10}.
(xi) first coordinates are at distance 2 in C_4 and second coordinates at distance 3 in C_6 is λ_{11}.

Further, we give an algorithm to obtain the cartesian product of two hypergraphs $\mathcal{H}_1(C_m)\Box\mathcal{H}_1(C_n)$ along with the association scheme for design arising from it.

Algorithm to obtain cartesian product of two hypergraphs and the association scheme of the design arising from it.	
Input:	Cycle graphs C_m and C_n. Vertex sets V_1, V_2, and edge sets E_1, E_2 of graphs C_m and C_n, respectively. Distance matrices D_1 and D_2 of graphs C_m and C_n, respectively.
Output:	Cartesian product of hypergraphs, that is, $\mathcal{H}_1(C_m)\Box\mathcal{H}_1(C_n)$. Associates of each vertex.
Algorithm 1	
Step 1:	Start
Step 2:	Initialize $G \leftarrow (0, 0)$ graph
Step 3:	Define function $\mathcal{H}_1(G) := neighbourhood\_hypergraph(G)$
Step 4:	Call function $\mathcal{H}_1()$ add C_m to G. Return $\mathcal{H}_1(C_m)$ add C_n to G. Return $\mathcal{H}_1(C_n)$
Step 5:	Initialize $H_1 \leftarrow (0, 0)$ hypergraph, $H_2 \leftarrow (0, 0)$ hypergraph
Step 6:	Define function $CP(H_1, H_2) := cartesian\_product(H_1, H_2)$
Step 7:	Call function $CP()$ add $\mathcal{H}_1(C_m)$ to H_1, add $\mathcal{H}_1(C_n)$ to H_2
Step 8:	Return edge set $E(CP(\mathcal{H}_1(C_m), \mathcal{H}_1(C_n)))$
Step 9:	For (a_i, b_j), (a_l, b_k) in $E(CP(\mathcal{H}_1(C_m), \mathcal{H}_1(C_n)))$, $i, l \leftarrow 1$ to m, $j, k \leftarrow 1$ to n, $s \leftarrow 1$ to $(\lfloor m/2\rfloor \times \lfloor n/2\rfloor) + (\lfloor m/2\rfloor \times \lfloor n/2\rfloor)$
Step 10:	Define $\lambda_s \leftarrow$ no of edges containing (a_i, b_j) and (a_l, b_k) for varying values of $i,\ l,\ j,\ k$
Step 11:	Return the values of $\lambda_1, \lambda_2, \ldots, \lambda_{(\lfloor m/2\rfloor \times \lfloor n/2\rfloor)+(\lfloor m/2\rfloor \times \lfloor n/2\rfloor)}$

Step 12: Stop

Remark 3.2. *Time complexity of the above Algorithm 1 can be proved in similar lines as in* [6]. *The order and size of cartesian product hypergraph $H = \mathcal{H}_1(C_m) \,\square\, \mathcal{H}_1(C_n)$ is mn and $2mn$, respectively. Therefore, time complexity of Algorithm 1 is $O(mn \times 2mn)$ which is equal to $O(m^2n^2)$.*

Output of the above algorithm when $m = 4$ and $n = 6$ can be verified from the table below.

TABLE 2. Table of association scheme for design arising from $\mathcal{H}_1(C_4) \,\square\, \mathcal{H}_1(C_6)$

Vertex	1^{st} associate	2^{nd} associate	3^{rd} associate	4^{th} associate	5^{th} associate	6^{th} associate	7^{th} associate	8^{th} associate	9^{th} associate	10^{th} associate	11^{th} associate
a_1b_1	a_1b_2, a_1b_6	a_1b_3, a_1b_5	a_2b_1, a_4b_1	a_3b_1	a_1b_4	$a_2b_2, a_2b_6, a_4b_2, a_4b_6$	$a_2b_3, a_2b_5, a_4b_3, a_4b_5$	a_2b_4, a_4b_4	a_3b_2, a_3b_6	a_3b_3, a_3b_5	a_3b_4
a_1b_2	a_1b_1, a_1b_3	a_1b_4, a_1b_6	a_2b_2, a_4b_2	a_3b_2	a_1b_5	$a_2b_1, a_2b_3, a_4b_1, a_4b_3$	$a_2b_4, a_2b_6, a_4b_4, a_4b_6$	a_2b_5, a_4b_5	a_3b_1, a_3b_3	a_3b_4, a_3b_6	a_3b_5
a_1b_3	a_1b_2, a_1b_4	a_1b_1, a_1b_5	a_2b_3, a_4b_3	a_3b_3	a_1b_6	$a_2b_2, a_2b_4, a_4b_2, a_4b_4$	$a_2b_1, a_2b_5, a_4b_1, a_4b_5$	a_2b_6, a_4b_6	a_3b_2, a_3b_4	a_3b_1, a_3b_5	a_3b_6
a_1b_4	a_1b_3, a_1b_5	a_1b_2, a_1b_6	a_2b_4, a_4b_4	a_3b_4	a_1b_1	$a_2b_3, a_2b_5, a_4b_3, a_4b_5$	$a_2b_2, a_2b_6, a_4b_2, a_4b_6$	a_2b_1, a_4b_1	a_3b_3, a_3b_5	a_3b_2, a_3b_6	a_3b_1
a_1b_5	a_1b_4, a_1b_6	a_1b_1, a_1b_3	a_2b_5, a_4b_5	a_3b_5	a_1b_2	$a_2b_4, a_2b_6, a_4b_4, a_4b_6$	$a_2b_1, a_2b_3, a_4b_1, a_4b_3$	a_2b_2, a_4b_2	a_3b_4, a_3b_6	a_3b_1, a_3b_3	a_3b_2
a_1b_6	a_1b_1, a_1b_5	a_1b_2, a_1b_4	a_2b_6, a_4b_6	a_3b_6	a_1b_3	$a_2b_1, a_2b_5, a_4b_1, a_4b_5$	$a_2b_2, a_2b_4, a_4b_2, a_4b_4$	a_2b_3, a_4b_3	a_3b_1, a_3b_5	a_3b_2, a_3b_4	a_3b_3
a_2b_1	a_2b_2, a_2b_6	a_2b_3, a_2b_5	a_1b_1, a_3b_1	a_4b_1	a_2b_4	$a_1b_2, a_1b_6, a_3b_2, a_3b_6$	$a_1b_3, a_1b_5, a_3b_3, a_3b_5$	a_1b_4, a_3b_4	a_4b_2, a_4b_6	a_4b_3, a_4b_5	a_4b_4
a_2b_2	a_2b_1, a_2b_3	a_2b_4, a_2b_6	a_1b_2, a_3b_2	a_4b_2	a_2b_5	$a_1b_1, a_1b_3, a_3b_1, a_3b_3$	$a_1b_4, a_1b_6, a_3b_4, a_3b_6$	a_1b_5, a_3b_5	a_4b_1, a_4b_3	a_4b_4, a_4b_6	a_4b_5
a_2b_3	a_2b_2, a_2b_4	a_2b_1, a_2b_5	a_1b_3, a_3b_3	a_4b_3	a_2b_6	$a_1b_2, a_1b_4, a_3b_2, a_3b_4$	$a_1b_1, a_1b_5, a_3b_1, a_3b_5$	a_1b_6, a_3b_6	a_4b_2, a_4b_4	a_4b_1, a_4b_5	a_4b_6
a_2b_4	a_2b_3, a_2b_5	a_2b_2, a_2b_6	a_1b_4, a_3b_4	a_4b_4	a_2b_1	$a_1b_3, a_1b_5, a_3b_3, a_3b_5$	$a_1b_2, a_1b_6, a_3b_2, a_3b_6$	a_1b_1, a_3b_1	a_4b_3, a_4b_5	a_4b_2, a_4b_6	a_4b_1
a_2b_5	a_2b_4, a_2b_6	a_2b_1, a_2b_3	a_1b_5, a_3b_5	a_4b_5	a_2b_2	$a_1b_4, a_1b_6, a_3b_4, a_3b_6$	$a_1b_1, a_1b_3, a_3b_1, a_3b_3$	a_1b_2, a_3b_2	a_4b_4, a_4b_6	a_4b_1, a_4b_3	a_4b_2
a_2b_6	a_2b_1, a_2b_5	a_2b_2, a_2b_4	a_1b_6, a_3b_6	a_4b_6	a_2b_3	$a_1b_1, a_1b_5, a_3b_1, a_3b_5$	$a_1b_2, a_1b_4, a_3b_2, a_3b_4$	a_1b_3, a_3b_3	a_4b_1, a_4b_5	a_4b_2, a_4b_4	a_4b_3
a_3b_1	a_3b_2, a_3b_6	a_3b_3, a_3b_5	a_2b_1, a_4b_1	a_1b_1	a_3b_4	$a_2b_2, a_2b_6, a_4b_2, a_4b_6$	$a_2b_3, a_2b_5, a_4b_3, a_4b_5$	a_2b_4, a_4b_4	a_1b_2, a_1b_6	a_1b_3, a_1b_5	a_1b_4
a_3b_2	a_3b_1, a_3b_3	a_3b_4, a_3b_6	a_2b_2, a_4b_2	a_1b_2	a_3b_5	$a_2b_1, a_2b_3, a_4b_1, a_4b_3$	$a_2b_4, a_2b_6, a_4b_4, a_4b_6$	a_2b_5, a_4b_5	a_1b_1, a_1b_3	a_1b_4, a_1b_6	a_1b_5
a_3b_3	a_3b_2, a_3b_4	a_3b_1, a_3b_5	a_2b_3, a_4b_3	a_1b_3	a_3b_6	$a_2b_2, a_2b_4, a_4b_2, a_4b_4$	$a_2b_1, a_2b_5, a_4b_1, a_4b_5$	a_2b_6, a_4b_6	a_1b_2, a_1b_4	a_1b_1, a_1b_5	a_1b_6
a_3b_4	a_3b_3, a_3b_5	a_3b_2, a_3b_6	a_2b_4, a_4b_4	a_1b_4	a_3b_1	$a_2b_3, a_2b_5, a_4b_3, a_4b_5$	$a_2b_2, a_2b_6, a_4b_2, a_4b_6$	a_2b_1, a_4b_1	a_1b_3, a_1b_5	a_1b_2, a_1b_6	a_1b_1
a_3b_5	a_3b_4, a_3b_6	a_3b_1, a_3b_3	a_2b_5, a_4b_5	a_1b_5	a_3b_2	$a_2b_4, a_2b_6, a_4b_4, a_4b_6$	$a_2b_1, a_2b_3, a_4b_1, a_4b_3$	a_2b_2, a_4b_2	a_1b_4, a_1b_6	a_1b_1, a_1b_3	a_1b_2
a_3b_6	a_3b_1, a_3b_5	a_3b_2, a_3b_4	a_2b_6, a_4b_6	a_1b_6	a_3b_3	$a_2b_1, a_2b_5, a_4b_1, a_4b_5$	$a_2b_2, a_2b_4, a_4b_2, a_4b_4$	a_2b_3, a_4b_3	a_1b_1, a_1b_5	a_1b_2, a_1b_4	a_1b_3
a_4b_1	a_4b_2, a_4b_6	a_4b_3, a_4b_5	a_1b_1, a_3b_1	a_2b_1	a_4b_4	$a_1b_2, a_1b_6, a_3b_2, a_3b_6$	$a_1b_3, a_1b_5, a_3b_3, a_3b_5$	a_1b_4, a_3b_4	a_2b_2, a_2b_6	a_2b_3, a_2b_5	a_2b_4
a_4b_2	a_4b_1, a_4b_3	a_4b_4, a_4b_6	a_1b_2, a_3b_2	a_2b_2	a_4b_5	$a_1b_1, a_1b_3, a_3b_1, a_3b_3$	$a_1b_4, a_1b_6, a_3b_4, a_3b_6$	a_1b_5, a_3b_5	a_2b_1, a_2b_3	a_2b_4, a_2b_6	a_2b_5
a_4b_3	a_4b_2, a_4b_4	a_4b_1, a_4b_5	a_1b_3, a_3b_3	a_2b_3	a_4b_6	$a_1b_2, a_1b_4, a_3b_2, a_3b_4$	$a_1b_1, a_1b_5, a_3b_1, a_3b_5$	a_1b_6, a_3b_6	a_2b_2, a_2b_4	a_2b_1, a_2b_5	a_2b_6
a_4b_4	a_4b_3, a_4b_5	a_4b_2, a_4b_6	a_1b_4, a_3b_4	a_2b_4	a_4b_1	$a_1b_3, a_1b_5, a_3b_3, a_3b_5$	$a_1b_2, a_1b_6, a_3b_2, a_3b_6$	a_1b_1, a_3b_1	a_2b_3, a_2b_5	a_2b_2, a_2b_6	a_2b_1
a_4b_5	a_4b_4, a_4b_6	a_4b_1, a_4b_3	a_1b_5, a_3b_5	a_2b_5	a_4b_2	$a_1b_4, a_1b_6, a_3b_4, a_3b_6$	$a_1b_1, a_1b_3, a_3b_1, a_3b_3$	a_1b_2, a_3b_2	a_2b_4, a_2b_6	a_2b_1, a_2b_3	a_2b_2

Continued on next page

Table continued from previous page

Vertex	1^{st} associate	2^{nd} associate	3^{rd} associate	4^{th} associate	5^{th} associate	6^{th} associate	7^{th} associate	8^{th} associate	9^{th} associate	10^{th} associate	11^{th} associate
a_4b_6	a_4b_1, a_4b_5	a_4b_2, a_4b_4	a_1b_6, a_3b_6	a_2b_6	a_4b_3	a_1b_1, a_1b_5, a_3b_1, a_3b_5	a_1b_2, a_1b_4, a_3b_2, a_3b_4	a_1b_3, a_3b_3	a_2b_1, a_2b_5	a_2b_2, a_2b_4	a_2b_3

Parameters of first kind are $v = 24$, $b = 48$, $r = 6$, $k = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 2$, $\lambda_i = 0$ for $1 \le i \le 11$ and parameters of second kind are $n_1 = 2$, $n_2 = 2$, $n_3 = 2$, $n_4 = 1$, $n_5 = 1$, $n_6 = 4$, $n_7 = 4$, $n_8 = 2$, $n_9 = 2$, $n_{10} = 2$ and $n_{11} = 1$ along with matrices P_1 to P_{11}, each of order 11×11 which can be obtained from Table 2 as explained in Definition 2.1.

Next we move on to another important standard product.

3.2. **Direct product of hypergraphs.** Let us start with an illustration which will give more clarity of the concept.

Illustration: Consider cycles C_3 and C_4 given in Figures 3 and 4, respectively. Consider hypergraph H, where $H = \mathcal{H}_1(C_3) \times \mathcal{H}_1(C_4)$. Then from Definition 2.7 [13],
$V(H) = \{a_1b_1, a_1b_2, a_1b_3, a_1b_4, a_2b_1, a_2b_2, a_2b_3, a_2b_4, a_3b_1, a_3b_2, a_3b_3, a_3b_4\}$ and
$E(H) = \Big\{\{a_1b_1, a_2b_2, a_3b_3\}, \{a_1b_1, a_2b_3, a_3b_2\}, \{a_1b_2, a_2b_1, a_3b_3\}, \{a_1b_2, a_2b_3, a_3b_1\},$
$\{a_1b_3, a_2b_1, a_3b_2\}, \{a_1b_3, a_2b_2, a_3b_1\}, \{a_1b_2, a_2b_3, a_3b_4\}, \{a_1b_2, a_2b_4, a_3b_3\}, \{a_1b_3, a_2b_2, a_3b_4\},$
$\{a_1b_3, a_2b_4, a_3b_2\}, \{a_1b_4, a_2b_2, a_3b_3\}, \{a_1b_4, a_2b_3, a_3b_2\}, \{a_1b_1, a_2b_3, a_3b_4\}, \{a_1b_1, a_2b_4, a_3b_3\},$
$\{a_1b_3, a_2b_1, a_3b_4\}, \{a_1b_3, a_2b_4, a_3b_1\}, \{a_1b_4, a_2b_1, a_3b_3\}, \{a_1b_4, a_2b_3, a_3b_1\}, \{a_1b_1, a_2b_2, a_3b_4\},$
$\{a_1b_1, a_2b_4, a_3b_2\}, \{a_1b_2, a_2b_1, a_3b_4\}, \{a_1b_2, a_2b_4, a_3b_1\}, \{a_1b_4, a_2b_1, a_3b_2\}, \{a_1b_4, a_2b_2, a_3b_1\}\Big\}.$

Now, let us define association scheme for the design arising from direct product of hypergraphs $\mathcal{H}_1$ of two cycles C_m and C_n.
The number of blocks containing a pair of vertices whose
(i) first coordinates are at distance 1 in C_m and second coordinates are at distance 1 in C_n is λ_1.
(ii) first coordinates are at distance 1 in C_m and second coordinates are at distance 2 in C_n is λ_2.
(iii) first coordinates are at distance 2 in C_m and second coordiantes are at distance 1 in C_n is λ_3.
(iv) first coordinates are at distance 2 in C_m and second coordinates are at distance 2 in C_n is λ_4.
(v) atleast one of the coordinates are at distance greater than or equal to 3 in their respective graphs or one of the coodinates are same is λ_5^*.

Using the above association scheme, we obtain following result.

Theorem 3.2. *The collection of all hyperedges of direct product, $H = \mathcal{H}_1(C_m) \times \mathcal{H}_1(C_n)$, of hypergraphs $\mathcal{H}_1$ of two cycles C_m and C_n forms a partially balanced incomplete block (PBIB)-design with 5-class association scheme having parameters (v, b, r, k, λ_i) for $1 \le i \le 5$ as follows*
(i) $(9, 6, 2, 3, 1, 0, 0, 0, 0)$ *when* $m = 3$ *and* $n = 3$.
(ii) $(12, 24, 6, 3, 2, 2, 0, 0, 0)$ *when* $m = 3$ *and* $n = 4$.
(iii) $(3n, 6n, 6, 3, 2, 1, 0, 0, 0)$ *when* $m = 3$ *and* $n \ge 5$.
(iv) $(12, 24, 6, 3, 2, 0, 2, 0, 0)$ *when* $m = 4$ *and* $n = 3$.
(v) $(3m, 6m, 6, 3, 2, 0, 1, 0, 0)$ *when* $m \ge 5$ *and* $n = 3$.
(vi) $(16, 96, 18, 3, 4, 4, 4, 4, 0)$ *when* $m = 4$ *and* $n = 4$.
(vii) $(4n, 24n, 18, 3, 4, 2, 4, 2, 0)$ *when* $m = 4$ *and* $n \ge 5$.
(viii) $(4m, 24m, 18, 3, 4, 4, 2, 2, 0)$ *when* $m \ge 5$ *and* $n = 4$.
(ix) $(mn, 6mn, 18, 3, 4, 2, 2, 1, 0)$ *when* $m \ge 5$ *and* $n \ge 5$.

Proof. Let $H = \mathcal{H}_1(C_m) \times \mathcal{H}_1(C_n)$ be the hypergraph obtained by taking direct product of hypergraphs $\mathcal{H}_1$ of two cycles C_m and C_n. Clearly, there are mn number of vertices in H.

Now, we count the number of hyperedges, $|E(H)|$, in hypergraph H. Vertices of H are the set of 2-tuples where first element belongs to $V(\mathcal{H}_1(C_m))$ and second element belongs to $V(\mathcal{H}_1(C_n))$. Vertices present in each hyperedge of H is such that the set of first elements form a

hyperedge in $\mathcal{H}_1(C_m)$ and the set of second elements form a hyperedge in $\mathcal{H}_1(C_n)$. Using combinatorics, we can easily prove that six distinct hyperedges in H can be obtained corresponding to a pair of hyperedges each in $\mathcal{H}_1(C_m)$ and $\mathcal{H}_1(C_n)$. There are $|E(\mathcal{H}_1(C_m))|$ and $|E(\mathcal{H}_1(C_n))|$ number of hyperedges in $\mathcal{H}_1(C_m)$ and $\mathcal{H}_1(C_n)$, respectively. Hence, total number of distinct hyperedges $|E(H)|$ in H is $|E(\mathcal{H}_1(C_m))| \times |E(\mathcal{H}_1(C_n))| \times 6$.
For $m \geq 4$ and $n \geq 4$, $|E(H)| = 6mn$.
When $m = 3$ and $n \geq 4$, $|E(H)| = 6n$ and for $m \geq 4$ and $n = 3$, $|E(H)| = 6m$.
When $m = 3$ and $n = 3$, $|E(H)| = 6$.

Considering hyperedges of H to be blocks of a design, we now find pararmeters of this design. H being a 3-uniform hypergraph from Remark 2.1, we get the block size k to be 3. Next, we find repetition number of the design. There are exactly two hyperedges in H containing a particular vertex, say, a_ib_j (where $1 \leq i \leq m$ and $1 \leq j \leq n$) corresponding to a particular pair of hyperedges in $E(\mathcal{H}_1(C_m))$ and $E(\mathcal{H}_1(C_n))$, respectively. For example, suppose $\{a_1, a_2, a_3\} \in E(\mathcal{H}_1(C_m))$ and $\{b_1, b_2, b_3\} \in E(\mathcal{H}_1(C_n))$ be any two hyperedges, then we get exactly two hyperedges containing the vertex a_1b_1 in H of the form $\{a_1b_1, a_2b_2, a_3b_3\}$ and $\{a_1b_1, a_2b_3, a_3b_2\}$. There are atmost three hyperedges in $\mathcal{H}_1(C_m)$ containing vertex a_i and similarly atmost three hyperedges in $\mathcal{H}_1(C_n)$ containing vertex b_i. Thus, there are atmost 18 hyperedges in H containing the vertex a_ib_j where $1 \leq i \leq m$ and $1 \leq j \leq n$. Equality is attained when both m and n are greater than 3. If $m = 3$ and $n \geq 4$ (or $m \geq 4$ and $n = 3$), repetition number of the design becomes 6 and when both m and n are equal to 3, repetition number reduces to 2.

To obtain the values of λ_i, for $1 \leq i \leq 5$, we consider different cases. Let $a_i \in V(\mathcal{H}_1(C_m))$ and $b_i \in V(\mathcal{H}_1(C_n))$. λ_1 gives the number of blocks in H containing a pair of vertices a_1b_1 and a_2b_2 where vertices a_1 and a_2 are at distance 1 in C_m and vertices b_1 and b_2 are at distance 1 in C_n. If b_1 and b_3 are vertices at distance 2 in C_n, then the number of blocks containing the pair of vertices a_1b_1 and a_2b_3 is λ_2. Similarly, if a_1 and a_3 are vertices at distance 2 in C_m, then number of blocks containing the pair of vertices a_1b_1 and a_3b_2 is λ_3. λ_4 gives the number of blocks containing a pair of vertices a_1b_1 and a_3b_3 where vertices a_1 and a_3 are at distance 2 in C_m and vertices b_1 and b_3 are at distance 2 in C_n. Number of blocks containing a pair of vertices which is different from those mentioned above gives the value of λ_5^*. Clearly, λ_5^* is always 0.

Case i) : $m = 3$ *and* $n = 3$.
Since $\mathcal{H}_1(C_3)$ has a unique hyperedge and all vertices are mutually adjacent to each other in C_3 we get the value of λ_1 as 1. Remaining all λ_i, $2 \leq i \leq 5$ goes to zero. Thus, design parameters are (9, 6, 2, 3, 1, 0, 0, 0, 0).

Case ii) : $m = 3$ *and* $n = 4$.
There are two hyperedges each in $\mathcal{H}_1(C_4)$ containing a pair of vertices at distance 1 in C_4 and at distance 2 in C_4. Thus, we get the value of λ_1 and λ_2 as 2. C_3 being a complete graph, λ_3 and λ_4 becomes 0. Thus, design parameters are (12, 24, 6, 3, 2, 2, 0, 0, 0).

Case iii) : $m = 3$ *and* $n \geq 5$.
There are two and one hyperedges each in $\mathcal{H}_1(C_n)$ containing a pair of vertices at distance 1 and 2 respectively in C_n. Thus, we get the values of λ_1 and λ_2 as 2 and 1 respectively. Hence, parameters of design obtained are $(3n, 6n, 6, 3, 2, 1, 0, 0, 0)$.

Case iv) : $m = 4$ *and* $n = 3$.
The parameters of design obtained in this case is similar to design parameters obtained in $Case\ ii)$ where only the values of λ_2 and λ_3 are interchanged as m and n values are interchanged. Thus, design parameters are (12, 24, 6, 3, 2, 0, 2, 0, 0).

Case v) : $m \geq 5$ *and* $n = 3$.
Here, only the values of λ_2 and λ_3 are interchanged from that in $Case\ iii)$. Remaining all parameters are same. Hence the parameters of design are $(3m, 6m, 6, 3, 2, 0, 1, 0, 0)$.

Case vi) : $m = 4$ *and* $n = 4$.
There are two hyperedges each in H containing a pair of vertices which are at distance 1 as well as at distance 2 in C_4. Hence, we get the values of λ_i for $1 \leq i \leq 4$ as 4. Therefore, design parameters are (16, 96, 18, 3, 4, 4, 4, 4, 0).

Case vii) : $m = 4$ *and* $n \geq 5$.
Clearly, there are 2 hyperedges in $\mathcal{H}_1(C_n)$ containing a pair of vertices at distance 1 in C_n and a single hyperedge containing a pair of vertices at distance 2 in C_n. Hence, λ_1 is 4 and λ_2 is 2. As $m = 4$, we get the value of λ_3 as 4 and λ_4 as 2. Thus, design parameters are $(4n, 24n, 18, 3, 4, 2, 4, 2, 0)$.

Case viii) : $m \geq 5$ *and* $n = 4$.
The parameters of design obtained in this case is similar to that of $Case\ vii)$ except that the values of λ_2 and λ_3 gets interchanged as m and n values are interchanged. Thus, we get the parameters as $(4m, 24m, 18, 3, 4, 4, 2, 2, 0)$.

Case ix) : $m \geq 5$ *and* $n \geq 5$.
This is the general case for direct product of hypergraphs $\mathcal{H}_1$ of cycles of order greater than 4. The design parameters are $(mn, 6mn, 18, 3, 4, 2, 2, 1, 0)$.

Parameters of second kind differ depending on diameters of C_m and C_n as explained in Remark 3.1. Each vertex will have $(\lfloor m/2 \rfloor \times \lfloor n/2 \rfloor) + (\lfloor m/2 \rfloor + \lfloor n/2 \rfloor)$ number of associates. For explicit values of m and n, we can get parameters of second kind in similar way as illustrated in Remark 3.1. □

Next subsection deals with yet another important graph product, the strong product, where edges are union of edges obtained from cartesian product and direct product.

3.3. **Strong product of hypergraphs.** We begin this subsection with an illustration showing strong product of hypergraphs.
Illustration: Consider cycles C_3 and C_4 given in Figures 3 and 4, respectively. From Definition 2.8 of strong product, we see that hyperedges of hypergraph H, where $H = \mathcal{H}_1(C_3) \boxtimes \mathcal{H}_1(C_4)$ is union of hyperedges present in hypergraphs obtained by taking cartesian product and direct product of $\mathcal{H}_1(C_3)$ and $\mathcal{H}_1(C_4)$. Therefore,
$V(H) = \{a_1b_1, a_1b_2, a_1b_3, a_1b_4, a_2b_1, a_2b_2, a_2b_3, a_2b_4, a_3b_1, a_3b_2, a_3b_3, a_3b_4\}$ and
$E(H) = \Big\{\{a_1b_1, a_1b_2, a_1b_3\}, \{a_1b_2, a_1b_3, a_1b_4\}, \{a_1b_1, a_1b_3, a_1b_4\}, \{a_1b_1, a_1b_2, a_1b_4\},$
$\{a_2b_1, a_2b_2, a_2b_3\}, \{a_2b_2, a_2b_3, a_2b_4\}, \{a_2b_1, a_2b_3, a_2b_4\}, \{a_2b_1, a_2b_2, a_2b_4\}, \{a_3b_1, a_3b_2, a_3b_3\},$
$\{a_3b_2, a_3b_3, a_3b_4\}, \{a_3b_1, a_3b_3, a_3b_4\}, \{a_3b_1, a_3b_2, a_3b_4\}, \{a_1b_1, a_2b_1, a_3b_1\}, \{a_1b_2, a_2b_2, a_3b_2\},$
$\{a_1b_3, a_2b_3, a_3b_3\}, \{a_1b_4, a_2b_4, a_3b_4\}, \{a_1b_1, a_2b_2, a_3b_3\}, \{a_1b_1, a_2b_3, a_3b_2\}, \{a_1b_2, a_2b_1, a_3b_3\},$
$\{a_1b_2, a_2b_3, a_3b_1\}, \{a_1b_3, a_2b_1, a_3b_2\}, \{a_1b_3, a_2b_2, a_3b_1\}, \{a_1b_2, a_2b_3, a_3b_4\}, \{a_1b_2, a_2b_4, a_3b_3\},$
$\{a_1b_3, a_2b_2, a_3b_4\}, \{a_1b_3, a_2b_4, a_3b_2\}, \{a_1b_4, a_2b_2, a_3b_3\}, \{a_1b_4, a_2b_3, a_3b_2\}, \{a_1b_1, a_2b_3, a_3b_4\},$
$\{a_1b_1, a_2b_4, a_3b_3\}, \{a_1b_3, a_2b_1, a_3b_4\}, \{a_1b_3, a_2b_4, a_3b_1\}, \{a_1b_4, a_2b_1, a_3b_3\}, \{a_1b_4, a_2b_3, a_3b_1\},$
$\{a_1b_1, a_2b_2, a_3b_4\}, \{a_1b_1, a_2b_4, a_3b_2\}, \{a_1b_2, a_2b_1, a_3b_4\}, \{a_1b_2, a_2b_4, a_3b_1\}, \{a_1b_4, a_2b_1, a_3b_2\},$
$\{a_1b_4, a_2b_2, a_3b_1\}\Big\}$.

Let us define the association scheme for the design arising from strong product of hypergraphs $\mathcal{H}_1$ of two cycles C_m and C_n. Since, the edges in strong product is union of edges of cartesian product and direct product which are disjoint, we get a 10 class association scheme for the design arising from strong product of hypergraphs.
The number of blocks containing a pair of vertices whose
(i) first coordinates are same and second coordinates are at distance 1 in C_n is λ_1.
(ii) first coordinates are same and second coordinates are at distance 2 in C_n is λ_2.
(iii) first coordinates are at distance 1 in C_m and second coordiantes are same is λ_3.
(iv) first coordinates are at distance 2 in C_m and second coordinates are same is λ_4.

(v) atleast one of the coordinates are at distance greater than or equal to 3 in their respective graphs and the other coordinates are same is λ_5^*.
(vi) first coordinates are at distance 1 in C_m and second coordinates are at distance 1 in C_n is λ_6.
(vii) first coordinates are at distance 1 in C_m and second coordinates are at distance 2 in C_n is λ_7.
$(viii)$ first coordinates are at distance 2 in C_m and second coordiantes at distance 1 in C_n is λ_8.
(ix) first coordinates are at distance 2 in C_m and second coordinates are at distance 2 in C_n is λ_9.
(x) atleast one of the coordinates are at distance greater than or equal to 3 in their respective graphs is λ_{10}^*.

The following result is obtained using the above association scheme.

Theorem 3.3. *The collection of all hyperedges of strong product, $H = \mathcal{H}_1(C_m) \boxtimes \mathcal{H}_1(C_n)$, of hypergraphs $\mathcal{H}_1$ of two cycles C_m and C_n forms a partially balanced incomplete block (PBIB)-design with 10-class association scheme having parameters (v, b, r, k, λ_i) for $1 \leq i \leq 10$ as follows.*

(i) $(9, 12, 4, 3, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0)$ *when* $m = 3$ *and* $n = 3$.
(ii) $(12, 40, 10, 3, 2, 2, 1, 0, 0, 2, 2, 0, 0, 0)$ *when* $m = 3$ *and* $n = 4$.
(iii) $(3n, 10n, 10, 3, 2, 1, 1, 0, 0, 2, 1, 0, 0, 0)$ *when* $m = 3$ *and* $n \geq 5$.
(iv) $(12, 40, 10, 3, 1, 0, 2, 2, 0, 2, 0, 2, 0, 0)$ *when* $m = 4$ *and* $n = 3$.
(v) $(3m, 10m, 10, 3, 1, 0, 2, 1, 0, 2, 0, 1, 0, 0)$ *when* $m \geq 5$ *and* $n = 3$.
(vi) $(16, 128, 24, 3, 2, 2, 2, 2, 0, 4, 4, 4, 4, 0)$ *when* $m = 4$ *and* $n = 4$.
(vii) $(4n, 32n, 24, 3, 2, 1, 2, 2, 0, 4, 2, 4, 2, 0)$ *when* $m = 4$ *and* $n \geq 5$.
(viii) $(4m, 32m, 24, 3, 2, 2, 2, 1, 0, 4, 4, 2, 2, 0)$ *when* $m \geq 5$ *and* $n = 4$.
(ix) $(mn, 8mn, 24, 3, 2, 1, 2, 1, 0, 4, 2, 2, 1, 0)$ *when* $m \geq 5$ *and* $n \geq 5$.

Proof. Let $H = \mathcal{H}_1(C_m) \boxtimes \mathcal{H}_1(C_n)$. Order of hypergraph H is mn. From Definition 2.8 [13], $E(H) = E\Big(\mathcal{H}_1(C_m)\square\mathcal{H}_1(C_n)\Big) \cup E\Big(\mathcal{H}_1(C_m) \times \mathcal{H}_1(C_n)\Big)$. That is, edge set of hypergraph H can be partitioned into two subsets E_1 and E_2 where E_1 contains cartesian product hyperedges and E_2 contains direct product hyperedges. These two sets , E_1 and E_2 are clearly disjoint. Therefore, $|E(H)| = |E_1| + |E_2|$.
$|E(H)| = |E(\mathcal{H}_1(C_m)\square\mathcal{H}_1(C_n))| + |E(\mathcal{H}_1(C_m) \times \mathcal{H}_1(C_n))|$.
For $m \geq 4$ and $n \geq 4$, $|E(H)| = 8mn$.
When $m = 3$ and $n \geq 4$, $|E(H) = 10n$ annd for $m \geq 4$ and $n = 3$, $|E(H)| = 10m$.
When $n = 3$ and $m = 3$, $|E(H)| = 12$.

We now find parameters of the design taking hyperedges of H as blocks. Since H is 3-uniform hypergraph from Remark 2.1, block size k is 3. As edge sets E_1 and E_2 are disjoint, repetition number of this design is sum of the repetition numbers of the designs obtained in Theorem 3.1 and Theorem 3.2. When both m and n are greater than 3, repetition number is 24. For $m = 3$ and $n \geq 4$ (or $m \geq 4$ and $n = 3$) repetition number becomes 10 and when both m and n are 3, it reduces to 4.

Since, edge sets E_1 and E_2 are disjoint, we get 10-class association scheme for this design, that is, we get 10 λ_i's. $\lambda_1 - \lambda_5^*$ are obtained from cartesian product edges of hypergraph H as explained in proof of Theorem 3.1 and $\lambda_6 - \lambda_{10}^*$ are obtained from direct product edges of H as explained in proof of Theorem 3.2. Thus nine different cases arise depending on the values of m nad n where design parameters follow from Theorem 3.1 and Theorem 3.2.

Each vertex will have $(\lfloor m/2 \rfloor \times \lfloor n/2 \rfloor) + (\lfloor m/2 \rfloor + \lfloor n/2 \rfloor)$ number of assocciates in H. Therefore, parameters of second kind differ depending on diameters of C_m and C_n as explained in Remark 3.1. For explicit values of m and n, parameters of second kind can be obtained in similar way as illustrated in Remark 3.1. □

Last subsection deals with another important product - the lexicographic product.

3.4. **Lexicographic product of hypergraphs.** This is one of the four standard products that differs from the other three in being non-commutative. Let us give an illustration to understand lexicographic product in detail.
Illustration: Consider cycles C_3 and C_4 given in Figures 3 and 4, respectively. Let $H = \mathcal{H}_1(C_3) \circ \mathcal{H}_1(C_4)$ be a hypergraph. Then from Definition 2.9 [13],
$V(H) = \{a_1b_1, a_1b_2, a_1b_3, a_1b_4, a_2b_1, a_2b_2, a_2b_3, a_2b_4, a_3b_1, a_3b_2, a_3b_3, a_3b_4\}$ and
$E(H) = \Big\{\{a_1b_1, a_2b_1, a_3b_1\}, \{a_1b_1, a_2b_1, a_3b_2\}, \{a_1b_1, a_2b_1, a_3b_3\}, \{a_1b_1, a_2b_1, a_3b_4\},$
$\{a_1b_1, a_2b_2, a_3b_1\}, \{a_1b_1, a_2b_2, a_3b_2\}, \{a_1b_1, a_2b_2, a_3b_3\}, \{a_1b_1, a_2b_2, a_3b_4\}, \{a_1b_1, a_2b_3, a_3b_1\},$
$\{a_1b_1, a_2b_3, a_3b_2\}, \{a_1b_1, a_2b_3, a_3b_3\}, \{a_1b_1, a_2b_3, a_3b_4\}, \{a_1b_1, a_2b_4, a_3b_1\}, \{a_1b_1, a_2b_4, a_3b_2\},$
$\{a_1b_1, a_2b_4, a_3b_3\}, \{a_1b_1, a_2b_4, a_3b_4\}, \{a_1b_2, a_2b_1, a_3b_1\}, \{a_1b_2, a_2b_1, a_3b_2\}, \{a_1b_2, a_2b_1, a_3b_3\},$
$\{a_1b_2, a_2b_1, a_3b_4\}, \{a_1b_2, a_2b_2, a_3b_1\}, \{a_1b_2, a_2b_2, a_3b_2\}, \{a_1b_2, a_2b_2, a_3b_3\}, \{a_1b_2, a_2b_2, a_3b_4\},$
$\{a_1b_2, a_2b_3, a_3b_1\}, \{a_1b_2, a_2b_3, a_3b_2\}, \{a_1b_2, a_2b_3, a_3b_3\}, \{a_1b_2, a_2b_3, a_3b_4\}, \{a_1b_2, a_2b_4, a_3b_1\},$
$\{a_1b_2, a_2b_4, a_3b_2\}, \{a_1b_2, a_2b_4, a_3b_3\}, \{a_1b_2, a_2b_4, a_3b_4\}, \{a_1b_3, a_2b_1, a_3b_1\}, \{a_1b_3, a_2b_1, a_3b_2\},$
$\{a_1b_3, a_2b_1, a_3b_3\}, \{a_1b_3, a_2b_1, a_3b_4\}, \{a_1b_3, a_2b_2, a_3b_1\}, \{a_1b_3, a_2b_2, a_3b_2\}, \{a_1b_3, a_2b_2, a_3b_3\},$
$\{a_1b_3, a_2b_2, a_3b_4\}, \{a_1b_3, a_2b_3, a_3b_1\}, \{a_1b_3, a_2b_3, a_3b_2\}, \{a_1b_3, a_2b_3, a_3b_3\}, \{a_1b_3, a_2b_3, a_3b_4\},$
$\{a_1b_3, a_2b_4, a_3b_1\}, \{a_1b_3, a_2b_4, a_3b_2\}, \{a_1b_3, a_2b_4, a_3b_3\}, \{a_1b_3, a_2b_4, a_3b_4\}, \{a_1b_4, a_2b_1, a_3b_1\},$
$\{a_1b_4, a_2b_1, a_3b_2\}, \{a_1b_4, a_2b_1, a_3b_3\}, \{a_1b_4, a_2b_1, a_3b_4\}, \{a_1b_4, a_2b_2, a_3b_1\}, \{a_1b_4, a_2b_2, a_3b_2\},$
$\{a_1b_4, a_2b_2, a_3b_3\}, \{a_1b_4, a_2b_2, a_3b_4\}, \{a_1b_4, a_2b_3, a_3b_1\}, \{a_1b_4, a_2b_3, a_3b_2\}, \{a_1b_4, a_2b_3, a_3b_3\},$
$\{a_1b_4, a_2b_3, a_3b_4\}, \{a_1b_4, a_2b_4, a_3b_1\}, \{a_1b_4, a_2b_4, a_3b_2\}, \{a_1b_4, a_2b_4, a_3b_3\}, \{a_1b_4, a_2b_4, a_3b_4\},$
$\{a_1b_1, a_1b_2, a_1b_3\}, \{a_1b_2, a_1b_3, a_1b_4\}, \{a_1b_1, a_1b_3, a_1, b_4\}, \{a_1b_1, a_1b_2, a_1b_4\}, \{a_2b_1, a_2b_2, a_2b_3\},$
$\{a_2b_2, a_2b_3, a_2b_4\}, \{a_2b_1, a_2b_3, a_2, b_4\}, \{a_2b_1, a_2b_2, a_2b_4\}, \{a_3b_1, a_3b_2, a_3b_3\}, \{a_3b_2, a_3b_3, a_3b_4\},$
$\{a_3b_1, a_3b_3, a_3b_4\}, \{a_3b_1, a_3b_2, a_3b_4\}\Big\}.$

Now we give association scheme for the design arising from lexicographic product of hypergraphs $\mathcal{H}_1$ of two cycles C_m and C_n.
The number of blocks containing a pair of vertices whose
(i) first coordinates are at distance 1 in C_m and no restrictions on the second coordinates is λ_1.
(ii) first coordinates are at distance 2 in C_m and no restrictions on the second coordinates is λ_2.
(iii) first coordinates are same and second coordinates are at distance 1 in C_n is λ_3.
(iv) first coordinates are same and second coordinates are at distance 2 in C_n is λ_4.
(v) atleast one of the coordinates are at distance greater than 2 in their respective graphs is λ_5^*.

Based on the above association scheme, following result is obtained.

Theorem 3.4. *The collection of all hyperedges of lexicographic product,* $H = \mathcal{H}_1(C_m) \circ \mathcal{H}_1(C_n)$*, of hypergraphs* $\mathcal{H}_1$ *of two cycles* C_m *and* C_n *forms a partially balanced incomplete block (PBIB)-design with 5-class association scheme having parameters* (v, b, r, k, λ_i) *for* $1 \le i \le 5$ *as follows.*

(i) $(9, 30, 10, 3, 3, 0, 1, 0, 0)$ *when* $m = 3$ *and* $n = 3$.
(ii) $(12, 76, 19, 3, 4, 0, 2, 2, 0)$ *when* $m = 3$ *and* $n = 4$.
(iii) $(3n, n^3 + 3n, n^2 + 3, 3, n, 0, 2, 1, 0)$ *when* $m = 3$ *and* $n \ge 5$.
(iv) $(12, 112, 28, 3, 6, 6, 1, 0, 0)$ *when* $m = 4$ *and* $n = 3$.
(v) $(3m, 28m, 28, 3, 6, 3, 1, 0, 0)$ *when* $m \ge 5$ *and* $n = 3$.
(vi) $(16, 272, 51, 3, 8, 8, 2, 2, 0)$ *when* $m = 4$ *and* $n = 4$.
(vii) $(4n, 4n^3 + 4n, 3n^2 + 3, 3, 2n, 2n, 2, 1, 0)$ *when* $m = 4$ *and* $n \ge 5$.
(viii) $(4m, 68m, 51, 3, 8, 4, 2, 2, 0)$ *when* $m \ge 5$ *and* $n = 4$.
(ix) $(mn, mn^3 + mn, 3n^2 + 3, 3, 2n, n, 2, 1, 0)$ *when* $m \ge 5$ *and* $n \ge 5$.

Proof. Let $H = \mathcal{H}_1(C_m) \circ \mathcal{H}_1(C_n)$.

In order to count the number of hyperedges in hypergraph H, we partition the set of hyperedges $E(H)$ into two subsets E_1 and E_2 where
$E_1 = \Big\{\{a_1b_1, a_2b_2, a_3b_3\} \mid \{a_1, a_2, a_3\} \in E(\mathcal{H}_1(C_m))$ and $(b_1, b_2, b_3) \in V(\mathcal{H}_1(C_n))\Big\}$ and

$E_2 = \Big\{\{ab_1, ab_2, ab_3\} \mid a \in V(\mathcal{H}_1(C_m))$ and $\{b_1, b_2, b_3\} \in E(\mathcal{H}_1(C_n))\Big\}$.

For a particular hyperedge $\{a_1, a_2, a_3\} \in E(\mathcal{H}_1(C_m))$, we can consider any combinations of $b_i \in V(\mathcal{H}_1(C_n))$ to form a hyperedge of H which fall in set E_1. Since $|V(\mathcal{H}_1(C_n))|$ is n, we get a total of n^3 hyperedges in H with a fixed hyperedge $\{a_1, a_2, a_3\} \in E(\mathcal{H}_1(C_m))$. Since there are $|E(\mathcal{H}_1(C_m))|$ number of hyperedges in $\mathcal{H}_1(C_m)$, we get $|E_1| = n^3 \times |E(\mathcal{H}_1(C_m))|$. Each hyperedge in E_2 is such that first coordinates of vertices are same and set of second coordinates form a hyperedge in $\mathcal{H}_1(C_n)$. Hence, there are $m \times |E(\mathcal{H}_1(C_n))|$ number of hyperedges in E_2. Therefore, $|E(H)| = |E_1| + |E_2|$, that is, $|E(H)| = \Big(n^3 \times |E(\mathcal{H}_1(C_m))|\Big) + \Big(m \times |E(\mathcal{H}_1(C_n))|\Big)$.

For $m \geq 4$ and $n \geq 4$, $|E(H)| = n^3 m + mn$.

When $m = 3$ and $n \geq 4$, $|E(H)| = n^3 + 3n$ and for $m \geq 4$ and $n = 3$, $|E(H)| = 28m$.

When $m = 3$ and $n = 3$, $|E(H)| = 30$.

Taking each hyperedge of H to be a block of a design, we now find parameters of the design obtained. From Remark 2.1, it is clear that block size k is 3 as H is 3-uniform. Next, we find repetition number of the design. With a fixed hyperedge $\{a_1, a_2, a_3\} \in E(\mathcal{H}_1 C_m)$ and any $b_i \in V(\mathcal{H}_1(C_n))$, there are n^2 number of hyperedges in H containing the vertex $a_1 b_1$. There are atmost three hyperedges in $\mathcal{H}_1(C_m)$ containing the vertex a_1. Hence, we get atmost $3n^2$ hyperedges in $E(H)$ containing the vertex $a_1 b_1$ and they lie in set E_1. Similarly, there are atmost three hyperedges in $E(\mathcal{H}_1(C_n))$ containing vertex b_1. Hence, we get atmost 3 hyperedges containing $a_1 b_1$ in E_2. Hence repetition number is atmost $3n^2 + 3$. When $m = 3$ and $n \geq 4$, repetition number is $n^2 + 3$ and when $m \geq 4$ and $n = 3$, repetition number becomes 28 as it depends only on the value of n. When both m and n are equal to 3, repetition number reduces to 10.

To obtain the values of λ_i, for $1 \leq i \leq 5$, we consider different cases. Let $a_i \in V(\mathcal{H}_1(C_m))$ and $b_i \in V(\mathcal{H}_1(C_n))$. λ_1 gives the number of blocks in H containing a pair of vertices $a_1 b_i$ and $a_2 b_j$ where a_1 and a_2 are vertices at distance 1 in C_m and b_i and b_j are random vertices of C_n. Clearly, n times the number of hyperedges containing a pair of vertices at distance 1 in C_m gives the value of λ_1. If vertices a_1 and a_3 are at distance 2 in C_m and b_i and b_j are random vertices of C_n, then the number of blocks with vertices $a_1 b_i$ and $a_3 b_j$ gives the value of λ_2. Similar to the above case, n times the number of hyperedges containing a pair of vertices at distance 2 in C_m gives the value of λ_2. Number of blocks containing the pairs of vertices of the form $a_1 b_1$ and $a_1 b_2$ where $a_1 \in V(\mathcal{H}_1(C_m))$ and vertices b_1 and b_2 are at distance 1 in C_n gives the value of λ_3. λ_4 is the number of blocks containing a pair of vertices $a_1 b_1$ and $a_1 b_3$ where $a \in V(\mathcal{H}_1(C_m))$ and vertices b_1 and b_3 are at distance 2 in C_n. Number of blocks containing a pair of vertices, say, $(a_p b_i, a_q b_j)$ (or $(a_i b_k, a_i b_l)$) where a_p and a_q are vertices at distance greater than 2 in C_m and b_i and b_j are random vertices in C_n (or b_k and b_l are vertices at distance greater than 2 in C_n) is λ_5. It is obvious that λ_5^* is 0 as there is no hyperedge in hypergraph $\mathcal{H}_1$ of cycles containing a pair of vertices at distance greater than 2.

Case i) : When $m = 3$ and $n = 3$.

As $\mathcal{H}_1(C_3)$ has a single hyperedge and all vertices are mutually adjacent to each other in C_3, we get design parameters as (9, 30, 10, 3, 3, 0, 1, 0, 0).

Case ii) : $m = 3$ and $n = 4$.

There are two hyperedges each in $\mathcal{H}_1(C_4)$ containing a pair of vertices at distance 1 as well as 2 in C_4. Hence, we get the value of λ_3 and λ_4 as 2. Therefore, design parameters are (12, 76, 19, 3, 4, 0, 2, 2, 0).

Case iii) : $m = 3$ and $n \geq 5$.

There are two hyperedges in hypergraph $\mathcal{H}_1$ of cycle C_n containing a pair of vertices at distance 1 in C_n and a single hyperedge containing a pair of vertices at distance 2 in C_n. Hence the values of λ_3 and λ_4 are 2 and 1 respectively. Thus, the design parameters are $(3n, n^3 + 3n, n^2 + 3, 3, n, 0, 2, 1, 0)$.

Case iv) : $m = 4$ *and* $n = 3$.
Since m is 4, values of λ_1 and λ_2 are 6 and 6, respectively, and hence, the design parameters are (12, 112, 28, 3, 6, 6, 1, 0, 0).

Case v) : $m \geq 5$ *and* $n = 3$.
As there are two hyperedges in hypergraph $\mathcal{H}_1$ of cycle C_m containing a pair of vertices at distance 1 in C_m and a single hyperedge containing a pair of vertices at distance 2 in C_m, the values of λ_1 and λ_2 are 6 and 3 respectively. Hence, the parameters of design are $(3m, 28m, 28, 3, 6, 3, 1, 0, 0)$.

Case vi) : $m = 4$ *and* $n = 4$.
Since n is 4, we get 2 as the values of both λ_3 and λ_4. Therefore, parameters of the design are (16, 272, 51, 3, 8, 8, 2, 2, 0).

Case vii) : $m = 4$ *and* $n \geq 5$.
In this case, we get the values of λ_1 and λ_2 as $2n$ as m is 4. Thus, design parameters are $(4n, 4n^3 + 4n, 3n^2 + 3, 3, 2n, 2n, 2, 1, 0)$.

Case viii) : $m \geq 5$ *and* $n = 4$,
Since $m \geq 5$, λ_1 is 8 and λ_2 is 4. Parameters of the design obtained is $(4m, 68m, 51, 3, 8, 4, 2, 2, 0)$.

Case ix) : $m \geq 5$ *and* $n \geq 5$.
This is the general case for lexicographic product of hypergraphs $\mathcal{H}_1$ of cycles of order greater than 4. Therefore, the design parameters are $(mn, mn^3 + mn, 3n^2 + 3, 3, 2n, n, 2, 1, 0)$.

In the association scheme given above, λ_5^* includes the number of blocks containing a pair of vertices where first coordinates are at distance greater than 2 upto $\lfloor m/2 \rfloor$ in C_m and no restrictions on second coordinates, or pair of vertices whose first coordinates are same and second coordinates are at distance greater than 2 upto $\lfloor n/2 \rfloor$ in C_n. Therefore, parameters of second kind differ depending on diameters of C_m and C_n as explained in Remark 3.1 and hence for explicit values of m and n, we can get parameters of second kind in a similar manner as illustrated in Remark 3.1. □

Here, we observe that the number of hyperedges in hypergraph $H = \mathcal{H}_1(C_m) \circ \mathcal{H}_1(C_n)$ depends on the cube of n value. Also the degree of each vertex in H depends only on the value of n. Thus, we can conclude that hypergraphs obtained from lexicographic product of hypergraphs of two cycles are not isomorphic, that is $\mathcal{H}_1(C_m) \circ \mathcal{H}_1(C_n)$ is not isomorphic to $\mathcal{H}_1(C_n) \circ \mathcal{H}_1(C_m)$.

3.5. **Application.** We now give one of the applications of designs arising from cartesian product of graphs. Suppose vertices of graph A represent dominant traits and vertices of graph B represent recessive traits of a plant, then their cartesian product represents all possible traits that can be observed in the offsprings based on Mendelian genetics. Taking these traits as experimental units, incomplete block designs can be constructed for crop sequence experiments for the next generation offsprings. By applying the above results and Algorithm 1, we readily get the applications in agricultural sciences. Similar applications can be envisaged for different products considered above, wherever cross property considerations are involved in multi element sets, for example groups of people from different backgrounds, ethnicity, culture, social network groups, etc. in demography studies.

4. Conclusions

Hypergraph theory finds a lot of applications in real world problems such as to model gene interactions, computer networks, visual classification and social media. Products of hypergraphs find their applications in chemistry, computer science and networking. Applications of graph products naturally extend to hypergraph products as well. In this paper, we have considered hypergraph $\mathcal{H}_1$ of a cycle which is obtained by taking closed neighbourhood of each vertex of the cycle as hyperedges. We have obtained fundamental product hypergraphs viz. cartesian product, direct product, strong product and lexicographic product of cycles using closed neighbourhoods. We have defined association scheme for each product explicitly and obtained PBIB-designs arising from these hypergraph products, where hyperedges are taken as blocks. We have also given an algorithm to construct cartesian product hypergraph along with the associates of each vertex which helps in determining the design parameters. Just by tweaking steps in this alogorithm, one can easily construct algorithmically product hypergraphs, association schemes and repective PBIB-designs. To maintain uniform block sizes for designs, we have used products of hypergraphs of cycles. This ensures $\mathcal{H}_1(C_n)$ to be 3-uniform as cycles are 2-regular. As some of the designs constructed in this paper have less replication number, they can be considered to be used in various experiments, because of their appropriateness in situations where experiments are constrained of resources. Such designs have high efficiency and can be beneficial in varietal trials in the field of agriculture where a large number of cultivars are being tested. Hence our results have paramount importance in real world applications with strong theoretical background. Further research can be carried out in obtaining new design parameters in similar way from other hypergraph constructions and their respective products over different classes of graphs.

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